# Tensor Complexes

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# Overview

1. Two examples where free resolutions appear in algebraic geometry.
2. Explicit construction of free resolutions from tensors.
3. Hyperdeterminantal varieties.

# Free resolutions

- Let $S = \Bbbk[x_1, \ldots, x_N]$ with $\deg(x_i) = 1$.
- A graded *free resolution* over $S$ is an acyclic complex

$$F_\bullet : \; 0 \longleftarrow F_0 \xleftarrow{\partial_0} F_1 \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_c} F_c \longleftarrow 0$$

with $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$ and all $\partial_i$ have degree 0.

- Further, $F_\bullet$ is *minimal* if $(\text{image}\, \partial_i) \subseteq \langle x_1, \ldots, x_N \rangle F_{i-1} \;\; \forall\, i$.

# Example 1: Determinantal varieties

- $S = \Bbbk[x_{ij}], \quad M = (x_{ij}) \in S^{p \times q}, \; p \geq q.$
- $I_X = \langle (q \times q)\text{-minors of } M \rangle \subseteq S.$
- $X = \mathrm{Spec}(S/I_X)$ has closed points
$$\{(p \times q)\text{-matrices of rank } < q\} \subseteq \Bbbk^{pq}.$$
- The Eagon–Northcott complex resolves $S/I_X$ with
$$\mathrm{pdim}_S(S/I_X) = \mathrm{codim}_{\mathbb{A}^{pq}}(X) = p - q + 1.$$

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# Example 2: Hyperdeterminantal hypersurfaces

- Let $\mathbb{P}^1 \times \mathbb{P}^1$ have coordinates $([u_0 : u_1], [v_0 : v_1])$.
- Choose scalars $a_{ijk}$ to define multilinear forms:

$$
\begin{aligned}
f_1 &= a_{100} u_0 v_0 + a_{110} u_1 v_0 + a_{101} u_0 v_1 + a_{111} u_1 v_1, \\
f_2 &= a_{200} u_0 v_0 + a_{210} u_1 v_0 + a_{201} u_0 v_1 + a_{211} u_1 v_1, \\
f_3 &= a_{300} u_0 v_0 + a_{310} u_1 v_0 + a_{301} u_0 v_1 + a_{311} u_1 v_1.
\end{aligned}
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- $V = \mathrm{Var}(f_1, f_2, f_3) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.
- Let $\Delta_{3\times 2\times 2}$ be a hyperdeterminant [Cayley, GKZ].

Fact: $V \neq \varnothing$ if and only if $(a_{ijk}) \in \mathrm{Var}(\Delta_{3\times 2\times 2})$.

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# Hyperdeterminants from free resolutions

- Set $S = \mathbb{k}[x_{ijk}]$ with $i \in \{1, 2, 3\}$ and $j, k \in \{0, 1\}$.
- Consider the complex $F_\bullet : S^6 \xleftarrow{\partial} S^6(-1) \leftarrow 0$, with

$$\partial = \begin{pmatrix} x_{100} & 0 & x_{200} & 0 & x_{300} & 0 \\ x_{101} & x_{100} & x_{201} & x_{200} & x_{301} & x_{300} \\ 0 & x_{101} & 0 & x_{201} & 0 & x_{301} \\ x_{110} & 0 & x_{210} & 0 & x_{310} & 0 \\ x_{111} & x_{110} & x_{211} & x_{210} & x_{311} & x_{310} \\ 0 & x_{111} & 0 & x_{211} & 0 & x_{311} \end{pmatrix}.$$

- Then $\det(\partial) = \Delta_{3\times 2\times 2}$.
- Thus $V \neq \varnothing$ if and only if $F_\bullet \otimes_S \frac{S}{\langle x_{ijk} - a_{ijk} \rangle}$ does not give an isomorphism of $\mathbb{k}$-vector spaces.

