Computational Complexity of Tensor Problems

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Joint work with Lek-Heng Lim

(with L-H Lim) *Most tensor problems are NP-hard*, Journal of the ACM, 2013, to appear.

Outline

Motivational examples / definitions

Computational complexity

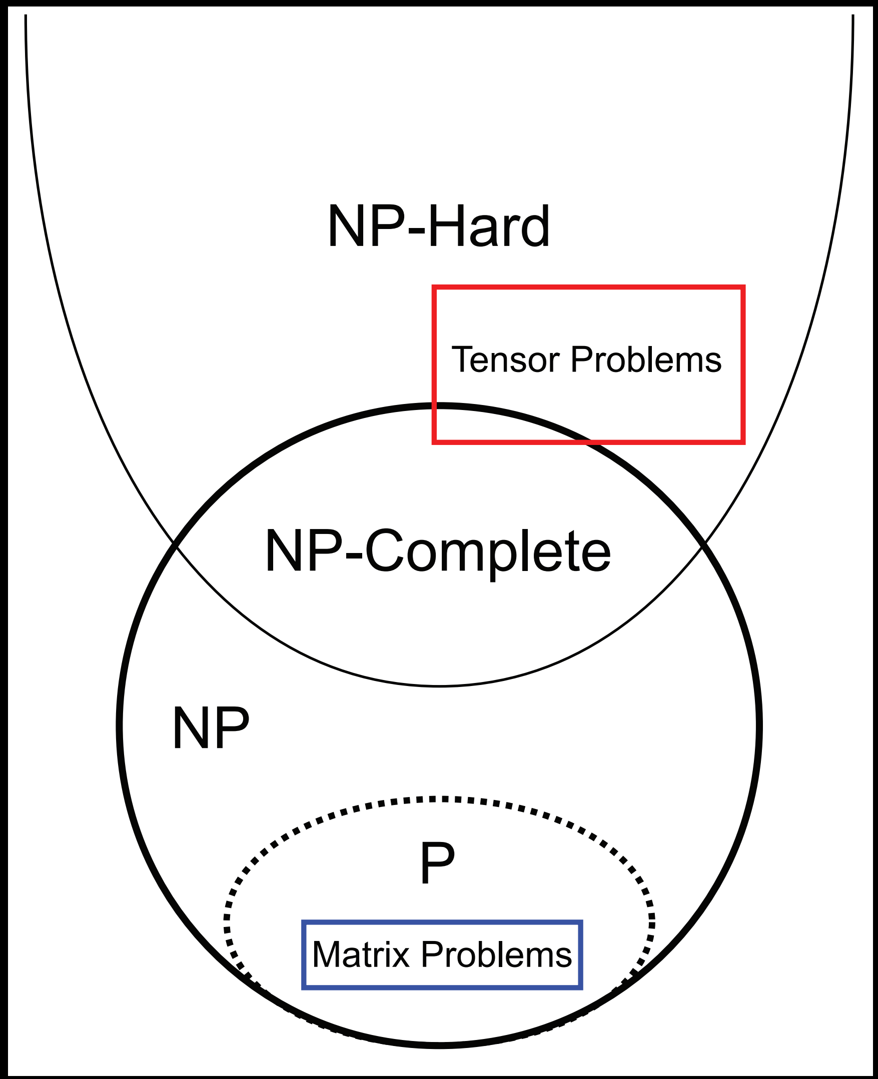
Tensor Problems

- Deciding eigenvalue
- Approximating eigenvalue
- Computing rank

Open Problems

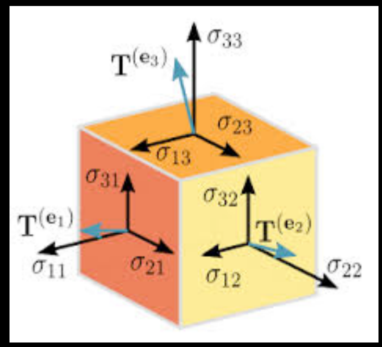
- Hyperdeterminant

Conclusions

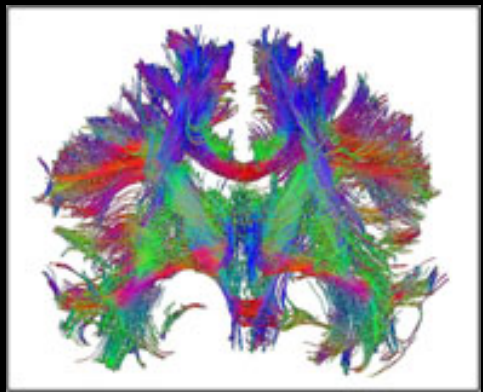


Applications of "tensor methods"

Approximation algorithms [De La Vega et al. 2005; Brubaker and Vempala 2009]
Computational biology [Cartwright et al. 2009]
Computer graphics [Vasilescu and Terzopoulos 2004]
Computer vision [Shashua and Hazan 2005; Vasilescu and Terzopoulos 2002]
Data analysis [Coppi and Bolasco 1989]
Graph theory [Friedman 1991; Friedman and Wigderson 1995]
Neuroimaging [Schultz and Seidel 2008]
Pattern recognition [Vasilescu 2002]
Phylogenetics [Allman and Rhodes 2008]
Quantum computing [Miyake and Wadati 2002]
Scientific computing [Beylkin and Mohlenkamp 1997]
Signal processing [Comon 1994; 2004; Kofidis and Regalia 2001/02]
Spectroscopy [Smilde et al. 2004]
Wireless communication [Sidiropoulos et al. 2000]
....



