Basic Results: - Joint w Buczynski, Landsberg

$X \subset \mathbb{P}(V)$ smooth, $v_d : \mathbb{P}(V) \to \mathbb{P}(Sym^d(V))$

$$\begin{tikzcd}
 & \mathbb{P}(V) \arrow[dr] & \\
X \arrow[ur, hook] \arrow[dr] & & \mathbb{P}(Sym^d(V)) \\
 & \langle v_d(X) \rangle \arrow[ur] &
\end{tikzcd}$$

always $\sigma_2(v_d(X)) \subset \langle v_d(X) \rangle \cap \sigma_2(v_d(\mathbb{P}^n))$

2) Have equality if $d \geq r-1+\mathrm{Cot}(X)$

) So eqns defining $\sigma_2(v_d(X))$ are "inherited"

from eqns defining $\sigma_2(v_d(\mathbb{P}^n))$

which are the determinantal eqns.

Key Lemma: All as above & R a degree $= r$

zero diml subscheme, then $v_d(R \cap X) = v_d(R) \cap v_d(X)$

Uniqueness: $p \in Sym^a\, \mathbb{P}(Sym^d(V))$ if $r_{v_d(\mathbb{P}^n)}(p) \leq \frac{d+1}{2}$

then $br(p) = r(p)$.

) ~~Fat~~ All fuels of X is singular.

2) Basic Notation $X \subset \mathbb{P}(V)$ a variety over $\mathbb{C}$

3) $\langle X \rangle$ = linear span of $X \subset \mathbb{P}(V)$ ie $\langle X \rangle$ is the proj space w/ lin fns given by $H^0(\mathcal{O}_X(1))$.

4) $\sigma_r(X) = \overline{\bigcup_{\substack{p_i \in X \\ \text{distinct}}} \langle p_1 \dots p_r \rangle}$

5) If $p \in \mathbb{P}(V)$ & $p \in \langle p_1 \dots p_r \rangle$ but $\nexists$ $q_1 \dots q_{r-1}$ ($p_i, q_i \in X$) s.t. $p \in \langle q_1, \dots, q_{r-1} \rangle$ then $r_X(p) = r$

If $p \in \sigma_r(X) \setminus \sigma_{r-1}(X)$ $br_X(p) = r$

note $r_X \leq br_X$ always.

6) Rmk: rank more intuitive, but br more computable

7) Goal: Find eqns for X & $\sigma_r(X) \subset \mathbb{P}(V)$

8) Impossible as stated in sense anything is possible. equations are not intrinsic to X

9) $v_d : \mathbb{P}(V) \to \mathbb{P}(Sym^d(V))$: Abbreviated

$\mathbb{P}^n \to \mathbb{P}^N$ $N = \binom{n+d}{d} - 1$

Cont   Fix a basis of $F_1, \dots, F_N \in \mathrm{Sym}^d(V^\vee)$ then

$v \in V \mapsto (F_1(v), \dots, F_N(v))$

Im Point to note: hyperplanes $H \subset \mathbb{P}^N \leftrightarrow$

$V(F) \subset \mathbb{P}^n$ ie hyperplanes $\subset \mathbb{P}^N \leftrightarrow$ hypersurfaces of deg $= d$

$\subset \mathbb{P}^n$

so $X \subset V(F) \Leftrightarrow v_d(X) \subset H$ a hyperplane

$v_d(\mathbb{P}^n) \subset \mathbb{P}^N$ is defined by quadratic eqns

From this (Mumford) $v_d(X)$ is defined by quadratic eqns in $\langle v_d(X) \rangle$

Generalize ~~to $\sigma_r(X)$~~: argument to $\sigma_r(X)$

Picture

$$\begin{tikzcd} & \mathbb{P}(V) \arrow[dr] & \\ X \arrow[ur, hook] \arrow[dr] & & \mathbb{P}(\mathrm{Sym}^d(V)) \\ & \langle v_d(X) \rangle \arrow[ur, hook] & \end{tikzcd}$$

So Always $\sigma_r(v_d(X)) \subset \sigma_r(v_d(\mathbb{P}^n)) \cap \langle v_d(X) \rangle$

so have 'determinantal eqns'

4

18) What are eqns of $\sigma_r(v_d(\mathbb{P}^n))$ & are they ones

19) Complete eqns for $\sigma_r(v_d(\mathbb{P}^n))$ not known

20) Hope they are in degree $r+1$: Buczynski & Buczynska show the obvious ones are not enough. These are the det eqns though.

21) But do we have equality $\sigma_r(v_d(X)) = \langle v_d(X)\rangle \cap \sigma_r(v_d(\mathbb{P}^n))$?

22) Our result: true set theoretically once $d \geq \mathrm{Cast}(X) + r - 1$

23) What is $\mathrm{Cast}(X)$: smallest integer so that have C-M. regularity of $\mathcal{O}_X$

24) If $d \geq \mathrm{Cast}(X)$ then $h_X(d) = h^0(\mathcal{O}_X(d)) = \langle v_d(X)\rangle - 1$ ($\mathcal{O}_X(d)$ is $\mathcal{O}_X(1)$ for v_d)

25) Recall our thms: Main th:

$d \geq r-1 + \mathrm{Cast}(X)$ then ~~$\sigma_r(\ )$~~

$\sigma_r(v_d(X)) = \langle v_d(X)\rangle \cap \sigma_r(v_d(\mathbb{P}^n))$.

26) Follows from key Lemma & proving that a subscheme that is smoothable on $\mathbb{P}^n$ is smoothable on X → This uses X smooth

27) Key Lemma $\langle v_d(R \cap X)\rangle = \langle v_d(X)\rangle \cap \langle v_d(R)\rangle$
(Pf is lin alg of Hilbert polys)

28) Small rank thm of $r(p) \leq \frac{d+1}{2} \Rightarrow$
$r(p) = br(p)$ also follows. In essence Lemma ⇒ uniqueness.

29) Results fail for gen'l sing X -
$\sigma_2(v_d(X)) \underset{\neq}{\subseteq} \sigma_2(v_d\mathbb{P}^n) \langle v_d(X)\rangle +$
Need eqns of large degree.