A Proof of the Set-theoretic Version of a Salmon Conjecture

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[1] 2011 SIAM Conference on Applied Algebraic Geometry, October 6

Summary

1. Statement of the problem
2. Known results
3. New conditions
4. Outline of the complete solution

Notations

$\mathbb{C}^{m\times n\times l} := \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^l$ consists of $\mathcal{T} = [t_{i,j,k}]_{i=j=k=1}^{m,n,l}$

$V_r(m,n,l) \subset \mathbb{C}^{m\times n\times l}$ the closure of 3-mode tensors of rank r at most

$\mathbb{P}V_r(m,n,l) = \mathrm{Sec}_r(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{l-1})$.

$I_r(m,n,l) \subset \mathbb{C}[\mathbb{C}^{m\times n\times l}]$ the ideal defining $V_r(m,n,l)$.

$\mathbf{T}_3(\mathcal{T}) \subset \mathbb{C}^{m\times n}$ subspace spanned by l frontal sections $[t_{i,j,k}]_{i=j=1}^{m,l}, k = 1, \ldots, l$.

Similarly $\mathbf{T}_1(\mathcal{T}) \subset \mathbb{C}^{n\times l}$, $\mathbf{T}_2(\mathcal{T}) \subset \mathbb{C}^{m\times l}$

$\mathrm{S}_n(\mathbb{C})$ - symmetric $n \times n$ matrices

A short history of the salmon conjecture

At the IMA workshop in March 2007, Elizabeth Allman offered an Alaskan speciality: smoked Copper river salmon for determining the generators of $I_4(4,4,4)$.

$V_4(4,4,4)$ appears as a basic bloc in molecular phylogenetics [3], in which DNA sequences are used to infer evolutionary trees describing the descent of species from a common ancestor. $\mathbb{C}^4$ comes from 4 nucleotides A;C; G; T. 3-mode tensor comes from an ancestor splitting two species, since all internal nodes of an evolutionary tree are of degree 3.

In Pachter-Sturmfels book [2, Conjecture 3.24] states $I_4(4,4,4)$ is generated by polynomials of degree 5 and 9. The degree 5 are coming from Strassen's commutative conditions [3, 1], degree 9 from Strassen's result: $V_4(3,3,3)$ is a hypersurface of degree 9.

In view of degree 6 polynomials in $I_4(4,4,4)$ found by Landsberg and Manivel [5] Sturmfels revised the Salmon conjecture:

$I_4(4,4,4)$ is generated by polynomials of degree $5,6,9$ [4, §2].

Tensors of rank m in $\mathbb{C}^{m\times m\times l}$ Strassen's commutative conditions

$\mathcal{T} \in \mathbb{C}^{m\times m\times l}$, $\operatorname{rank} \mathcal{T} = m$, $\mathbf{W} = \operatorname{span}(T_{1,3}, \ldots, T_{l,3}) \in \mathbb{C}^{m\times m}$
spanned by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_m\mathbf{v}_m^\top$.

generic case: $\exists P, Q \in \mathbf{GL}(m, \mathbb{C})$ $P\mathbf{W}Q$
subspace of commuting of diagonal matrices.
If $\mathbf{W}$ contains invertible Z then
$(PXQ)(PZQ)^{-1}(PYQ) = (PYQ)(PZQ)^{-1}(PXQ) \Rightarrow$
$X(\mathrm{adj}Z)Y = Y(\mathrm{adj}Z)X$
for all $X, Y \in \mathbf{W}$
equations of degree 5 for $m = 4$

similarly $\mathrm{C}_r(X)\widetilde{\mathrm{C}_{m-r}(Z)}\mathrm{C}_r(Y) = \mathrm{C}_r(Y)\widetilde{\mathrm{C}_{m-r}(Z)}\mathrm{C}_r(Y)$
equations of degree $m + r$ for $r = 1, \ldots, \lfloor \frac{m}{2} \rfloor$.

For $m = 4, r = 2$ polynomials of degree 6 but no new info.

Strassen and Manivel-Landsberg conditions

Strassen 1983 $V_4(3,3,3)$ is a hypersurface of degree 9

$$\frac{1}{\det\ Z}\det\left(X(\mathrm{adj}\,Z)Y - Y(\mathrm{adj}\,Z)X\right) = 0$$

X, Y, Z are three sections of $\mathcal{T} = [t_{i,j,k}] \in \mathbb{C}^{3\times3\times3}$

Landsberg-Manivel 2004: $I_4(3,3,4)$ contains polynomials of degree 6.
Study the action of $\mathbf{GL}(3,\mathbb{C}) \times \mathbf{GL}(3,\mathbb{C}) \times \mathbf{GL}(4,\mathbb{C})$ on $\mathbb{C}[\mathbb{C}^{3\times3\times4})]$
use Schur duality and symbolic computations to conclude existence of polynomials of degree 6.

Bates and Oeding[4] constructed explicitly using symbolic computations 10 polynomials of degree 6 in $I_4(3,3,4)$.

Symmetrization conditions for $V_{m+1}(m,m,l)$ [1]

For a generic $\mathcal{T} = [x_{i,j,k}] \in \mathbb{C}^{m\times n\times l}$, $X_k = [t_{i,j,k}]_{i=j=1}^{mn}$ of rank $m+1$
$\mathbf{T}_3(\mathcal{T}) \in \mathbb{C}^{m\times n}$ generated by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_{m+1}\mathbf{v}_{m+1}^\top$,
any m vectors out of $\mathbf{u}_1, \ldots, \mathbf{u}_{m+1}$ or $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$ linearly independent

$\exists P, Q \in \mathbf{GL}(m, \mathbb{C}) \Rightarrow P\mathbf{u}_i\mathbf{v}_i^\top Q^\top = \mathbf{e}_i\mathbf{e}_i^\top$ for $i = 1, \ldots, m$
and $P\mathbf{u}_{m+1}\mathbf{v}_{m+1}^\top Q^\top = \mathbf{w}\mathbf{w}^\top$.

