Algorithms for Tensor Decomposition via Numerical Homotopy

Session on "Algebraic Geometry of Tensor Decompositions"

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Overview

- Join Varieties
- Numerics and Homotopy Continuation
- Special Structure - Parameter Homotopies/Monodromy
- Closing remarks

Join Varieties

Given points $Q_1, \ldots, Q_k \in \mathbb{P}^n$, we let $< Q_1, \ldots, Q_k >$ denote their linear span.

Let $V_1, \ldots, V_k \subseteq \mathbb{P}^n$ be irreducible projective varieties of dimensions $d_1, \ldots, d_k$.

The *open join* of the varieties, $J^O(V_1, \ldots, V_k)$, is the union of the linear span of k-tuples of points $(Q_1, \ldots, Q_k)$ where $Q_i \in V_i$.

The *join* of the varieties is $J(V_1, \ldots, V_k) = \overline{J^O(V_1, \ldots, V_k)}$.

The expected dimension (and the maximum possible dimension) of $J(V_1, \ldots, V_k)$ is $\min\{m, k - 1 + \sum d_i\}$.

There can be repeated factors among the V_i (I.e. this formulation includes secant varieties).

Let $P \in J^O(V_1, \ldots, V_k)$.

A $(V_1, \ldots, V_k)$-decomposition of P is a k-tuple of points $(Q_1, \ldots, Q_k)$ with $Q_i \in V_i$ and $P \in < Q_1, \ldots, Q_k >$.

Suppose $k - 1 + \sum d_i = n$ and that $J(V_1, \ldots, V_k)$ has the expected dimension.

If $P \in \mathbb{P}^n$ is general then $P \in J^O(V_1, \ldots, V_k)$ and P will have a finite number of $(V_1, \ldots, V_k)$-decompositions.

Problem Formulation:

The goal of this talk is to describe a couple of tools from numerical algebraic geometry and how they can be used to

1) Compute a single $(V_1, \ldots, V_k)$-decomposition of P.

2) Compute all of the $(V_1, \ldots, V_k)$-decompositions of P.

The tools that will be used are:

i) Parameter Homotopy

ii) Monodromy

Homotopy Continuation:

In homotopy continuation, a polynomial ideal, I, is cast as a member of a parameterized family of polynomial ideals one of which has known isolated solutions.

Each of the known isolated solutions is tracked, as the parameter is varied, through a predictor/corrector method.

Some of these paths lead to points which lie numerically close to the algebraic set $V(I)$ determined by I.

The points that are close to $V(I)$ can be refined to lie within any prescribed tolerance of $V(I)$.

Example:

$F(x,t) = \gamma t(x^2 - 1) + (1 - t)(x^2 + 8x + 13)$.

When $t = 1$, we have the equation $x^2 - 1$ which has roots $1, -1$.

When $t = 0$, we have $x^2 + 8x + 13$ which has roots $-4 \pm 5.1235I$.

As t varies from $t = 1$ to $t = 0$, we follow the two solutions $\{1, -1\}$ to the two solutions $\{-4 + 5.1235I, -4 - 5.1235I\}$.

γ is a random complex number of magnitude 1. It ensures that the paths don't run into any problems along the way.

Example:

One can study the twisted cubic $I = (x^2 - wy, y^2 - xz, xy - wz)$ via the complete intersection $J = (x^2 - w^2, y^2 - w^2, z^2 - w^2)$.

This is done through the parametrized family

$$I_t := (1-t)(x^2-wy, y^2-xz, xy-wz)+t\gamma(x^2-w^2, y^2-w^2, z^2-w^2)$$

Where t is a real parameter and γ is a random complex number.

The ideal I_1 is equal to J. It has 8 isolated solutions that are easy to write down.

For each value of t in $(0, 1]$, I_t also has 8 isolated solutions.

Each of the known isolated solutions for I_t can be tracked through a predictor/corrector method as t varies from 1 to 0.

This leads to points which lie close to the algebraic set $V(I)$ determined by I.

