Decomposition of Tensors of Small Rank

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Tensor Decomposition and Rank

Let $V_1, \ldots, V_k$ be complex vector spaces. A *decomposition* of $f \in V_1 \otimes \ldots \otimes V_k$ is

$$f = \sum_{i=1}^{r} c_i v_{i,1} \otimes \ldots \otimes v_{i,k} \qquad \text{with } c_i \in \mathbb{C}, \quad v_{i,j} \in V_j$$

Definition

$\mathrm{rk}(f)$ is the minimum number of summands in a decomposition of f. A minimal decomposition has $\mathrm{rk}(f)$ summands and it is called CANDECOMP or PARAFAC.

We may assume $c_i = 1$, although in practice it is more convenient to determine $v_{i,k}$ up to scalars, and then solve for c_i.

Symmetric Tensor Decomposition (Waring)

In the case $V_1 = \ldots = V_k = V$ we may consider symmetric tensors $f \in S^d V$. A *Waring decomposition* of $f \in S^d V$ is

$$f = \sum_{i=1}^{r} c_i (l_i)^d \qquad \text{with } l_i \in V$$

with minimal r.

Example: $7x^3 - 30x^2y + 42xy^2 - 19y^3 = (-x + 2y)^3 + (2x - 3y)^3$
$\mathrm{rk}\left(7x^3 - 30x^2y + 42xy^2 - 19y^3\right) = 2$

Applications

- Signal Processing, [Comon],...
- Phylogenetics, [Allman-Rhodes], [Chang],...
- ....

Recent Updates and Questions

- There are MATLAB packages computing *a* decomposition, ([Sidiropolous - Bro],[Kolda et al.]...), working very well when the rank is small, efficiently storing very large tensors, but when the rank is high they often compute a not minimal decomposition.
- Very interesting new techniques, by extending the moment matrices, by [Bernardi-Brachat-Comon-Mourrain-Tsigaridas]
- **[Main Questions]** Is there only one minimal decomposition ? How to compute it ? The problem of efficiently computing a minimal decomposition in any rank is still open.

Weak Defectivity and Uniqueness

Weak defectivity goes back to classical papers by Terracini. Also studied extensively by Chiantini and Ciliberto.
We realized recently that it allows computer experiments regarding the tensor decomposition.

Given $t = \sum_{i=1}^{r} x_i$ the uniqueness of the decomposition of t is implied by the equality

$$\{x \in X | T_x X \subset T_t \sigma_r(X)\} = \{x_1, \ldots, x_r\}$$

The left hand side is called the contact locus and always contains the right hand side by Terracini's Lemma. When equality fails for some t, we say X is weakly defective.

Not weakly defective $\Longrightarrow$ unique decomposition.

The symmetric case: uniqueness in the subgeneric case

Theorem (Sylvester[1851], Chiantini-Ciliberto, Mella, Ballico, [2002-2005])

The general $f \in S^d\mathbb{C}^{n+1}$ of rank s smaller than the generic rank has a unique Waring decomposition, with the only exceptions

- *rank $s = \binom{n+2}{2} - 1$ in $S^4\mathbb{C}^{n+1}$, $2 \leq n \leq 4$: infinitely many decompositions*
- *rank 7 in $S^3\mathbb{C}^5$: infinitely many decompositions*
- *rank 9 in $S^6\mathbb{C}^3$: exactly two decompositions*
- *rank 8 in $S^4\mathbb{C}^4$: exactly two decompositions*

The cases listed in red are the *defective cases*.
The cases listed in blue are the *weakly defective cases*.

The case of $V_1 \otimes V_2 \otimes V_3$

Defective and Weakly Defective examples

Only known examples when the decomposition of the general $f \in V_1 \otimes V_2 \otimes V_3$ ($\dim V_i = n_i + 1$) of subgeneric rank s is NOT UNIQUE are

- unbalanced case, $n_3 \geq n_1 n_2 + 2$, $n_1 n_2 + 2 \leq s \leq \min(n_3 + 1, (n_1 + 1)(n_2 + 1))$ [Catalisano-Geramita-Gimigliano]
- $k = 3$, $(n_1, n_2, n_3) = (2, m, m)$ with m even [Strassen],
- $k = 3$, $(n_1, n_2, n_3) = (2, 3, 3)$, sporadic case [Abo-O-Peterson]
- unbalanced case, rank $s = n_1 n_2 + 1$, $n_3 \geq n_1 n_2 + 1$
- rank 6 $(n_1, n_2, n_3) = (3, 3, 3)$: two decompositions
- rank 8 $(n_1, n_2, n_3) = (2, 5, 5)$, sporadic case [Chiantini-O]: maybe six decompositions

Results on uniqueness, I

Theorem (Chiantini-O. [2011])

The exceptions to uniqueness listed in the previous slide are the only ones in the cases

- *unbalanced*
- $n_i \leq 6$
- $s \leq 6$

Results on uniqueness, II

Main Theorem

- *There is a unique decomposition for tensor of rank s in* $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$
  - *if* $s \leq \frac{3n+1}{2}$ *[Kruskal, 1977] (it may be applied to tensors satisfying the Kruskal's condition)*
  - *if* $s \leq \frac{(n+2)^2}{16}$ *[Chiantini-O., 2011] (it holds for general tensors of given rank, application to specific tensors requires weak defectivity)*
- *Decomposition is unique for general tensor of rank s in* $\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ *(k times) unless* $k = 4$ *or* 5 *(two decompositions). The result is proved for all possible values of s except one. [Bocci-Chiantini-O., 2011]*

Proof uses a generalization of the inductive technique in [AOP] to the weak defectivity setting.

