Forms as sums of powers of lower degree forms

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Let $H_d(\mathbb{C}^n)$ denote the vector space of complex forms in n variables with degree d. How can a form of degree $m = rt$ be written as a sum of t-th powers of forms of degree r? More specifically, given $p \in H_m(\mathbb{C}^n)$, what is the smallest number N so that there exist forms $f_j \in H_r(\mathbb{C}^n)$ satisfying

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- I haven't found much literature on the subject. Please enlighten me.

It might be worth mentioning that if $t = 2$, the Gram matrix method used in studying Hilbert's 17th problem and sums of squares still applies; the matrix still needs to lie in a certain subspace based on the coefficients of p, but it no longer needs to be semidefinite.

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The Motzkin polynomial $x^6 + y^4z^2 + y^2z^4 - 3x^2y^2z^2$ is famously not a sum of squares over $\mathbb{R}$, but, as it stands, it a sum of 4 monomial squares over $\mathbb{C}$, and it is a sum of no fewer if the only allowable monomials are $\{x^3, xyz, y^2z, yz^2\}$, as in the real case. However, in the absence of "order" there is no reason to *a priori* exclude, for example, "y^3" from a summand.

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We restrict our attention in this talk to binary forms, in part because the rank case was completely settled there by Sylvester. The detailed results for powers of linear forms are a goal for the study of powers of higher degree forms. We begin with an auto-plagiaristic look at Sylvester's algorithm. Apologies to anyone who has seen the next few pages before at previous talks.

Theorem (Sylvester, 1851)

Suppose $p(x,y) = \sum_{j=0}^{d} \binom{d}{j} a_j x^{d-j} y^j \in F[x,y] \subset \mathbb{C}[x,y]$ *and* $h(x,y) = \sum_{t=0}^{r} c_t x^{r-t} y^t = \prod_{j=1}^{r} (\beta_j x - \alpha_j y)$ *is a product of pairwise distinct linear factors,* $\alpha_j, \beta_j \in F$*. Then there exist* $\lambda_k \in F$ *so that*

$$p(x,y) = \sum_{k=1}^{r} \lambda_k (\alpha_k x + \beta_k y)^d$$

if and only if

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-r} & a_{d-r+1} & \cdots & a_d \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

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- Since $(\beta \frac{\partial}{\partial x} - \alpha_j \frac{\partial}{\partial y})$ kills $(\alpha x + \beta y)^d$, if $h(D)$ is defined to be $\prod_{j=1}^r (\beta_j \frac{\partial}{\partial x} - \alpha_j \frac{\partial}{\partial y})$, then

$$h(D)p = \sum_{m=0}^{d-r} \frac{d!}{(d-r-m)!m!} \left(\sum_{i=0}^{d-r} a_{i+m} c_i \right) x^{d-r-m} y^m$$

The coefficients of $h(D)p$ are, up to multiple, the rows in the matrix product, so the matrix condition is $h(D)p = 0$. Each linear factor in $h(D)$ kills a different summand, and dimension counting takes care of the rest.

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- If h has repeated factors, see Gundelfinger's Theorem (1886). A factor $(\beta x - \alpha y)^\ell$ gives a summand $(\alpha x + \beta y)^{d+1-\ell}q$, where q is an arbitrary form of degree $\ell - 1$.

Here is an example of Sylvester's Theorem in action. Let

$$p(x,y) = x^3 + 12x^2y - 6xy^2 + 10y^3 =$$

$$\binom{3}{0}\cdot 1\ x^3 + \binom{3}{1}\cdot 4\ x^2y + \binom{3}{2}\cdot(-2)xy^2 + \binom{3}{3}\cdot 10\ y^3$$

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and $2x^2 - xy - y^2 = (2x+y)(x-y)$, so that

$$p(x,y) = \lambda_1(x-2y)^3 + \lambda_2(x+y)^3.$$

In fact, $p(x,y) = -(x-2y)^3 + 2(x+y)^3$.

The next simple example is $p(x,y) = 3x^2y$. Note that

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies c_0 = c_1 = 0$$

so that h would have to have repeated factors, and p is not a sum of two cubes. Similarly, $x^{d-1}y$ requires d d-th powers.

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It can be proved in a similar way that a cubic is a sum of two cubes, unless it has a square factor and isn't a cube. We'll use this later.

For later reference, it is easy to check if p has rank two over $\mathbb{C}$.

Corollary

Suppose $p(x,y) = \sum_{j=0}^{d} \binom{d}{j} a_j x^{d-j} y^j$. *Then* p *is a sum of two* d*-th powers of linear forms over* $\mathbb{C}$ *if and only if*

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{d-2} & a_{d-1} & a_d \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

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As a side-note, Sylvester's Theorem allows one to compute the rank of a form over different fields: for example, the quintic $3x^5 - 20x^3y^2 + 10xy^4$ is a sum of three 5-th powers over $\mathbb{Q}[i]$, four 5-th powers over $\mathbb{Q}[\sqrt{-2}]$ and five 5-th powers over any real field.