Generalize: $Ax = \mathbf{b}$



Computational complexity

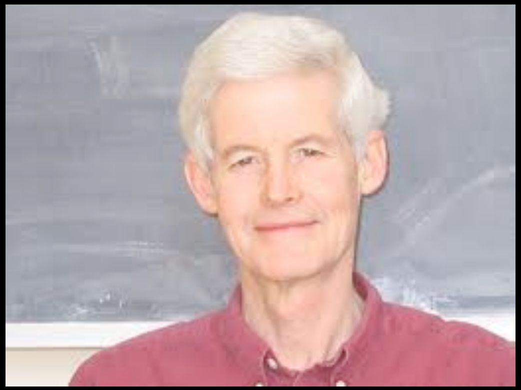
I. Model of computation
- What are inputs / outputs?
- What is a computation?

II. Model of complexity
- Cost of computation?

III. Model of reducibility
- What are equivalent problems?

Alan Turing

Dick Karp



Stephen Cook

Leonid Levin

I. Model of computation:

Turing Machine [Turing 1936]
Inputs: rational tensors
Outputs: YES/NO or rational vectors

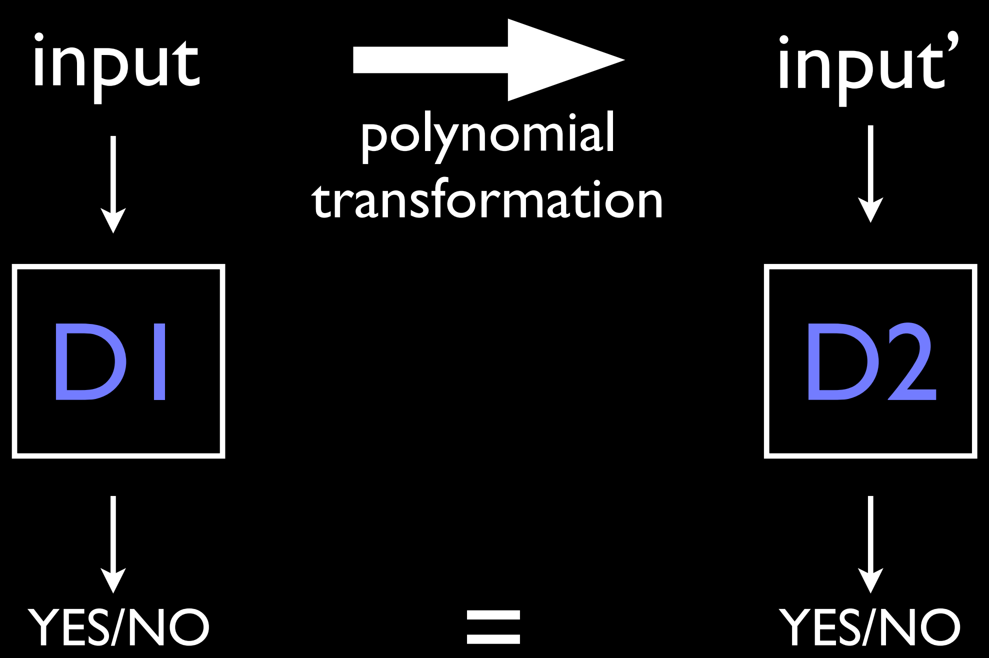
Turing Machine (Mike Davey)

