Toward a Salmon Conjecture

Daniel J. Bates[1] and Luke Oeding[2]

[1]Colorado State University, USA
[2]Universitá Degli Studi di Firenze, Italy --→ UC Berkeley, USA

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...and why even a vegetarian might care...

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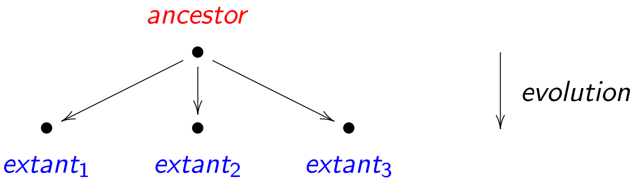
The salmon prize

In 2007, E. Allman offered a prize of Alaskan salmon (!) to whoever finds the defining ideal of

$$\sigma_4\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right).$$

This algebraic variety may be viewed as a statistical model for evolution.



Nucleotides $\{A, C, G, T\}$
Independent extant species
Unknown (hidden) Ancestor
Invariants of this statistical model

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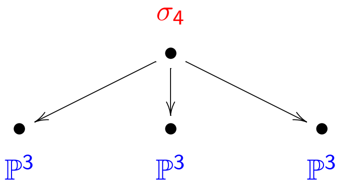

$\Delta \subset \mathbb{PR}^{64} \subset \mathbb{PC}^{64}$

Nucleotides $\{A, C, G, T\} \leftrightarrow \mathbb{PC}^4$.
Independent extant species $\leftrightarrow$ $Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$.
Unknown (hidden) Ancestor $\leftrightarrow$ 4^{th} secant variety.
Invariants of this statistical model $\leftrightarrow$ ideal of the algebraic variety.

The salmon prize

Allman's Motivation: Work of Allman-Rhodes'03 implies that solving the salmon problem would provide all phylogenetic invariants for a whole class of binary evolutionary tree models!

As in this example, nice varieties in spaces of tensors (like secant varieties) appear in several fields outside of mathematics, such as

- algebraic statistics (other problems like this one)
- computational complexity theory (bounding the complexity of algorithms via ranks of tensors)
- signal processing (CDMA protocol for mobile phones)
- physics (quantum information theory and measures of entanglement)
- computer vision (multi-view geometry)
- ... your favorite variety?

Recent history and current status

- [Landsberg–Manivel 2004]: Some equations of $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$ in degrees 5,6,9 in representation theoretic language.
- [Landsberg–Manivel 2008]: Reduced set-theoretic problem for $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^n \times \mathbb{P}^m)$, $n, m \geq 3$ to $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$.
- October 2008 Sturmfels asked for explicit (M2) version of degree 6 equations. (Now an ancillary file on ArXiv [Bates–O. 2011]).
- December 2008 (O–) MSRI Algebraic Statistics Workshop: Conjecture about zero set of degree 6 equations, if confirmed would prove set-theoretic result.
- March 2010 (Friedland): set-theoretic result using degrees 5, 9, 16. Second version corrects proof of Landsberg-Manivel reduction.
- July 2010 (Bates): Numerical Algebraic Geometry (NAG) calculation in Bertini for deg. 6 equations, MSRI conjecture $\Rightarrow$ numerical, set-theoretic result using degrees 5, 6, 9, [Bates–O 2011].
- April 2011 (Friedland–Gross): Explicit equations + previous proof of Friedland: confirm NAG result without numerical methods.
- Ideal theoretic problem is still open.

Secant varieties

Let $A = \{a_i\}, B = \{b_j\}, C = \{c_k\}$, be $\mathbb{C}$-vector spaces, then the tensor product is $A \otimes B \otimes C = \{a_i \otimes b_j \otimes c_k\}$, with coordinates p_{ijk}.

- Segre variety (rank 1 tensors): (Independence model) Defined by

$$\begin{array}{rcl} Seg : \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C & \longrightarrow & \mathbb{P}\big(A \otimes B \otimes C\big) \\ ([a],[b],[c]) & \longmapsto & [a \otimes b \otimes c]. \end{array}$$

- The r^{th} secant variety of a variety $X \subset \mathbb{P}^n$: (Mixture model)

$$\sigma_r(X) = \overline{\bigcup_{x_1,\dots,x_r \in X} \mathbb{P}(\operatorname{span}\{x_1,\dots,x_r\})} \subset \mathbb{P}^n.$$

A useful reduction

Theorem (Landsberg–Manivel '08, Friedland'10)

$\sigma_4\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right)$ *is the zero set of:*

1. $M_5 = \{$ *(Strassen's [1983] degree* 5 *commutation conditions)* $\}$
2. *Equations inherited from* $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$

- Key point: It remains to find the equations of $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$!
- Note: M_5 is a 1728 dimensional irreducible G-module, for $G = GL(4) \times GL(4) \times GL(4) \rtimes \mathfrak{S}_3$ with a natural basis of polynomials with 180 or 360 or 540 monomials (see also [Allman-Rhodes '03]).

