Tensor decomposition and moment matrices

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Collaboration with
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Blind identification

- Observing $\mathbf{x}_t$ with

$$\mathbf{x}_t = H\, \mathbf{s}_t$$

☞ **find H and $\mathbf{s}_t$??**

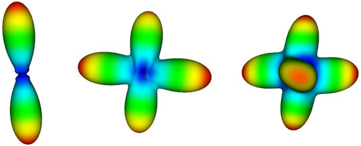
- If the sources are statistically independent, using high order statistics $E(x_i\, x_j\, x_k \cdots)$ of the signal $\mathbf{x}$, identifying H reduces to **decompose the symmetric tensor**

$$T_{i,j,k,\dots} = E(x_i\, x_j\, x_k \cdots)$$

as a sum of s tensors of rank 1.

Recovering branching structures

- From some measurements,

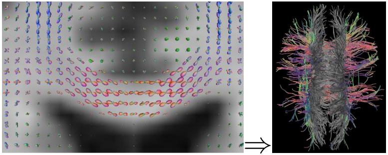


☞ **identify the main directions at the crossing points**.

- Approximate the set of measurements on the sphere by a (symmetric) tensor $T = (T_{i,j,k,l})$.
- Decompose/approximate it as a minimal sum of tensors of rank 1:

$$T = \sum_{i=1}^{r} \lambda_i \, v_i \otimes v_i \otimes v_i \otimes v_i.$$

- From the decomposition of tensors, deduce the geometric structure:



cf. [T. Schultz, H.P. Seidel'08], [A. Ghosh, R. Deriche, ...'09]

Generalized Waring problem (1770)

Given a homogeneous polynomial T of degree d in the variables $\mathbf{x} = (x_0, x_1, \ldots, x_n)$:

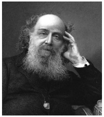
$$T(\mathbf{x}) = \sum_{|\alpha|=d} T_\alpha \, \mathbf{x}^\alpha,$$

find a minimal decomposition of T of the form

$$T(\mathbf{x}) = \sum_{i=1}^{r} \gamma_i (\zeta_{i,0} x_0 + \zeta_{i,1} x_1 + \cdots + \zeta_{i,n} x_n)^d$$

for $\zeta_i = (\zeta_{i,0}, \zeta_{i,1}, \ldots, \zeta_{i,n}) \in \mathbb{C}^{n+1}$, $\gamma_i \in \mathbb{C}$.

Sylvester approach (1886)



For all $f = \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha$, $g = \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha \in S_d$,

$$\langle f, g \rangle = \sum_{|\alpha|=d} f_\alpha \, g_\alpha \binom{d}{\alpha}.$$

For homogeneous polynomials $g(x_0, \ldots, x_n)$ of degree d and $\mathbf{k}(\mathbf{x}) = \mathbf{k}_0 \, x_0 + \cdots + \mathbf{k}_n x_n$,

$$\langle g(\mathbf{x}), \mathbf{k}(\mathbf{x})^d \rangle = g(\mathbf{k}_0, \ldots, \mathbf{k}_n) = g(\mathbf{k}).$$

☞ **If** $T = \mathbf{k}_1(\mathbf{x})^d + \cdots + \mathbf{k}_r(\mathbf{x})^d$ **and** $g(\mathbf{k}_i) = 0$ **for** $i = 1, \ldots, r$**, then** $\forall h$ **with** $\deg(h) = d - \deg(g)$

$$\langle g \, h, T \rangle = 0.$$

☞ **Find the polynomials apolar to** T **and their roots.**

Sylvester'method for binary forms

Theorem

The binary form $T(x_0, x_1) = \sum_{i=0}^{d} c_i \binom{d}{i} x_0^{d-i} x_1^i$ can be decomposed as a sum of r distinct powers of linear forms

$$T = \sum_{k=1}^{r} \lambda_k (\alpha_k x_0 + \beta_k x_1)^d$$

iff there exists a polynomial q such that

$$\begin{bmatrix} c_0 & c_1 & \dots & c_r \\ c_1 & & & c_{r+1} \\ \vdots & & & \vdots \\ c_{d-r} & \dots & c_{d-1} & c_d \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_r \end{bmatrix} = 0$$

and of the form

$$q(x_0, x_1) := \mu \prod_{k=1}^{r} (\beta_k x_0 - \alpha_k x_1).$$

Geometric point of view

Definition (Veronese variety)

$$\begin{array}{rcl}\nu_d : \mathbb{P}(E) & \rightarrow & \mathbb{P}(S_d(E)) \\ \mathbf{v} & \mapsto & \mathbf{v}(\mathbf{x})^{\mathbf{d}}\end{array}$$

Its image is the **Veronese** Variety, denoted $\Xi(S_d(E))$.