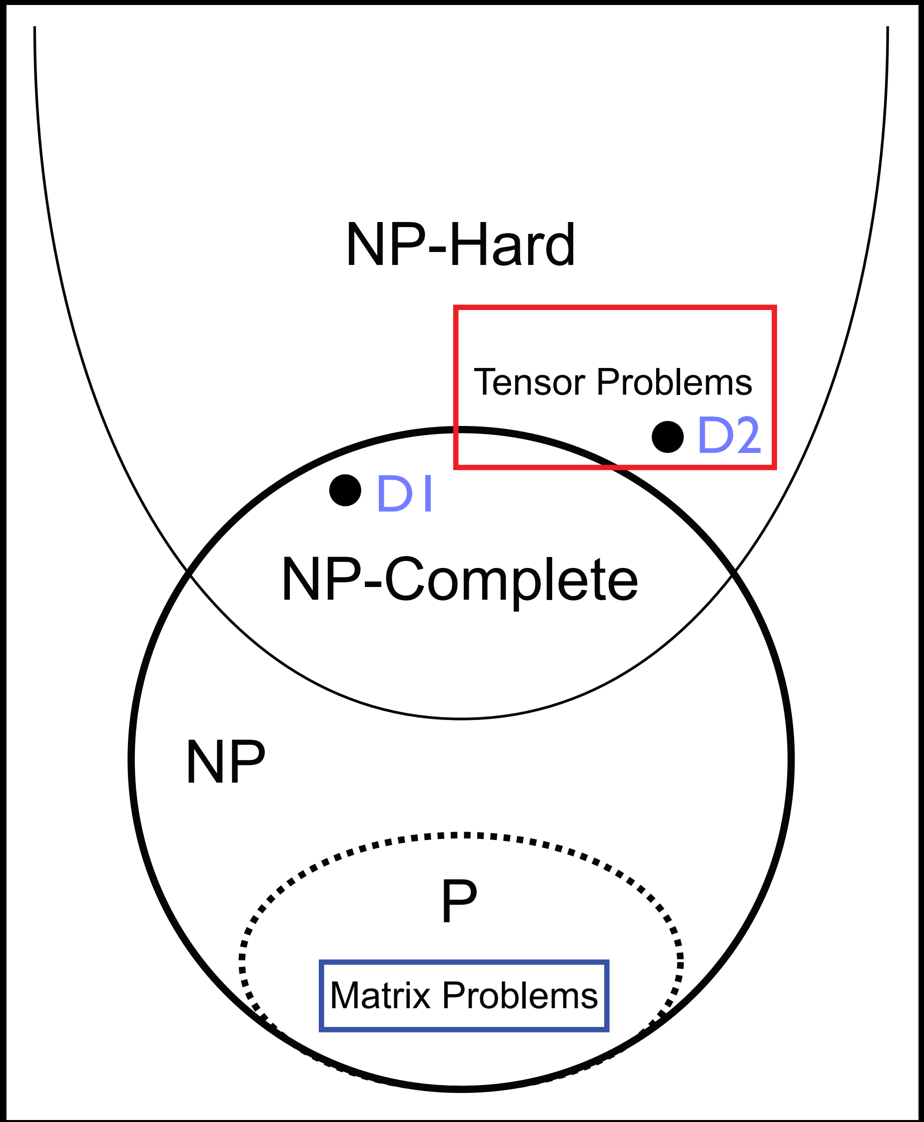
II. Model of complexity:

Time complexity
Number of Tape-Level moves

III. Model of reducibility:

P, NP, NP-complete, NP-hard, ...





Complexity of tensor problems

Problem	Complexity
Bivariate Matrix Functions over $\mathbb{R}$, $\mathbb{C}$	Undecidable (Proposition 12.2)
Bilinear System over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorems 2.6, 3.7, 3.8)
Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 1.3)
Approximating Eigenvector over $\mathbb{R}$	NP-hard (Theorem 1.5)
Symmetric Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 9.3)
Approximating Symmetric Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 9.6)
Singular Value over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorem 1.7)
Symmetric Singular Value over $\mathbb{R}$	NP-hard (Theorem 10.2)
Approximating Singular Vector over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorem 6.3)
Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 1.10)
Symmetric Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 10.2)
Approximating Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 1.11)
Nonnegative Definiteness	NP-hard (Theorem 11.2)
Best Rank-1 Approximation	NP-hard (Theorem 1.13)
Best Symmetric Rank-1 Approximation	NP-hard (Theorem 10.2)
Rank over $\mathbb{R}$ or $\mathbb{C}$	NP-hard (Theorem 8.2)
Enumerating Eigenvectors over $\mathbb{R}$	#P-hard (Corollary 1.16)
Combinatorial Hyperdeterminant	NP-, #P-, VNP-hard (Theorems 4.1 , 4.2, Corollary 4.3)
Geometric Hyperdeterminant	Conjectures 1.9, 13.1

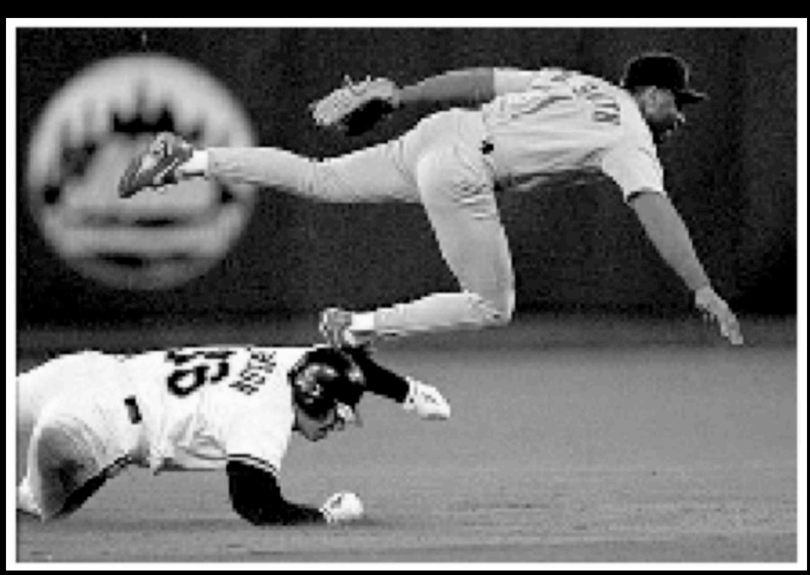
Eigenvalues

Problem: Given $A = [a_{ij}] \in \mathbb{Q}^{n \times n}$

find $(\mathbf{x}, \lambda)$ with $\mathbf{x} \neq 0$ s.t. :