If $d = 2s - 1$ and $r = s$, then the matrix in Sylvester's Theorem is $s \times (s + 1)$ and has a non-trivial null-vector. The corresponding h has distinct factors unless its discriminant vanishes. If $d = 2s$ and $r = s$, then the matrix is square, and for fixed $\ell = \alpha_0 x + \beta_0 y$, there exists λ so that $p(x, y) - \lambda \ell^{2s}$ has a matrix with a non-trivial null-vector, generally corresponding to h with distinct factors.

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Theorem (Sylvester's Theorem, canonical form version)

(i) A general binary form p of odd degree $2s-1$ can be written as

$$p(x,y) = \sum_{j=1}^{s} (\alpha_j x + \beta_j y)^{2s-1}.$$

(ii) A general binary form p of even degree $2s$ can be written as

$$p(x,y) = \lambda(\alpha_0 x + \beta_0 y)^{2s} + \sum_{j=1}^{s} (\alpha_j x + \beta_j y)^{2s}.$$

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Theorem

The representations of $(x^2+y^2)^t$ as a sum of $t+1$ $2t$-th powers are given by

$$\binom{2t}{t}(x^2+y^2)^t$$

$$= \frac{1}{t+1}\sum_{j=0}^{t}\left(\cos\left(\tfrac{j\pi}{t+1}+\theta\right)x+\sin\left(\tfrac{j\pi}{t+1}+\theta\right)y\right)^{2t},$$

$$\theta \in \mathbb{C}.$$

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The only powers which never appear above are $(x \pm iy)^{2t}$. The earliest version I have found of this identity is for real θ, by Avner Friedman, from the 1950s.

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On the next page, the various versions of this for binary forms of even degree and powers of quadratics are worked out.

Corollary

(i) A general binary form of degree $d = 6s$ *can be written as*

$$(\lambda x)^{6s} + \sum_{j=1}^{2s} (\alpha_j x^2 + \beta_j xy + \gamma_j y^2)^{3s}$$

(ii) A general binary form of degree $d = 6s + 2$ *can be written as*

$$\sum_{j=1}^{2s+1} (\alpha_j x^2 + \beta_j xy + \gamma_j y^2)^{3s+1}.$$

(iii) A general binary form of degree $d = 6s + 4$ *can be written as*

$$(\lambda_1 x^2 + \lambda_2 y^2)^{3s+2} + \sum_{j=1}^{2s+1} (\alpha_j x^2 + \beta_j xy + \gamma_j y^2)^{3s+2}$$

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The simplest examples are even forms and symmetric forms. But if p is even, then $p(x, y) = q(x^2, y^2)$, where $\deg q = d$ and one expects q to be a sum of around $\frac{1}{2}d$ d-th powers of linear forms, from which p inherits a representation as a sum of $\frac{1}{2}d$ d-th powers of even quadratic forms, so that's going to be smaller than average. If p is symmetric, then $p = q(xy, (x + y)^2)$, and the same argument applies. We do not yet have a good candidate to be the poster child for forms which require a lot of d-th powers of quadratic forms, let alone a nice parameterization of any set of solutions.

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$$p^2 + q^2 = (p + iq)(p - iq)$$

$$fg = \left(\frac{f+g}{2}\right)^2 - \left(\frac{f-g}{2}\right)^2 = \left(\frac{f+g}{2}\right)^2 + \left(\frac{if-ig}{2}\right)^2$$

$$p^2 + q^2 = (\cos\theta \cdot p + \sin\theta \cdot q)^2 + (-\sin\theta \cdot p + \cos\theta \cdot q)^2.$$

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A binary form of degree $2d$ has in general $2d$ distinct linear factors, and these can be divided into a pair of forms of degree d in $\binom{2d-1}{d-1}$ ways. Each of these leads to a sum of two squares. (Repeated factors reduce this number and, unlike the real case, conjugate factors do not have to be split up.) The action of the orthogonal group on sums of two squares plays in too.

The coefficients of the sums of two cubes $(\alpha_i x^2 + \beta_i xy + \gamma_i y^2)^3$ give seven forms in the six variables, and so satisfy a non-trivial polynomial, probably an invariant. Until a highbrow condition can be given explicitly, we present two simple criteria.

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Theorem

Suppose $p \in H_6(\mathbb{C}^2)$. Here are two necessary and sufficient conditions for p to be sum of two cubes of quadratics:

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(i) $p = f_1 f_2 f_3$, where the f_i's are linearly dependent but non-proportional quadratic forms.
(ii) There either exists a linear change of variables so that $p(ax + by, cx + dy) = g(x^2, y^2)$, or $p = \ell^3 g$ for some linear form ℓ. Here, g is a cubic which is a sum of two cubes (not $\ell_1^2 \ell_2$.)

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The proofs of each of these criteria give more general results.

The first case is actually a theorem about sums of two cubes.

Theorem

Suppose $F \in \mathbb{C}[x_1, \dots, x_n]$. Then F is a sum of two cubes in $\mathbb{C}[x_1, \dots, x_n]$ if and only if it is itself a cube, or has a factorization $F = G_1 G_2 G_3$, into pairwise non-proportional factors.

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Proof.