Definition (Secant of the Veronese variety)

$\sigma_r^{0,d} = \{[T] \in \mathbb{P}(S_d) \mid \exists\, \mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{K}^{n+1} \text{ s.t. } T = \sum_{i=1}^r \mathbf{v}_i(x)^d\}$

$\sigma_r^d = \overline{\sigma_r^{0,d}}$

- ☞ $r(T)$ = smallest r s.t. $T \in \sigma_r^{0,d}$, called the **rank** of T.
- ☞ $r_\sigma(T)$ = smallest r s.t. $T \in \sigma_r^d$, called the **border rank** of T.

Example: $T_\epsilon := \frac{1}{d\,\varepsilon}\left((x_0 + \varepsilon x_1)^d - x_0^d\right)$, $T_0 := x_0^{d-1} x_1$.

$$r(T_\epsilon) = r_\sigma(T_0) = 2 \text{ but } r(T_0) = d.$$

Algebraic point of view

$S := \mathbb{K}[x_0, \ldots, x_n]$; $S_d := \{f \in S; \deg(f) = d\}$;

Definition (Apolar ideal)

$(T^\perp) = \{g \in S \mid \forall h \in S_{d-\deg(g)},\ \langle g\, h, T\rangle = 0\} \supset S_{d+1}$.

Problem: find an ideal $I \subset S$ such that

- $I \subset (T^\perp)$;
- **I is saturated zero dimensional;**
- **I defines a minimal number r of simple points.**

☞ **necessary and sufficient conditions.**

Schematic rank

Definition (Punctual Hilbert Scheme)

$\mathrm{Hilb}^r(\mathbb{P}^n)$ is the set of saturated ideals of r points (counted with mult.).

Property: $\mathrm{Hilb}^r(\mathbb{P}^n)$ is a subvariety of the Grassmannian $\mathrm{Gr}^r(S_r^*)$ defined by quadratic equations in the Plücker coordinates [ABM'10].

Definition (Cactus variety)

$$\mathcal{K}_r^d = \{[T] \in \mathbb{P}(S_d) \mid \exists I \in \mathrm{Hilb}^r(\mathbb{P}^n) \text{ s.t. } I \subset (T^\perp)\}$$

(def. with closure in [Buczynska-Buczynski'11]).

☞ $r_{sch}(T)$ = smallest r s.t. $T \in \mathcal{K}_r^d$, called the **schematic rank** of T.

(see [Iarrobino-Kanev'99])

Duality

- $S = \mathbb{K}[x_0, \ldots, x_n] = \mathbb{K}[\mathbf{x}]$, $R = \mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[\underline{\mathbf{x}}]$.
- Dual space: $E^* = \mathrm{Hom}_{\mathbb{K}}(E, \mathbb{K})$.
- $R^* \sim \mathbb{K}[[\underline{\mathbf{d}}]] = \mathbb{K}[[\mathbf{d}_1, \ldots, \mathbf{d}_n]] = \mathbb{K}[[\partial_1, \ldots, \partial_n]]$:

$$\Lambda = \sum_{\alpha \in \mathbb{N}^n} \Lambda(\mathbf{x}^\alpha)\, \underline{\mathbf{d}}^\alpha = \sum_{\alpha \in \mathbb{N}^n} \Lambda(\mathbf{x}^\alpha)\, \frac{1}{\alpha!}\, \mathbf{1_0} \circ \partial^\alpha.$$

where $(\underline{\mathbf{d}}^\alpha)_{\alpha \in \mathbb{N}^n}$ is the dual basis in R^*.
Example: for $\zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_k}$,

$$\begin{aligned} \mathbf{1}_\zeta \quad : \quad & R \to \mathbb{K} \\ & p \mapsto p(\zeta) \end{aligned}$$

with

$$\mathbf{1}_\zeta = \sum_{\alpha \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_k}} \zeta^\alpha\, \underline{\mathbf{d}}^\alpha = \sum_{\alpha \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_k}} \prod_{i=1}^{k} \zeta_i^{\alpha_i}\, \underline{\mathbf{d}}_i^{\alpha_i}.$$