Symmetry

- The symmetry group of the salmon variety

$$\sigma_4\big(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C\big)$$

is change of coordinates in each factor,

$$GL(A) \times GL(B) \times GL(C)$$

(or $GL(A) \times GL(B) \times GL(C) \rtimes \mathfrak{S}_3$ when $A \cong B \cong C$).

- Good news: A large group acts and we can use tools from Representation Theory!

- This symmetry is a powerful tool and we *should* exploit it!

Representation Theory notation

- Module notation: $S^d(A \otimes B \otimes C) = \mathbb{C}[p_{ijk}]_d$.
- Fact: $S^d(A \otimes B \otimes C)$ is a $GL(A) \times GL(B) \times GL(C)$-module.
- The irreducible submodules of $S^d(A \otimes B \otimes C)$ are isomorphic to Schur modules indexed by certain partitions π_1, π_2, π_3 of d:

$$S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C,$$

  and usually occur with multiplicity - this makes us work harder.
- Given π_1, π_2, π_3 and the multiplicity, there is a combinatorial algorithm for constructing polynomials!

An ideal membership test

Apply [Landsberg–Manivel'04] ideal membership test:
For each d,

- decompose $S^d(A^* \otimes B^* \otimes C^*)$ as a $GL(A) \times GL(B) \times GL(C)$-module.
- for each module (isotypic component), test a highest weight vector (highest weight space) for vanishing on $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.
- output: $\mathcal{I}_d(\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ as a list of modules.

Works well for small degree and produced the following results:

- $\mathcal{I}_5(\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)) = 0$.
- $\mathcal{I}_6(\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)) = S_{2,2,2}\mathbb{C}^3 \otimes S_{2,2,2}\mathbb{C}^3 \otimes S_{3,1,1,1}\mathbb{C}^4$
  $= \{$ *ten degree 6 polynomials on 36 variables* $\}$.

Inheritance via an example

Proposition (example of Proposition 4.4 Landsberg–Manivel'04)

$$\tilde{M}_6 := S_{(2,2,2)}\mathbb{C}^4 \otimes S_{(2,2,2)}\mathbb{C}^4 \otimes S_{(3,1,1,1)}\mathbb{C}^4 \subset \mathcal{I}\big(\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)\big)$$

if and only if

$$M_6 := S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(3,1,1,1)}\mathbb{C}^4 \subset \mathcal{I}\big(\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)\big).$$

Note: $\dim(\tilde{M}_6) = 10^3$ but $\dim(M_6) = 10$, and has basis of polynomials, each with 576 or 936 monomials.

At every stage we study the smallest module possible. This is a significant dimension reduction.

For $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ we only need to consider $S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C$ where π_1, π_2, π_3 have 4 parts, and those equations we get from inheritance.

What is a flattening?

Express a tensor $T = \sum_{i,j,k} p_{ijk} a_i \otimes b_j \otimes c_k \in A \otimes B \otimes C$ as a matrix:

$T = \sum_i a_i \otimes (\sum_{j,k} p_{ijk} b_j \otimes c_k) \in A \otimes (B \otimes C)$

For example: $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \cong \mathbb{C}^3 \otimes (\mathbb{C}^3 \otimes \mathbb{C}^3) \cong \mathbb{C}^3 \otimes \mathbb{C}^9$:

$T = [p_{ijk}] = \sum_i a_i \otimes (\sum_{j,k} p_{ijk} b_j \otimes c_k) = \sum_i a_i \otimes X_i$

$$\psi_{0,T} = \left(\begin{array}{ccc|ccc|ccc} p_{111} & p_{121} & p_{131} & p_{112} & p_{122} & p_{132} & p_{113} & p_{123} & p_{133} \\ p_{211} & p_{221} & p_{231} & p_{212} & p_{222} & p_{232} & p_{213} & p_{223} & p_{233} \\ p_{311} & p_{321} & p_{331} & p_{312} & p_{322} & p_{332} & p_{313} & p_{323} & p_{333} \end{array}\right)$$

$$= (X_1 \mid X_2 \mid X_3)$$

When they exist, $(r+1) \times (r+1)$ minors of $\psi_{0,T}$ are (some) equations of $\sigma_r(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.