First observe that, with ω denoting a primitive cube root of unity,

$$F = G^3 + H^3 = (G + H)(G + \omega H)(G + \omega^2 H),$$
$$(G + H) + \omega(G + \omega H) + \omega^2(g + \omega^2 H) = 0.$$

If two of the factors $G + \omega^j H$ are proportional, then so are G and H, and hence F is a cube. Conversely, if F has such a factorization, write $F = G_1 G_2(\alpha G_1 + \beta G_2)$, where $\alpha\beta \neq 0$. An application of Sylvester's Theorem shows that $xy(\alpha x + \beta y)$ is always a sum of two cubes of linear forms. Plug in G_1 and G_2 to get F. $\square$

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Lemma

Two quadratic forms $q_1(x, y)$ and $q_2(x, y)$ either have a common linear factor, or can be simultaneously diagonalized; that is, $q_j(ax + by, cx + dy) = \rho_j x^2 + \sigma_j y^2$.

In any particular case, if $\deg F = 3r$, there are only finitely many ways to write F as a product of three factors of degree r, so verifying this condition is algorithmic. Applied algebraic geometry! The second case uses a simple lemma whose proof is omitted.

Lemma

Two quadratic forms $q_1(x, y)$ and $q_2(x, y)$ either have a common linear factor, or can be simultaneously diagonalized; that is, $q_j(ax + by, cx + dy) = \rho_j x^2 + \sigma_j y^2$.

Thus, if $p = q_1^t + q_2^t$, where q_j is quadratic, then either the q_j's have a common linear factor (and $p = \ell^t g$, where g is a sum of two linear t-th powers), or after a linear change of variables,

$$p(ax + by, cx + dy) = \sum_{j=1}^{2} (\rho_j x^2 + \sigma_j y^2)^t;$$

That is, $p(ax + by, cx + dy) = g(x^2, y^2)$, where g again is a sum of two linear t-th powers (typical for $t = 3$, not for $t > 3$.)

Finding if p is even after a change of variables is also algorithmic.

$$p(x,y) = \prod_{j=0}^{2d-1} (x - \lambda_j y) \implies$$

$$p(ax+by, cx+dy) = p(a,-c) \prod_{j=0}^{2d-1} \left(x - \left(\frac{\lambda_j d - b}{a - \lambda_j c} \right) y \right)$$

$$:= p(a,-c) \prod_{j=0}^{2d-1} (x - \mu_j).$$

Thus, the roots of p (taking ∞ if $y \mid p$) are mapped by a Möbius transformation. If $\tilde{p}(x,y) = p(ax+by, cx+dy)$ is even, then $T(z) = -z$ is an involution on the roots, say $T(\mu_{2j}) = \mu_{2j+1}$. It follows that there is a Möbius transformation U which is also an involution permuting the d pairs of roots of p, to be specific:

$$\lambda_{2j+1} = \frac{2ad - (ad+bc)\lambda_{2j}}{(ad+bc) - 2cd\lambda_{2j}}.$$

Given p, find the roots λ_j, and for each quadruple $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}, \lambda_{i_4}$, define the Möbius transformation U so that $U(\lambda_{i_1}) = \lambda_{i_2}$, $U(\lambda_{i_2}) = \lambda_{i_1}$ and $U(\lambda_{i_3}) = \lambda_{i_4}$ and see if it permutes the others. There are instances in which more than one U may work; for example, if p is both even and symmetric.

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Don't get me wrong. Complications abound. Here's a simple one. Consider the even sextic

$$p(x,y) = 2x^6 - 2x^4y^2 - 2x^2y^4 + 2y^6 = 2(x^2 - y^2)^2(x^2 + y^2).$$

Here, $p(x,y) = g(x^2, y^2)$, where $g(x,y) = 2(x-y)^2(x+y)$ is unfortunately not a sum of two cubes. On the other hand, if $\gamma = \frac{2}{\sqrt{3}} i$, then

$$(x^2 + \gamma xy + y^2)^3 + (x^2 - \gamma xy + y^2)^3 = 2x^6 - 2x^4y^2 - 2x^2y^4 + 2y^6.$$

There can be multiple representations of $p = q_1^t + q_2^t$ for $t \in \{3, 4, 5\}$, but that's really for another talk. I will note that $p(x, y) = x^5y + xy^5$ is some kind of champion, having six different representations as a sum of two cubes. (Twenty-fourth roots of unity play a central role in this.)

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Why is the quadratic case so much harder than the linear case? Here's one reason: in the linear case, $(\alpha x + \beta y)^d$ is killed by $\beta\frac{\partial}{\partial x} - \alpha\frac{\partial}{\partial y}$, and two operators of this shape commute. Although each $(\alpha x^2 + \beta xy + \gamma y^2)^d$ is killed by the non-constant-coefficient $(\beta x + 2\gamma y)\frac{\partial}{\partial x} - (2\alpha x + \beta y)\frac{\partial}{\partial y}$, two operators of this kind do not usually commute. The smallest constant-coefficient differential operator which kills $(\alpha x^2 + \beta xy + \gamma y^2)^d$ has degree $d + 1$; the product of any two of these would kill every form of degree $2d$ and so provide no information.

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Thanks to the organizers for the invitation and to the audience!