- $(S_d)^* \sim \mathbb{K}[[\mathbf{d}]]_d = \mathbb{K}[\mathbf{d}_0, \ldots, \mathbf{d}_n]_d$:

Duality cont'd

For $T = T(\mathbf{x}) = \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha|=d}} T_\alpha \, \mathbf{x}^\alpha \in S_d$, let

- $T^*(\mathbf{d}) = \sum_{\substack{\alpha \in \mathbb{N}^{n+1}; \\ |\alpha|=d}} \binom{d}{\alpha}^{-1} T_\alpha \, \mathbf{d}^\alpha \in (S_d)^*$ such that $\forall T' \in S_d$,

$$\langle T | T' \rangle = T^*(T').$$

- $\underline{T}^*(\underline{\mathbf{d}}) = \sum_{\substack{\alpha \in \mathbb{N}^{n+1}; \\ |\alpha|=d}} \binom{d}{\alpha}^{-1} T_\alpha \, \underline{\mathbf{d}}^\alpha \in (R_{\leq d})^*$ such that $\forall \underline{T}' \in R_{\leq d}$,

$$\langle \underline{T} | \underline{T}' \rangle = \underline{T}^*(\underline{T}').$$

Hankel operators

$$\begin{array}{rcl} H_\Lambda : R & \rightarrow & R^* \\ p & \mapsto & p \cdot \Lambda \end{array}$$

where $p \cdot \Lambda : q \mapsto \Lambda(p\,q)$.

Properties:

- $I_\Lambda := \ker H_\Lambda$ is an ideal of R.
- $\mathrm{rank} H_\Lambda = r$ iff $\mathcal{A}_\Lambda = R/I_\Lambda$ is an algebra of dimension r over $\mathbb{K}$.
- If $\mathrm{rank} H_\Lambda = r$, $\mathcal{A}_\Lambda$ is a zero-dimensional Gorenstein algebra:
  1. $\mathcal{A}_\Lambda^* = \mathcal{A}_\Lambda \cdot \Lambda$ (free module of rank 1).
  2. $(a, b) \mapsto \Lambda(ab)$ is non-degenerate in $\mathcal{A}_\Lambda$.
  3. $\mathrm{Hom}_{\mathcal{A}_\Lambda}(\mathcal{A}_\Lambda^*, \mathcal{A}_\Lambda) = \mathcal{D} \cdot \mathcal{A}_\Lambda$ where $\mathcal{D} = \sum_{i=1}^r b_i \otimes \omega_i$ for $(b_i)_{1 \leqslant i \leqslant r}$ a basis of $\mathcal{A}_\Lambda$ and $(\omega_i)_{1 \leqslant i \leqslant r}$ its dual basis for Λ.

- If $\mathrm{rank} H_\Lambda = r$, then

$$\Lambda : p \mapsto \sum_{i=1}^{r'} \mathbf{1}_{\zeta_i} \cdot \theta_i(\partial_1, \ldots, \partial_n)(p)$$

  for some $\zeta_i \in \mathbb{C}^n$ and some differential polynomials θ_i with
  - $r = \sum_{i=1}^{r'} \dim(\langle \frac{d^{|\alpha|}}{d\partial^\alpha}(\theta_i), \alpha \in \mathbb{N}^N \rangle)$,
  - $V_{\mathbb{C}}(I_\Lambda) = \{\zeta_1, \ldots, \zeta_{r'}\}$.

- If rank $H_\Lambda = r$, $(b_i)_{1 \leqslant i \leqslant r}$ a basis of $\mathcal{A}_\Lambda$ and $(\omega_i)_{1 \leqslant i \leqslant r}$ its dual basis for Λ then

$$\sqrt{I_\Lambda} = \ker H_{\Delta \cdot \Lambda}$$

  where $\Delta = \sum_{i=1}^{r} b_i \omega_i$.