Flattenings and subspace varieties

Tensors that can be written using fewer variables:

$$Sub_{p,q,r} := \left\{ [T] \in \mathbb{P}(A \otimes B \otimes C) \mid \begin{array}{c} \exists \mathbb{C}^p \subseteq A, \exists \mathbb{C}^q \subseteq B, \exists \mathbb{C}^r \subseteq C, \\ \text{and } [T] \in \mathbb{P}(\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r) \end{array} \right\}$$

Theorem (3.1, Landsberg–Weyman '07)

$Sub_{p,q,r}$ is normal with rational singularities. Its ideal is generated by the minors of flattenings;

$$\left(\bigwedge^{p+1} A \otimes \bigwedge^{p+1}(B \otimes C)\right) \oplus \left(\bigwedge^{q+1} B \otimes \bigwedge^{q+1}(A \otimes C)\right) \oplus \left(\bigwedge^{r+1}(A \otimes B) \otimes \bigwedge^{r+1} C\right)$$

Fact: $Sub_{r,r,r} \supseteq \sigma_r(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$

Key Point: The subspace varieties contain secant varieties, and therefore they give some of the equations of the secant varieties.

A result of Strassen

Theorem (Strassen 1988 (reinterpreted by Landsberg–Manivel))

The ideal of the hypersurface $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^{26}$ *is generated in degree* 9 *by a nonzero vector in the* 1 *dimensional module*

$$S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3$$

Since $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$, inheritance implies that $M_9 := S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^4 \subset \mathcal{I}(\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$

Strassen's polynomial *only* has 9,216 monomials on 27 variables.
$\dim(M_9) = 20$, with natural basis of polynomials with 9,216 or 25,488 or 43,668 monomials on 36 variables! 23 Mb file of polynomials... :-(

Strassen's equation: A useful reformulation by Ottaviani

$T = [p_{ijk}] = \sum_i a_i \otimes (\sum_{j,k} p_{ijk} b_j \otimes c_k) = \sum_i a_i \otimes X_i$
Strassen's equation is the determinant of the 9×9 matrix:

$$\psi_T = \begin{pmatrix} 0 & X_3 & -X_2 \\ -X_3 & 0 & X_1 \\ X_2 & -X_1 & 0 \end{pmatrix}$$

Basic idea:

$\psi_{1,T+T'} = \psi_T + \psi_{T'}$	construction is linear in T
$Rank(T) = 1 \Rightarrow Rank(\psi_T) = 2$	base case
$\therefore Rank(T) = r \Rightarrow Rank(\psi_T) \leq 2r$	upper bound on rank

Numerical Algebraic Geometry: Bertini

Theorem*

The zero set of M_6 (ten polynomials on 36 variables) has precisely two components of codimensions 4 and 6 and degrees 345 and 84 respectively. $\mathcal{V}(M_6) = \sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right) \cup Sub_{3,3,3}$.

Using numerical homotopy continuation (Bertini):

- Original computation (July 2010) 2 weeks of computational time on 8 processors: $2.66 GHz \times 8p \times 336h = 7150 GHzh$
- Regeneration -Hauenstein, Sommese & Wampler, May 2011. $2.33 GHz \times 65p \times 20h = 3029 GHzh$ – confirmed same result
- Small tracking and final tolerances (10^{-10} or smaller)
- Adaptive precision numerical methods
- Checks and error controls built into Bertini such as checking at t = 0.1 that no paths have crossed.

Confirms conjecture from MSRI 2008.

Numerical Algebraic Geometry: Bertini

Theorem*

$\mathcal{V}(M_6 + M_9) = \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$.

Suppose $x \in \mathcal{V}(M_6) = \sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right) \cup Sub_{3,3,3}$.
If $x \notin \sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$, then use M_9 and consider $x \in Sub_{3,3,3} \cap \mathcal{V}(M_9)$
$\Rightarrow x$ is in some $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2\right) \subset \sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$.

Theorem* (Corollary to Landsberg–Manivel 2008, Friedland 2010)

The salmon variety is cut out set-theoretically in degrees 5, 6, 9:

$$\mathcal{V}\left(M_5 + \tilde{M}_6 + \tilde{M}_9\right) = \sigma_4\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right)$$

Resolves the salmon problem set-theoretically.
Provides a more efficient set of equations than [Friedland 2010].
Sharpens the conjecture for the ideal-theoretic question.
Friedland–Gross 2011 make Theorem into Theorem.*

A template for finding equations of varieties coming from applications

The salmon variety has been studied via the following:

1. Start: statistical model, space of special tensors, etc.
2. Find the corresponding algebraic variety X.
3. Find the largest symmetry group G acting X.
4. Study $\mathcal{I}(X)$ as a G-module using Representation Theory.
5. Compute all modules in $\mathcal{I}_d(X)$ for small degree (nec. conditions).
6. Use Numerical Algebraic Geometry to compute unknown zero-sets.
7. Try to make geometric reductions to show that the known invariants suffice.
8. Try to prove what you know* is true.