$$A\mathbf{x} = \lambda\mathbf{x}$$

- Image Segmentation [Shi-Malik, 2000]



efficient algorithms
(in P)

top eigenvector

Tensor eigenvalues

Problem: Given $\mathcal{A} = [[a_{ijk}]] \in \mathbb{Q}^{n \times n \times n}$

find $(\mathbf{x}, \lambda)$ with $\mathbf{x} \neq 0$ s.t. :

$$\sum_{i,j=1}^{n} a_{ijk} x_i x_j = \lambda x_k, \quad k = 1, \ldots, n$$

[Lim 2005], [Qi 2005], [Ni, et al 2007], [Qi 2007], [Cartwright and Sturmfels 2012]

Facts: Generic or random tensors over complex numbers have a finite number of eigenvalues and eigenvectors (up to scaling equivalence), although their count is exponential.

Still, it is possible for a tensor to have an infinite number of non-equivalent eigenvalues, but in that case they comprise a cofinite set of complex numbers

Another important fact is that over the reals, every 3-tensor has a real eigenpair.

Decision problem

Problem: Given $\mathcal{A} = [[a_{ijk}]] \in \mathbb{Q}^{n\times n\times n}$ and $\lambda \in \mathbb{Q}$, does there exist $\mathbf{0} \neq \mathbf{x} \in \mathbf{C}^n$:

$$\sum_{i,j=1}^{n} a_{ijk} x_i x_j = \lambda x_k, \quad k = 1, \ldots, n$$

Decidable (Computable on a Turing machine):
- Quantifier elimination
- Buchberger's algorithm and Groebner bases
- Multivariate resultants

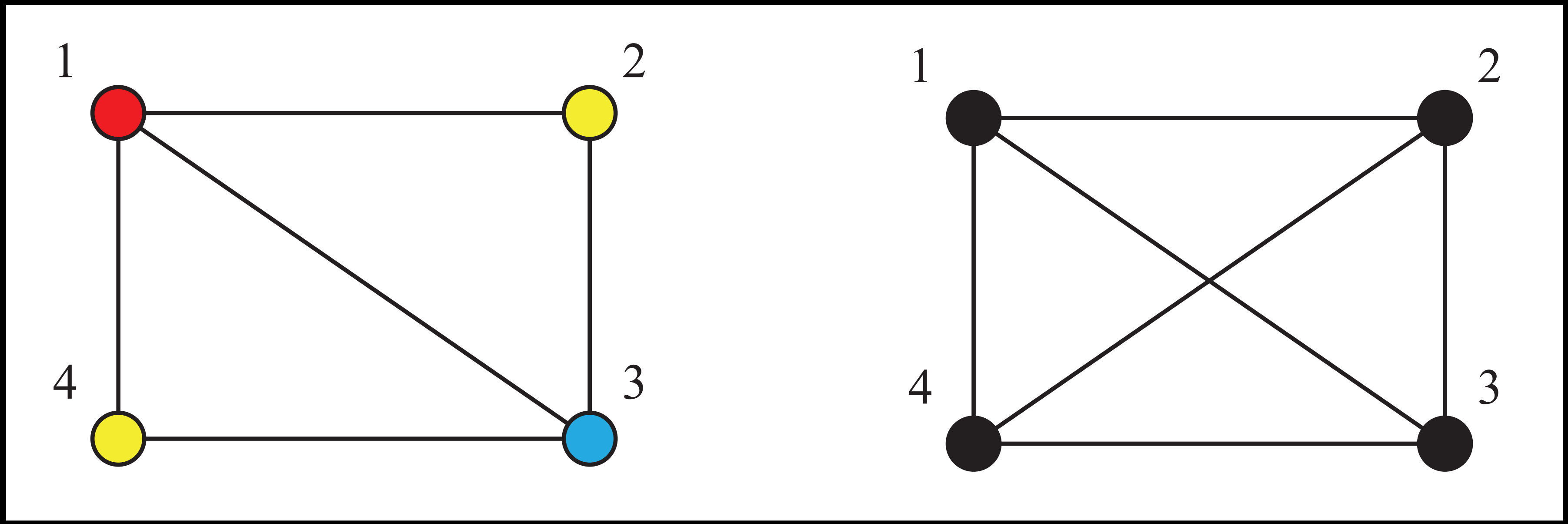
All quickly become inefficient
Is there an efficient algorithm?

NP-complete / NP-hard problems

[Cook-Karp-Levin 1971/2]

Graph coloring: Given a graph G, is there a 3-coloring?