$\exists L, R \in \mathbf{GL}(m, \mathbb{C})$ such that $L\mathbf{T}_3(\mathcal{T}), \mathbf{T}_3(\mathcal{T})R \in \mathrm{S}_n(\mathbb{C})$ (Symcon)
$LX_i - (LX_i)^\top = 0, i = 1, \ldots, l$ (Lsymcon): $(\frac{l(m(m-1)}{2})$ linear equation in entries of L
$X_iR - (X_iR)^\top = 0, i = 1, \ldots, l$ (Rsymcon): $(\frac{l(m(m-1)}{2})$ linear equation in entries of R
and $LR^\top = R^\top L = \frac{1}{n}\mathrm{tr}(LR^\top)I_n$ - (LRcond)
Existence of nonzero L, R: entries of $\mathcal{T}$ satisfy polynomial equations of degree m^2
(LRcond) yield polynomial equations of degree $2(m^2-1)$.

Characterization of $V_4(3,3,4)$

Generic subspace $\mathbf{W} \subset S_m(\mathbb{C}), \dim \mathbf{W} = \frac{m(m-1)}{2} + 1$ intersects variety of symmetric matrices of rank 1 at least at $\frac{m(m-1)}{2} + 1$ lin. ind. mat.

generic $\mathcal{T} \in \mathbb{C}^{3\times 3\times 4}$ symmetric in the first is in $V_4(3,3,4)$

Thm[1]: $V_4(3,3,4)$ characterized by (Lsymcon) - (Rsymcon) and (LRcond) degrees $9, 16$

Outline of proof: 1) assume only (Lsymcon) - (Rsymcon).
$R, L \in \mathbf{GL}(3,\mathbb{C})$ for generic $\mathcal{T}$ hence $\mathcal{T} \in V_4(3,3,4)$

Rest of proof: analyze cases where L, R are nonzero singular
All cases except the following are fine
A.I.3[1]: R, L are rank one matrices

(LRcond) (degree 16) yield $LR^\top = R^\top L = 0 \Rightarrow \mathcal{T} \in V_4(3,3,4)$

Friedland-Gross simplification [2]

Assume A.I.3 and either $LR^\top \neq 0$ or $R^\top L \neq 0$.
Change bases to get $L = \mathbf{e}_3\mathbf{e}_3^\top$ and $R \in \{\mathbf{e}_2\mathbf{e}_2^\top, \mathbf{e}_2\mathbf{e}_3^\top, \mathbf{e}_3\mathbf{e}_2^\top, \mathbf{e}_3\mathbf{e}_3^\top\}$
in first 3 cases for $R\ \mathcal{T} \in V_4(3,3,4)$

$$R = \mathbf{e}_3\mathbf{e}_3^\top \Rightarrow X_k = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} & 0 \\ x_{2,1,k} & x_{2,2,k} & 0 \\ 0 & 0 & x_{3,3,k} \end{bmatrix}, \quad k = 1,2,3,4,$$

It is shown in [1] that most $\mathcal{T}$ in $V_5(3,3,4) \setminus V_4(3,3,4)$
10 polynomials [4] are $x_{3,3,k}x_{3,3,l}f(\mathcal{X}) = 0, 1 \leq k \leq l \leq 4$

$$\det \begin{bmatrix} x_{1,1,1} & x_{1,2,1} & x_{2,1,1} & x_{2,2,1} \\ x_{1,1,2} & x_{1,2,2} & x_{2,1,2} & x_{2,2,2} \\ x_{1,1,3} & x_{1,2,3} & x_{2,1,3} & x_{2,2,3} \\ x_{1,1,4} & x_{1,2,4} & x_{2,1,4} & x_{2,2,4} \end{bmatrix}$$

So $\mathcal{T} \in V_4(3,3,4)$ since for $\mathcal{X} \in \mathbb{C}^{2\times2\times2}$: $\mathrm{rank}\ \mathcal{X} \leq 4$
and $\mathrm{rank}\ \mathcal{X} \leq 3$ if $\dim \mathbf{T}_3(\mathcal{X}) \leq 3$.

From $V_4(3,3,4)$ to $V_4(4,4,4)$

Manivel-Landsberg[1]: Cor. 5.6: Let $\mathcal{T} \in \mathbb{C}^{4\times4\times4}$ satisfies Strassen's commutative conditions of degree 5. Then either $\mathcal{T} \in V_4(4,4,4)$ or there exists $p \in \{1,2,3\}, \mathbf{u}, \mathbf{v} \in \mathbb{C}^4 \setminus \{\mathbf{0}\}$ such that
$\mathbf{u}^\top \mathbf{T}_p(\mathcal{T}) = \mathbf{0}^\top, \mathbf{T}_p(\mathcal{T})\mathbf{v} = \mathbf{0}$.
I.e. after change of bases and permuting the factors of $\mathbb{C}^4$ $\mathcal{T} \in \mathbb{C}^{3\times3\times4}$

Prf. is wrong as Prop. 5.4 wrong.

7 pages of [1] devoted to proof of Corollary 5.6.

Nice characterization of subspace $\mathbf{U} \subset \mathbb{C}^{m\times m}$ where most of the matrices are of rank $m-1$ which satisfy Strassen's commutative condition.

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