Generalized additive decomposition

Definition

A tensor $T \in \mathcal{T}$ has a **generalized additive decomposition** of size $\leq r$ iff there exists points $\zeta_i \in \mathbb{K}^N$ and differential polynomials θ_i s.t. after a change of coordinates,

$$\underline{T}^* \equiv \sum_{i=1}^{r'} \mathbf{1}_{\zeta_i} \cdot \theta_i(\partial) \text{ on } R_{\leq d}$$

and $\sum_{i=1}^{r'} \dim(\langle \frac{d^{|\alpha|}}{d\partial^{\alpha}}(\theta_i), \alpha \in \mathbb{N}^n \rangle) \leq r$.

Definition

$\mathcal{G}_r^d = \{[T] \in \mathbb{P}(S_d)$ with a generalized additive decomposition of size $\leq r\}$.

☞ $r_{\mathcal{G}}(T) =$ smallest r such that $T \in \mathcal{G}_r^d$, called the **generalized additive rank** of T.

Truncated moment matrices

Given $(\lambda_\alpha)_{\alpha \in A}$, let $\tilde{\Lambda} \in \langle \mathbf{x}^A \rangle^* := \mathrm{Hom}_{\mathbb{K}}(\langle \mathbf{x}^A \rangle, \mathbb{K})$ be

$$\tilde{\Lambda} : p = \sum_{\alpha \in A} p_\alpha \mathbf{x}^\alpha \mapsto \sum_{\alpha \in A} p_\alpha \lambda_\alpha$$

For $B, B' \subset A$ with $B \cdot B' \subset A$, we define

$$\begin{aligned} H_{\tilde{\Lambda}}^{B',B} : \langle \mathbf{x}^{B'} \rangle &\to \langle \mathbf{x}^B \rangle^* \\ p = \sum_{\beta \in B'} p_\beta \mathbf{x}^\beta &\mapsto p \cdot \tilde{\Lambda} \end{aligned}$$

with $p \cdot \tilde{\Lambda} : q \in \langle B \rangle \mapsto \tilde{\Lambda}(p\, q) \in \mathbb{K}$.

The matrix of $H_{\tilde{\Lambda}}^{B',B} := [\lambda_{\beta'+\beta}]_{\beta' \in B', \beta \in B}$.

☞ **Given $\tilde{\Lambda} \in \langle \mathbf{x}^A \rangle^*$, find $\Lambda \in R^*$ such that Λ extends $\tilde{\Lambda}$ and H_Λ of minimal rank.**

Flat extension

Definition

$T \in S_d$ has a **flat extension** of rank r iff there exists $\tilde{T} \in S_{m+m'}$ and $\mathbf{u} \in S_1 - \{0\}$ with $m = \max\{r, \lceil \frac{d}{2} \rceil\}$, $m' = \max\{r-1, \lfloor \frac{d}{2} \rfloor\}$ s.t.

$$\mathbf{u}^{m+m'-d} \cdot \tilde{T}^* = T^*$$

and

$$\begin{array}{rcl} H_{\tilde{T}^*}^{m,m'} : S_m & \rightarrow & (S_{m'})^* \\ p & \mapsto & p \cdot T^* \end{array}$$

is of rank r.

After a change of variables ($\mathbf{u} = x_0$), we have $\underline{\tilde{T}}_\alpha = \underline{T}_\alpha$ for $\mid \alpha \mid \leq d$ (truncated moments).

Definition (Flat extension variety)

$\mathcal{H}_r^d = \{[T] \in \mathbb{P}(S_d)$ s.t. T has a flat extension of rank $\leq r\}$.

☞ $r_{\mathcal{H}}(T) =$ smallest r such that $T \in \mathcal{H}_r^d$, called the **extension rank** of T.

Catalecticant operators

$$\begin{array}{rcl} H_{T^*}^{i,d-i}: S_i & \to & (S_{d-i})^* := \mathrm{Hom}(S_{d-i}, \mathbb{K}) \\ p & \mapsto & p \cdot T^* \end{array}$$

where

$$T^* \in (S_d)^* \text{ s.t. } \forall g \in (S_d),\ \ T^*(g) = \langle T, g \rangle;$$

$$\begin{array}{rcl} \forall p \in S_i, \ \ p \cdot T^* : S_{d-i} & \to & \mathbb{K} \\ q & \mapsto & p \cdot T^*(q) = T^*(p\,q) = \langle T, p\,q \rangle \end{array}$$

Remark: $\ker H_{T^*}^{i,d-i} = (T^{\perp})_i$.