NP-complete problem

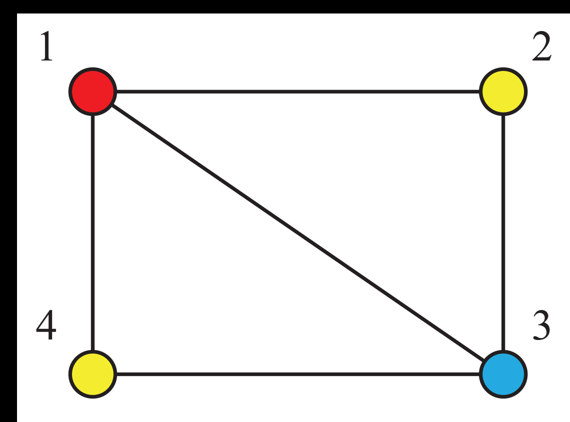


YES NO

Million $$$ prize (Clay Math)

Graph coloring: Given a graph G, is there a 3-coloring?

Theorem [Bayer 1982]: Whether or not a graph is 3-colorable can be encoded as whether a system of quadratic equations over **C** has a nonzero solution



$$
\begin{aligned}
&a_1c_1 - b_1d_1 - u^2,\ b_1c_1 + a_1d_1,\ c_1u - a_1^2 + b_1^2,\ d_1u - 2a_1b_1,\ a_1u - c_1^2 + d_1^2,\ b_1u - 2d_1c_1,\\
&a_2c_2 - b_2d_2 - u^2,\ b_2c_2 + a_2d_2,\ c_2u - a_2^2 + b_2^2,\ d_2u - 2a_2b_2,\ a_2u - c_2^2 + d_2^2,\ b_2u - 2d_2c_2,\\
&a_3c_3 - b_3d_3 - u^2,\ b_3c_3 + a_3d_3,\ c_3u - a_3^2 + b_3^2,\ d_3u - 2a_3b_3,\ a_3u - c_3^2 + d_3^2,\ b_3u - 2d_3c_3,\\
&a_4c_4 - b_4d_4 - u^2,\ b_4c_4 + a_4d_4,\ c_4u - a_4^2 + b_4^2,\ d_4u - 2a_4b_4,\ a_4u - c_4^2 + d_4^2,\ b_4u - 2d_4c_4,\\
&a_1^2 - b_1^2 + a_1a_3 - b_1b_3 + a_3^2 - b_3^2,\ a_1^2 - b_1^2 + a_1a_4 - b_1b_4 + a_4^2 - b_4^2,\ a_1^2 - b_1^2 + a_1a_2 - b_1b_2 + a_2^2 - b_2^2,\\
&a_2^2 - b_2^2 + a_2a_3 - b_2b_3 + a_3^2 - b_3^2,\ a_3^2 - b_3^2 + a_3a_4 - b_3b_4 + a_4^2 - b_4^2,\ 2a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2,\\
&2a_2b_2 + a_2b_3 + a_3b_2 + 2a_3b_3,\ 2a_1b_1 + a_1b_3 + a_2b_1 + 2a_3b_3,\ 2a_1b_1 + a_1b_4 + a_4b_1 + 2a_4b_4,\\
&2a_3b_3 + a_3b_4 + a_4b_3 + 2a_4b_4,\ w_1^2 + w_2^2 + \cdots + w_{17}^2 + w_{18}^2.
\end{aligned}
$$

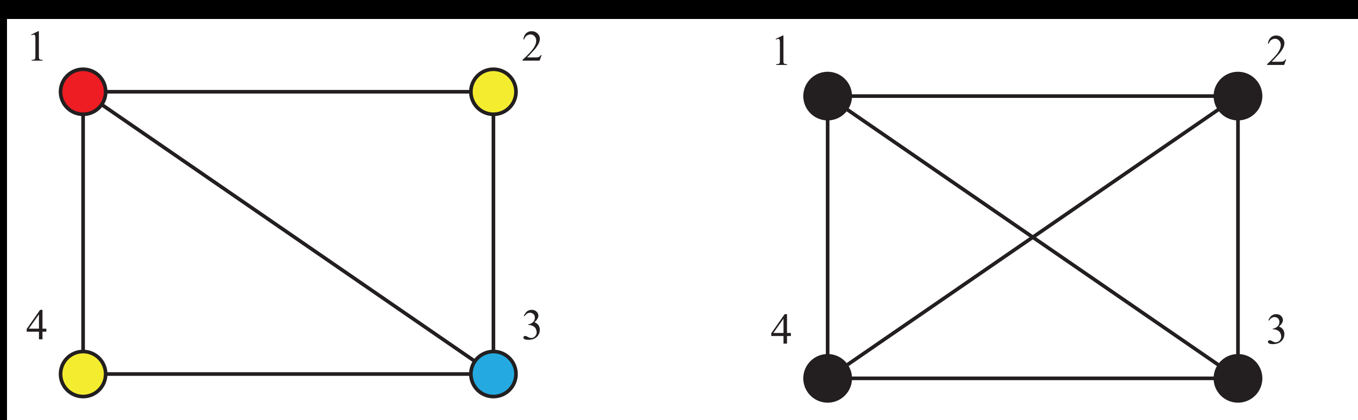
Using symbolic algebra or numerical algebraic geometry software[7] (see the Appendix for a list), one can solve these equations to find six real solutions (without loss of generality, we may take $u = 1$ and all $w_j = 0$), which correspond to the proper 3-colorings of the graph G as follows. Fix one such solution and define $x_k := a_k + \mathrm{i}b_k \in \mathbb{C}$ for $k = 1, \ldots, 4$ (we set $\mathrm{i} := \sqrt{-1}$). By construction, these x_k are one of the three cube roots of unity $\{1, \alpha, \alpha^2\}$ where $\alpha = \exp(2\pi\mathrm{i}/3) = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$ (see also Fig. 2).