Comparison with Kruskal bound, for general tensors of given rank

a		2	3	4	5	6	7	8	9	10
gen.rank ($a \neq 3$)	$\lceil \frac{a^3}{3a-2} \rceil$	2	4	7	10	14	19	24	30	36
Kruskal bound	$\lfloor \frac{3a-2}{2} \rfloor$	2	3	5	6	8	9	11	12	14
Chiantini-O	$k(a)$	2	3	5	9	13	18	22	27	32

Non Abelian Apolarity, I

Let E be a vector bundle on X, embedded by the very ample line bundle L in $\mathbb{P}(H^0(L)^\vee)$.
Consider the natural morphism

$$H^0(E) \quad \otimes \quad H^0(L)^\vee \quad \xrightarrow{A} \quad H^0(E^\vee \otimes L)^\vee$$

which induces $\forall f \in H^0(L)^\vee$ the linear map

$$A_f \colon H^0(E) \to H^0(E^\vee \otimes L)^\vee$$

Non Abelian Apolarity, II

Theorem (Landsberg-O)

Let $Z = \{x_1, \ldots x_k\} \subset X$ *such that* $H^0(E^\vee \otimes L) \to H^0(E^\vee \otimes L_{|Z})$ *is surjective.*
Let $f = \sum_{i=1}^{k} x_i \in H^0(L)^\vee$. *Then*

$$Z \subseteq \textit{ base locus of } \ker A_f$$

$$rk\ A_f = rk\ E \cdot rk\ f$$

The catalecticant map

When $E = \mathcal{O}(a)$ is a line bundle we get the classical apolarity of XIX century. If $f \in S^d V$ the map A_f becomes the catalecticant map
$C_f \colon S^a\mathbb{C}^{n+1*} \longrightarrow S^{d-a}\mathbb{C}^{n+1}$ [Sylvester, Iarrobino-Kanev]

It is convenient to set $a = \lceil \frac{d}{2} \rceil$. The base locus of $\ker C_f$ is cut out by polynomials of degree a.

Explicit construction of the minors of A_f from a presentation of E

We have a presentation of E

$$\begin{array}{ccccccccc}
\ldots \longrightarrow L_2 \longrightarrow L_1 & & \overset{p_E}{\longrightarrow} & & L_0 \longrightarrow L_{-1} \longrightarrow \ldots \\
 & \searrow & & \nearrow & \\
 & & E & & \\
 & \nearrow & & \searrow & \\
0 & & & & 0
\end{array}$$

We get that A_f factors through the map P_f obtained by differentiating with respect to p_E the catalecticant matrix. Minors of P_f and minors of A_f coincide. The presentation of $E = Q(m)$ on $\mathbb{P}^2$ is

$$\begin{bmatrix} & x_2 & -x_1 \\ -x_2 & & x_0 \\ x_1 & -x_0 & \end{bmatrix}$$

Non Abelian Apolarity for odd degree plane curves

Consider $X = \mathbb{P}^2$, $L = \mathcal{O}(2\delta + 1)$, $f \in S^{2\delta+1}\mathbb{C}^3$.
Set $E = Q(\delta)$ where Q is the quotient bundle (rank two) on $\mathbb{P}^2$.

$$A_f \colon H^0(E) = H^0(Q(\delta)) \to H^0(Q(\delta))^\vee = H^0(E^\vee \otimes L)^\vee$$

is now skew-symmetric.

Block structure of P_f for odd degree plane curves

P_f is represented by the following $3\binom{\delta+2}{2} \times 3\binom{\delta+2}{2}$ matrix, where each depicted block is the $\binom{\delta+2}{2} \times \binom{\delta+2}{2}$ catalecticant of $f_i = \frac{\partial f}{\partial x_i}$.

$$\begin{bmatrix} 0 & C_{f_2} & -C_{f_1} \\ -C_{f_2} & 0 & C_{f_0} \\ C_{f_1} & -C_{f_0} & 0 \end{bmatrix}$$

If (g_0, g_1, g_2) is in the kernel and the rank of f is $\leq \binom{\delta+2}{2}$ then the base locus of the 2-minors of $\begin{bmatrix} x_0 & x_1 & x_2 \\ g_0 & g_1 & g_2 \end{bmatrix}$ give the Waring decomposition of f.

Explicit form of the Sylvester Pentahedral Theorem [Oeding-O]

Theorem (Sylvester)

Given a general $f \in S^3\mathbb{C}^4$ there exist unique $l_1, \dots l_5$ such that $f = \sum_{i=1}^5 l_i^3$.

Let $E = \wedge^2 Q(1)$. We have $A_f \colon H^0(\wedge^2 Q(1)) \to H^0(Q(1))^\vee$.

Algorithm (Oeding-O)

The sections in $\ker A_f$ vanish on $\{l_1, \dots l_5\}$.
Note that $c_3(\wedge^2 Q(1)) = 5$.

Let $f = \sum_{x_i \in Z} x_i$ be general.

[Landsberg-O., [2010]]

If

$$H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^\vee \otimes L) \to H^0(I_{Z^2} \otimes L)$$

is surjective, then the locus $\{f | \mathrm{rk}\ A_f \leq k \cdot \mathrm{rk}\ E\}$ contains $\sigma_k(X)$ as irreducible component.

Application of the Infinitesimal Criterion

Theorem

Let $d = 2\delta + 1$, $a = \lfloor \frac{n}{2} \rfloor$. Let $Z \subset \mathbb{P}^n$ of length $k \leq \binom{\delta+n}{n}$. Then the map

$$H^0(I_Z \otimes \wedge^a Q(\delta)) \otimes H^0(I_Z \otimes \wedge^{n-a} Q(\delta)) \rightarrow H^0(I_{Z^2}(d))$$

is surjective. Then we may apply the Infinitesimal Criterion and we get local equations for $\sigma_k(v_d(\mathbb{P}^n))$.

Thanks !!

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