The parametrized family can be written

$$(1-t)\begin{bmatrix} x^2 - wy \\ y^2 - xz \\ xy - wz \end{bmatrix} + t\gamma \begin{bmatrix} x^2 - w^2 \\ y^2 - w^2 \\ z^2 - w^2 \end{bmatrix}.$$

In this form, the homotopy can be seen to be a deformation of a regular section of the sheaf

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(2)$$

to a non-regular section of $\mathcal{E}$.

Side Comment:

One commonly used homotopy is to to deform from a regular section to a specific section in $H^0(\mathbb{P}^r, \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^r}(n_i))$.

A general regular section vanishes on $\prod n_i$ isolated points.

If Z is a component of the zero locus of the special section then a certain number of these points lead to Z in the deformation.

The number of points leading to Z is the "equivalence of Z" in the complete intersection.

We can think of this as a function $F(n_1, \ldots, n_r, Z)$.

Let $\dim(Z) = n$, and let $c_0, \ldots, c_n$ be the Chern classes of Z. The k^{th} elementary symmetric function in $n_1, \ldots, n_r$ will be denoted by σ_k. If we let

$$a_k = \sum_{i=0}^{n-k} (-1)^i \binom{r+i}{i} \sigma_{n-k-i}$$

then we get the following numerical formula:

Corollary

$F(n_1, \ldots, n_r, Z) = \sum_{i=0}^{n} a_i \, deg \, c_i.$

Each choice of the n_i leads to a linear constraint on the Chern numbers. Thus a side benefit of homotopy continuation is a method for numerically computing the Chern numbers of a variety.

More generally:

Let $\mathcal{E}$ be a locally free, globally generated, rank n sheaf on $\mathbb{P}^n$.

Let s_1, s_0 be sections of $\mathcal{E}$.

If the zero locus of s_1 is a set of distinct points then for $t \in (0,1]$, the zero locus of $t\gamma s_1 + (1-t)s_0$ is also a set of distinct points.

Each of the points in the zero locus of s_1 can be tracked as t varies.

One can derive formulas relating the Chern numbers of $\mathcal{E}$ to the Chern numbers of the irreducible components of $(s_0)_0$ and the number of paths leading to the component.

Parameter Homotopy

Parameter homotopy is a closely related type of deformation.

You have a family of ideals such that the structure of an ideal in the family is constant on an open set.

You have a method of varying along a real path in the family such that for $t \in (0, 1]$, you are on this open set.

Deformation through the space of sections of a sheaf is an example of parameter homotopy.

From a polynomial ideal I one can produce a collection of points such that:

- The points are in one to one correspondence with the irreducible components of the algebraic set $V(I)$.
- The points are numerical approximations for general points on the component.
- From an approximation to a general point on a component, approximations to other general points can be produced.
- The numerical approximations can be refined to arbitrary numerical accuracy.

Furthermore, it can be determined if two given points lie on the same irreducible component.

Overview of Approach:

Suppose $k - 1 + \sum d_i = n$ and that $J(V_1, \ldots, V_k)$ has the expected dimension.

Recall that if $P \in \mathbb{P}^n$ is general then $P \in J^O(V_1, \ldots, V_k)$ and P will have a finite number of $(V_1, \ldots, V_k)$-decompositions.

Approach for finding one decomposition:

- Find general points $Q_1, \ldots, Q_k$ with $Q_i \in V_i$.
- Find a general point $Q \in < Q_1, \ldots, Q_k >$.
- Equations can be written down for the entry locus of Q.
- Given $P \in J^O(V_1, \ldots, V_k)$, homotopy the $(V_1, \ldots, V_k)$-decomposition for Q to a $(V_1, \ldots, V_k)$-decomposition for P.

Monodromy:

From one $(V_1, \ldots, V_k)$-decomposition for Q, use monodromy to find all of the other $(V_1, \ldots, V_k)$-decompositions for Q. I.e. find the entry locus of Q via monodromy.

Deform the entire collection of $(V_1, \ldots, V_k)$-decompositions for Q to the entire collection of $(V_1, \ldots, V_k)$-decompositions for P through a parameter homotopy.

Closing comments:

1) What about tensors not on $J^O(V_1, \ldots, V_k)$?

2) How big a problem can be solved?

3) What if $J(V_1, \ldots, V_k)$ does not have dimension n?

4) What if $J(V_1, \ldots, V_k)$ is "defective"?

Thank you!