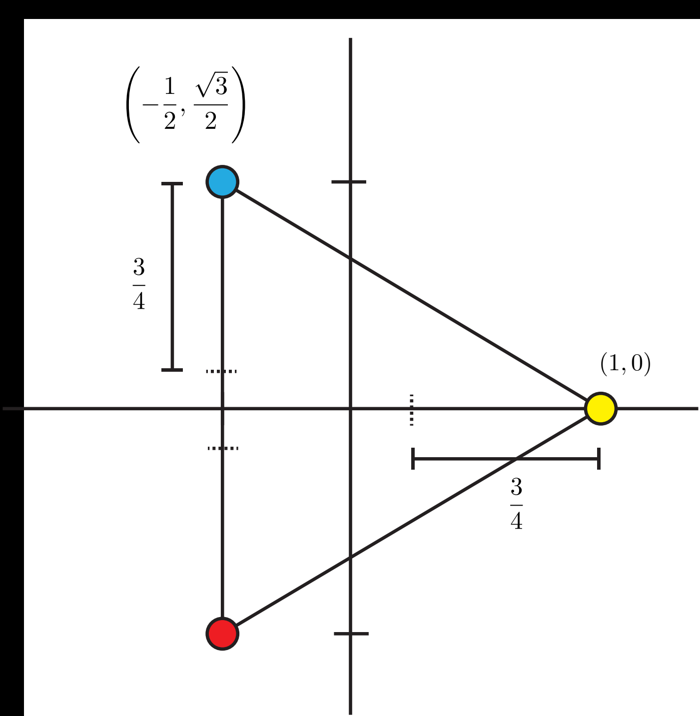
To determine a 3-coloring from this solution, one "colors" each vertex i by the root of unity that equals x_i. It can be checked that no two adjacent vertices share the same color in a coloring; thus, they are proper 3-colorings. For example, one solution is:

$$x_1 = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}, \quad x_2 = 1, \quad x_3 = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}, \quad x_4 = 1.$$

Polynomials for the right-hand side graph in Fig. 1 are the same as (4) except for two additional ones encoding a new restriction for colorings, the extra edge $\{2, 4\}$:

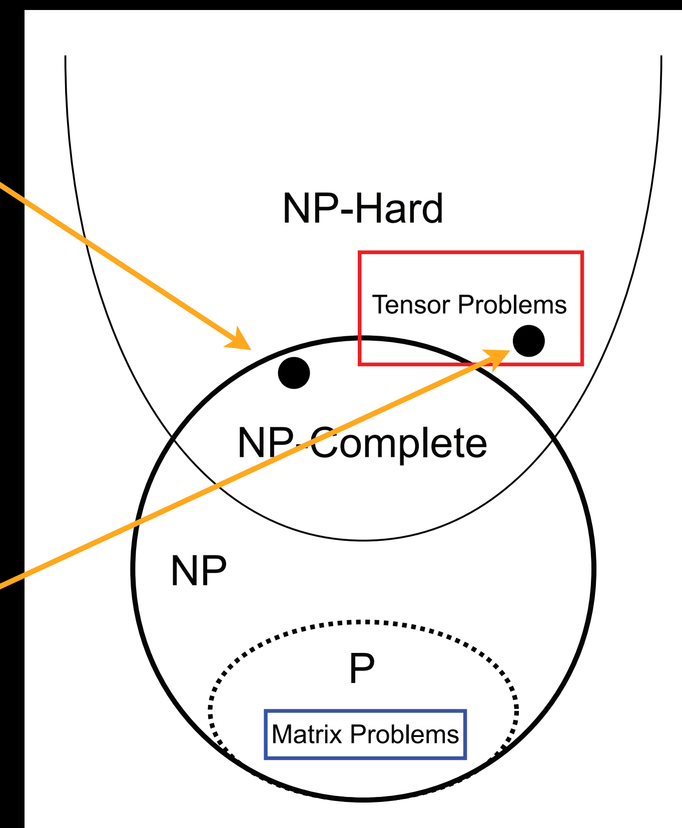
$$a_2^2 - b_2^2 + a_2a_4 - b_2b_4 + a_4^2 - b_4^2, \quad 2a_2b_2 + a_2b_4 + a_4b_2 + 2a_4b_4.$$





Quadratic equations are hard to solve

[Bayer 1982], [Lovasz 1994], [Grenet et al 2010], ...