Definition (Catalecticant variety)

$\Gamma_r^{i,d-i} := \{[T] \in \mathbb{P}(S_d) \mid rank\, H_{T^*}^{i,d-i} \leq r\}$.

☞ $r_{Cat}(T) :=$ maximal rank of $H_{T^*}^{i,d-i}$ for $i = 0, \ldots, d$, called the **catalecticant rank** of T.

The rank hierarchy

Theorem (Brachat-M.-B.'11)

1. $\mathcal{K}_r^d = \mathcal{H}_r^d = \mathcal{G}_r^d$.
2. *It is a closed variety of* $\mathbb{P}(S_d)$.

Theorem (Buczynska-Buczynski'11, Brachat'11)

For $d \geq 2r$ and for any $r \leq i \leq d - r$, we have

$$\mathcal{K}_r^{0,d} = \mathcal{K}_r^d = \Gamma_r^{i,d-i}.$$

(See also [Buczynski-Ginensky-Landsberg'10] and talk).
For $0 \leq i \leq d$,

$$\sigma_r^{0,d} \hookrightarrow \sigma_r^d \hookrightarrow \mathcal{K}_r^d = \mathcal{H}_r^d = \mathcal{G}_r^d \hookrightarrow \Gamma_r^{i,d-i} \hookrightarrow \mathbb{P}(S_d)$$

Consequently,

$$r(f) \geq r_\sigma^d(f) \geq r_{sch}^d(f) = r_{\mathcal{G}}^d(f) = r_{\mathcal{H}}^d(f) \geq r_{Cat}^d(f)$$

Computing a flat extension

Theorem (LM'09, BBCM'10)

If B, B' are connected to 1 ($m \in B$ implies $m = 1$ or $m = x_{i_0} m'$ with $m' \in B$) and $\tilde{\Lambda}$ known on $B'^+ \cdot B^+$ and $H_{\tilde{\Lambda}}^{B',B}$ is invertible then $\tilde{\Lambda}$ extends uniquely to R iff

$$M_i^{B',B} \circ M_j^{B',B} = M_j^{B',B} \circ M_i^{B',B} \quad (1 \leq i, j \leq n),$$

where $B^+ = \cup_i x_i \cdot B \cup B$, $M_i^{B',B} := H_{\tilde{\Lambda}}^{B',x_i B} (H_{\tilde{\Lambda}}^{B',B})^{-1}$.

Equivalently

Theorem (LM'09, BBCM'10)

Let B, B' be connected to 1 of size r and $\tilde{\Lambda}$ known on $B'^+ \cdot B^+$. If

$$\mathrm{rank} H_{\tilde{\Lambda}}^{B',B} = \mathrm{rank} H_{\tilde{\Lambda}}^{B'^+,B^+} = r,$$

then there exists a unique $\Lambda \in R^* = \mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K})$ which extends $\tilde{\Lambda}$.

Recovering the decomposition

If the truncated moment problem has a solution Λ, then:

- $r = \mathrm{rank} H_\Lambda = \dim R/(\ker H_\Lambda)$ where $r = |B| = |B'|$;
- Let B, B' be maximal sets of monomials $\subset R$ s.t. $H_\Lambda^{B',B}$ invertible, then
$$M_i^{B',B} := H_\Lambda^{B',x_i B}(H_\Lambda^{B',B})^{-1}.$$
is the matrix of multiplication by x_i in $\mathcal{A}_\Lambda$.
- If the decomposition is of size r, the eigenvectors of the operators $(M^t_{x_{i,j}})_{i,j}$ are simple and equal (up to scalar) to $\{\mathbf{1}_{\zeta_1}, \ldots, \mathbf{1}_{\zeta_r}\}$.
- $\ker H_\Lambda = (\ker H_\Lambda^{B'^+,B^+})$ where $B^+ = B \cup x_{1,1}\, B \cup \cdots x_{n_k,k} B$;
- For each $x^\alpha \in \partial B = B^+ \setminus B$, there exists a unique
$$f_\alpha = x^\alpha - \Sigma_\beta z_{\alpha,\beta}\, x^\beta \in \ker H_\Lambda^{B'^+,B^+}.$$
The $(f_\alpha)_{\alpha \in \partial B}$ form a **border basis** of $\ker H_\Lambda$ with respect to B.