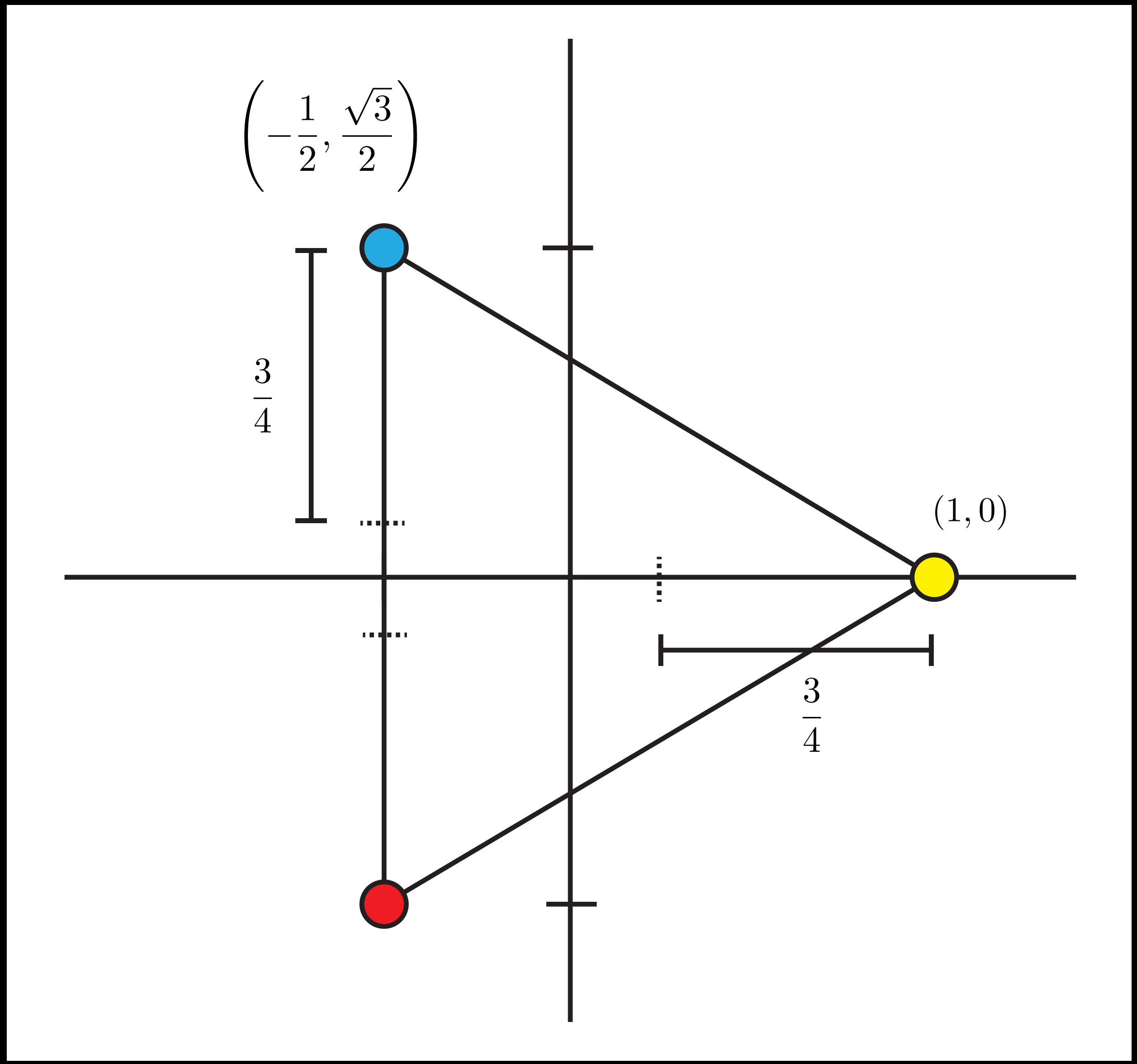
Corollary: Deciding tensor eigenvalue is NP-hard

Quadratic equations undecidable over the integers!
(there is no Turing machine that can solve them)

[Davis-Putnam-Robinson, Matijasevic]

Approximating tensor eigenvector is NP-hard

Corollary: Unless P = NP, there is no polynomial time approximation scheme for finding tensor eigenvectors



Rank

rank-1 tensors: $\mathcal{A} = [[x_i y_j z_k]] = x \otimes y \otimes z$

- Segre variety

Definition: Tensor rank over F is

$$\text{rank}_{\mathbb{F}}(\mathcal{A}) = \min_r \{\mathcal{A} = \sum_{i=1}^{r} x_i \otimes y_i \otimes z_i\}.$$

Theorem [Hastad]: Tensor rank is NP-hard over Q

Note: rank can change over changing fields (not true linear algebra)

Theorem: There are rational tensors with different rank over rationals versus the reals

$$\operatorname{rank}_{\mathbb{R}}(\mathcal{A}) < \operatorname{rank}_{\mathbb{Q}}(\mathcal{A})$$