The results extend to **general multihomogeneous tensors**:

- $\mathbf{x}_i = (x_{0,i}, \ldots, x_{n_i,i})$, $\underline{\mathbf{x}}_i = (x_{1,i}, \ldots, x_{n_i,i})$, $i = 1, \ldots, k$;
- $E_i = \langle x_{0,i}, \ldots, x_{n_i,i} \rangle$, $i = 1, \ldots, k$;
- $\mathcal{T} := S_{\delta_1}(E_1) \otimes \cdots \otimes S_{\delta_k}(E_k)$;
- $R = \mathbb{K}[\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_k]$;
- $\mathcal{T} \sim R_\delta := R_{\delta_1,\ldots,\delta_k} = \{f \in R; \deg_{\underline{\mathbf{x}}_i}(f) \le \delta_i, i = 1, \ldots, k\}$;

$$f = \sum_{\alpha=(\alpha_1,\ldots,\alpha_k);|\alpha_i|\le\delta_i} f_\alpha\, \underline{\mathbf{x}}_1^{\alpha_1} \cdots \underline{\mathbf{x}}_k^{\alpha_k}.$$

- Apolar inner product:

$$\langle f|g\rangle = \sum_{|\alpha_i|\le\delta_i} f_\alpha\, g_\alpha \binom{\delta_1}{\alpha_1} \cdots \binom{\delta_k}{\alpha_k}.$$

Algorithm

For $r = 1, \ldots,$

1. Choose B, B' of size r, connected to 1;
2. Find an extension $H_\Lambda^{B'^+, B^+}$ s.t. the operators $M_i = H_\Lambda^{x_i B', B} (H_\Lambda^{B', B})^{-1}$ commute.
3. If this is not possible, start again with $r := r + 1$.
4. Compute the $n \times r$ eigenvalues s.t. $M_i^t \mathbf{v}_j = \zeta_{i,j} \mathbf{v}_j$, $i = 1, \ldots, n$, $j = 1, \ldots, r$.
5. Solve the linear system in $(\gamma_j)_{j=1,\ldots,k}$: $\Lambda = \sum_{j=1}^r \gamma_j \mathbf{1}_{\zeta_j}$ where $\zeta_j \in \mathbb{C}^n$ are the vectors of eigenvalues found in (4).

Example in $\Xi_6(S_4(\mathbb{K}^3))$

The tensor:

$$T = 79\, x_0 x_1^3 + 56\, x_0^2 x_2^2 + 49\, x_1^2 x_2^2 + 4\, x_0 x_1 x_2^2 + 57\, x_0^3 x_1.$$

The 15×15 Hankel matrix:

	1	x_1	x_2	x_1^2	x_1x_2	x_2^2	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	x_1^4	$x_1^3x_2$	$x_1^2x_2^2$	$x_1x_2^3$	x_2^4
1	0	$\frac{57}{4}$	0	0	0	$\frac{28}{3}$	$\frac{79}{4}$	0	$\frac{1}{3}$	0	0	0	$\frac{49}{6}$	0	0
x_1	$\frac{57}{4}$	0	0	$\frac{79}{4}$	0	$\frac{1}{3}$	0	0	$\frac{49}{6}$	0	h_{500}	h_{410}	h_{320}	h_{230}	h_{140}
x_2	0	0	$\frac{28}{3}$	0	$\frac{1}{3}$	0	0	$\frac{49}{6}$	0	0	h_{410}	h_{320}	h_{230}	h_{140}	h_{050}
x_1^2	0	$\frac{79}{4}$	0	0	0	$\frac{49}{6}$	h_{500}	h_{410}	h_{320}	h_{230}	h_{600}	h_{510}	h_{420}	h_{330}	h_{240}
x_1x_2	0	0	$\frac{1}{3}$	0	$\frac{49}{6}$	0	h_{410}	h_{320}	h_{230}	h_{140}	h_{510}	h_{420}	h_{330}	h_{240}	h_{150}
x_2^2	$\frac{28}{3}$	$\frac{1}{3}$	0	$\frac{49}{6}$	0	0	h_{320}	h_{230}	h_{140}	h_{050}	h_{420}	h_{330}	h_{240}	h_{150}	h_{060}
x_1^3	$\frac{79}{4}$	0	0	h_{500}	h_{410}	h_{320}	h_{600}	h_{510}	h_{420}	h_{330}	h_{700}	h_{610}	h_{520}	h_{430}	h_{340}
$x_1^2x_2$	0	0	$\frac{49}{6}$	h_{410}	h_{320}	h_{230}	h_{510}	h_{420}	h_{330}	h_{240}	h_{610}	h_{520}	h_{430}	h_{340}	h_{250}
$x_1x_2^2$	$\frac{1}{3}$	$\frac{49}{6}$	0	h_{320}	h_{230}	h_{140}	h_{420}	h_{330}	h_{240}	h_{150}	h_{520}	h_{430}	h_{340}	h_{250}	h_{160}
x_2^3	0	0	0	h_{230}	h_{140}	h_{050}	h_{330}	h_{240}	h_{150}	h_{060}	h_{430}	h_{340}	h_{250}	h_{160}	h_{070}
x_1^4	0	h_{500}	h_{410}	h_{600}	h_{510}	h_{420}	h_{700}	h_{610}	h_{520}	h_{430}	h_{800}	h_{710}	h_{620}	h_{530}	h_{440}
$x_1^3x_2$	0	h_{410}	h_{320}	h_{510}	h_{420}	h_{330}	h_{610}	h_{520}	h_{430}	h_{340}	h_{710}	h_{620}	h_{530}	h_{440}	h_{350}
$x_1^2x_2^2$	$\frac{49}{6}$	h_{320}	h_{230}	h_{420}	h_{330}	h_{240}	h_{520}	h_{430}	h_{340}	h_{250}	h_{620}	h_{530}	h_{440}	h_{350}	h_{260}
$x_1x_2^3$	0	h_{230}	h_{140}	h_{330}	h_{240}	h_{150}	h_{430}	h_{340}	h_{250}	h_{160}	h_{530}	h_{440}	h_{350}	h_{260}	h_{170}
x_2^4	0	h_{140}	h_{050}	h_{240}	h_{150}	h_{060}	h_{340}	h_{250}	h_{160}	h_{070}	h_{440}	h_{350}	h_{260}	h_{170}	h_{080}

Extract a (6×6) principal minor of full rank:

$$\mathtt{H}_{\Lambda}^{B} = \begin{bmatrix} 0 & \frac{57}{4} & 0 & 0 & 0 & \frac{28}{3} \\ \frac{57}{4} & 0 & 0 & \frac{79}{4} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{28}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{79}{4} & 0 & 0 & 0 & \frac{49}{6} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{28}{3} & \frac{1}{3} & 0 & \frac{49}{6} & 0 & 0 \end{bmatrix}$$

The columns (and the rows) of the matrix correspond to the monomials $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$.

The shifted matrix $\mathtt{H}^B_{x_1\cdot\Lambda}$ is

$$\mathtt{H}_{x_1\cdot\Lambda} = \begin{bmatrix} \frac{57}{4} & 0 & 0 & \frac{79}{4} & 0 & \frac{1}{3} \\ 0 & \frac{79}{4} & 0 & 0 & 0 & \frac{49}{6} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{79}{4} & 0 & 0 & h_{500} & h_{410} & h_{320} \\ 0 & 0 & \frac{49}{6} & h_{410} & h_{320} & h_{230} \\ \frac{1}{3} & \frac{49}{6} & 0 & h_{320} & h_{230} & h_{140} \end{bmatrix}$$

The columns of the matrix correspond to the monomials $\{x_1, x_1^2, x_1x_2, x_1^3, x_1^2x_2, x_1x_2^2\} = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\} \times x_1$.
Similarly,

$$\mathtt{H}^B_{x_2\cdot\Lambda} = \begin{bmatrix} 0 & 0 & \frac{28}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{28}{3} & \frac{1}{3} & 0 & \frac{49}{6} & 0 & 0 \\ 0 & 0 & \frac{49}{6} & h_{410} & h_{320} & h_{230} \\ \frac{1}{3} & \frac{49}{6} & 0 & h_{320} & h_{230} & h_{140} \\ 0 & 0 & 0 & h_{230} & h_{140} & h_{050} \end{bmatrix}$$