PROOF OF THEOREM 1.14. We explicitly construct a rational tensor $\mathcal{A}$ with $\operatorname{rank}_{\mathbb{R}}(\mathcal{A}) < \operatorname{rank}_{\mathbb{Q}}(\mathcal{A})$. Let $\mathbf{x} = [1,0]^\top$ and $\mathbf{y} = [0,1]^\top$. First observe that

$$\overline{\mathbf{z}} \otimes \mathbf{z} \otimes \overline{\mathbf{z}} + \mathbf{z} \otimes \overline{\mathbf{z}} \otimes \mathbf{z} = 2\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} - 4\mathbf{y} \otimes \mathbf{y} \otimes \mathbf{x} + 4\mathbf{y} \otimes \mathbf{x} \otimes \mathbf{y} - 4\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{y} \in \mathbb{Q}^{2\times2\times2},$$

where $\mathbf{z} = \mathbf{x} + \sqrt{2}\mathbf{y}$ and $\overline{\mathbf{z}} = \mathbf{x} - \sqrt{2}\mathbf{y}$. Let $\mathcal{A}$ be this tensor; thus, $\operatorname{rank}_{\mathbb{R}}(\mathcal{A}) \leq 2$. We claim that $\operatorname{rank}_{\mathbb{Q}}(\mathcal{A}) > 2$. Suppose not and that there exist $\mathbf{u}_i = [a_i, b_i]^\top$, $\mathbf{v}_i = [c_i, d_i]^\top \in \mathbb{Q}^2$, $i = 1,2,3$, with

$$\mathcal{A} = \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \mathbf{u}_3 + \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3. \tag{29}$$

Identity (29) gives eight equations found in (30). Thus, by Lemma 8.1, $\operatorname{rank}_{\mathbb{Q}}(\mathcal{A}) > 2$. □

LEMMA 8.1. *The system of* 8 *equations in* 12 *unknowns:*

$$\begin{aligned} &a_1a_2a_3 + c_1c_2c_3 = 2,\; a_1a_3b_2 + c_1c_3d_2 = 0,\; a_2a_3b_1 + c_2c_3d_1 = 0,\\ &a_3b_1b_2 + c_3d_1d_2 = -4,\; a_1a_2b_3 + c_1c_2d_3 = 0,\; a_1b_2b_3 + c_1d_2d_3 = -4,\\ &a_2b_1b_3 + c_2d_3d_1 = 4,\; b_1b_2b_3 + d_1d_2d_3 = 0 \end{aligned} \tag{30}$$

has no solution in rational numbers a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, *and* d_1, d_2, d_3.

PROOF. One may verify in exact symbolic arithmetic (see the Appendix) that the following two equations are polynomial consequences of (30):

$$2c_2^2 - d_2^2 = 0 \quad \text{and} \quad c_1d_2d_3 - 2 = 0.$$

Since no rational number when squared equals 2, the first equation implies that any rational solution to (30) must have $c_2 = d_2 = 0$, an impossibility by the second. Thus, no rational solutions to (30) exist. □

Theorem: There are rational tensors with different rank over rationals versus the reals

$$\mathrm{rank}_{\mathbb{R}}(\mathcal{A}) < \mathrm{rank}_{\mathbb{Q}}(\mathcal{A})$$

Thus, Hastad's result doesn't necessarily apply.

Nevertheless, Hastad's proof shows that tensor rank is NP-hard over R and C

Theorem: Approximating rank-1 tensor is NP-hard

Hyperdeterminant

- Defining equation for the dual variety to Segre variety

[Cayley 1845]:

Example 3.2 (2 × 2 × 2 *hyperdeterminant*). For $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{C}^{2\times2\times2}$, define

$$\mathrm{Det}_{2,2,2}(\mathcal{A}) := \frac{1}{4}\left[\det\left(\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{bmatrix} + \begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{bmatrix}\right) - \det\left(\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{bmatrix} - \begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{bmatrix}\right)\right]^2 - 4\det\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{bmatrix}\det\begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{bmatrix}.$$

Given a matrix $A \in \mathbb{C}^{n\times n}$, the pair of linear equations $\mathbf{x}^\top A = \mathbf{0}$, $A\mathbf{y} = \mathbf{0}$ has a nontrivial solution ($\mathbf{x}, \mathbf{y}$ both nonzero) if and only if $\det(A) = 0$. Cayley proved a multilinear version that parallels the matrix case. The following system of bilinear equations:

$$\begin{aligned}
a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 = 0, &\quad a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 = 0,\\
a_{000}x_0z_0 + a_{001}x_0z_1 + a_{100}x_1z_0 + a_{101}x_1z_1 = 0, &\quad a_{010}x_0z_0 + a_{011}x_0z_1 + a_{110}x_1z_0 + a_{111}x_1z_1 = 0,\\
a_{000}y_0z_0 + a_{001}y_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 = 0, &\quad a_{100}y_0z_0 + a_{101}y_0z_1 + a_{110}y_1z_0 + a_{111}y_1z_1 = 0,
\end{aligned}$$

has a nontrivial solution ($\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^2$ all nonzero) if and only if $\mathrm{Det}_{2,2,2}(\mathcal{A}) = 0$. □

Conjecture: NP-hard to compute hyperdeterminant

Conclusions

"All interesting problems are NP-hard"

- Bernd Sturmfels

Conclusions

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+

"Most tensor problems are NP-hard"

Conclusions

"All interesting problems are NP-hard"

- Bernd Sturmfels

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"Most tensor problems are NP-hard"

↓

"Most tensor problems are interesting!"

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