We form (all) the possible matrix equations:

$$\mathtt{M}_{x_i}\mathtt{M}_{x_j} - \mathtt{M}_{x_j}\mathtt{M}_{x_i} = \mathtt{H}^B_{x_1\cdot\Lambda}(\mathtt{H}^B_\Lambda)^{-1}\mathtt{H}^B_{x_2\cdot\Lambda}(\mathtt{H}^B_\Lambda)^{-1} - \mathtt{H}^B_{x_2\cdot\Lambda}(\mathtt{H}^B_\Lambda)^{-1}\mathtt{H}^B_{x_1\Lambda}(\mathtt{H}^B_\Lambda)^{-1} = 0.$$

Many of the resuling equations are trivial. We have 6 unknonws: $h_{500}, h_{410}, h_{320}, h_{230}, h_{140}, h_{050}$ and 15 non-trivial equations.

A solution of the system is:

$$h_{500} = 1, h_{410} = 2, h_{320} = 3, h_{230} = 1.5060, h_{140} = 4.960, h_{050} = 0.056.$$

We subsitute these values to $\mathtt{H}_{x_1\cdot\Lambda}$ and solve the generalized eigenvalue problem $(\mathtt{H}_{x_1\cdot\Lambda} - \zeta\, \mathtt{H}_\Lambda)\, \mathbf{v} = 0$.The normalized eigenvectors are

$$\begin{bmatrix} 1 \\ -0.830 + 1.593\, i \\ -0.326 - 0.0501\, i \\ -1.849 - 2.645\, i \\ 0.350 - 0.478\, i \\ 0.103 + 0.0327\, i \end{bmatrix}, \begin{bmatrix} 1 \\ -0.830 - 1.593\, i \\ -0.326 + 0.050\, i \\ -1.849 + 2.645\, i \\ 0.350 + 0.478\, i \\ 0.103 - 0.032\, i \end{bmatrix}, \begin{bmatrix} 1 \\ 1.142 \\ 0.836 \\ 1.305 \\ 0.955 \\ 0.699 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 0.956 \\ -0.713 \\ 0.914 \\ -0.682 \\ 0.509 \end{bmatrix}, \begin{bmatrix} 1 \\ -0.838 + 0.130\, i \\ 0.060 + 0.736\, i \\ 0.686 - 0.219\, i \\ -0.147 - 0.610\, i \\ -0.539 + 0.089\, i \end{bmatrix}, \begin{bmatrix} 1 \\ -0.838 - 0.130\, i \\ 0.060 - 0.736\, i \\ 0.686 + 0.219\, i \\ -0.147 + 0.610\, i \\ -0.539 - 0.089\, i \end{bmatrix}.$$

As the coordinates of the eigenvectors correspond to the evaluations of $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, we can recover the values of (x_1, x_2).

After solving the over-constrained linear system obtained by expansion and looking coefficient-wise, we deduce the decomposition:

$$
\begin{aligned}
&(0.517 + 0.044\, i)\,(x_0 - (0.830 - 1.593\, i)x_1 - (0.326 + 0.050\, i)x_2)^4 \\
&+(0.517 - 0.044\, i)\,(x_0 - (0.830 + 1.593\, i)x_1 - (0.326 - 0.050\, i)x_2)^4 \\
&+2.958\,(x_0 + (1.142)x_1 + 0.836x_2)^4 \\
&+4.583\,(x_0 + (0.956)x_1 - 0.713x_2)^4 \\
&-(4.288 + 1.119\, i)\,(x_0 - (0.838 - 0.130\, i)x_1 + (0.060 + 0.736\, i)x_2)^4 \\
&-(4.288 - 1.119\, i)\,(x_0 - (0.838 + 0.130\, i)x_1 + (0.060 - 0.736\, i)x_2)^4
\end{aligned}
$$

Concluding remarks/questions

- What about **nearest r-decomposition(s)** and SVD-like properties ?
- What are the equations of $\mathcal{G}_r^d = \mathcal{H}_r^d = \mathcal{K}_r^d$ and σ_r^d?
- Extension of geometric properties to any type of tensors?

Thanks for your attention