# The framework

- The underlying tensors in
  - Ex. 1: $S = \Bbbk[x_{ij}]$, $M = (x_{ij}) = \phi^\flat$ for $\phi = (x_{ij}) \in S^p \otimes (S^q)^*$.
  - Ex. 2: $S = \Bbbk[x_{ijk}]$, $\phi = (x_{ijk}) \in S^3 \otimes (S^2)^* \otimes (S^2)^*$.
- Fix $a, b_1, \ldots, b_n \in \mathbb{N}$.
- $S = \mathbb{Z}[x_{i,J}]$ with $1 \leq i \leq a$, $J = (j_1, \ldots, j_n)$ with $1 \leq j_k \leq b_k$.
- Tensor: $\phi = (x_{i,J}) \in S^a \otimes (S^{b_1})^* \otimes \cdots \otimes (S^{b_n})^*$

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# Main properties

**Theorem [B.–Erman–Kummini–Sam]:**

1. $F(\phi, w)_\bullet$ is a graded minimal pure free resolution of a CM module $M(\phi, w)$.
2. $F(\phi, w)_\bullet$ is uniformly minimal over $\mathbb{Z}$.
3. $F(\phi, w)_\bullet$ is $G = \mathrm{GL}_a \times \mathrm{GL}_{b_1} \times \cdots \times \mathrm{GL}_{b_n}$-equivariant.
4. There is an explicit presentation of $F(\phi, w)_\bullet$.

Tensor complexes provide a unifying view of:

- Koszul complexes,
- Buchsbaum–Eisenbud matrix complexes, including the Eagon–Northcott and Buchsbaum–Rim complexes,
- Gelfand–Kapranov–Zelevinsky discriminantal complexes,
- Eisenbud–Schreyer pure free resolutions.

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# The geometric technique of Kempf–Lascoux–Weyman

- Note that $\mathbb{A}^{a \times b_1 \times \cdots \times b_n} = \text{Spec}(S)$.
- $\pi \colon \mathbb{A}^{a \times b_1 \times \cdots \times b_n} \times \mathbb{P}(\mathbb{Z}^{b_1}) \times \cdots \times \mathbb{P}(\mathbb{Z}^{b_n}) \twoheadrightarrow \mathbb{A}^{a \times b_1 \times \cdots \times b_n}$.
- $\mathcal{K}(\phi)_\bullet$: a Koszul complex of multilinear forms.
- Tensor with $\mathcal{O}(w)$ and apply $\mathbf{R}\pi_*$ to obtain $F(\phi, w)_\bullet$.
- This implies acyclicity and $G$-equivariance of $F(\phi, w)_\bullet$.

# Explicit differentials: the benefits of symmetry

- The $G$-action on $F(\phi, w)_\bullet$ allows us to understand its differentials.
- After fixing distinguished bases in

$$\cdots \leftarrow F(\phi, w)_{i-1} \xleftarrow{\partial_i} F(\phi, w)_i \leftarrow \cdots ,$$

we explicitly obtain the matrix $\partial_i$ (unique up to sign).

- These entries are given by minors of the flattening

$$\phi^\flat \in S^a \otimes \left( (S^{b_1})^* \otimes \cdots \otimes (S^{b_k})^* \right) .$$

# $(4 \times 2 \times 2)$-example

- Let $A = \mathbb{Z}^4$, $B_1 = B_2 = \mathbb{Z}^2$, and $S = \mathbb{Z}[x_{i,J}]$.
- Let $\phi = (x_{i,J}) \in S \otimes (A \otimes B_1^* \otimes B_2^*)$ and $w = (0, 0, 2)$.

$$F(\phi, w)_\bullet : \quad \begin{bmatrix} S \\ \bigwedge^0 A \\ \mathrm{Sym}^0 B_1 \\ \mathrm{Sym}^2 B_2 \end{bmatrix} \xleftarrow{\partial_1} \begin{bmatrix} S(-2) \\ \bigwedge^2 A \\ \mathrm{Div}^0 B_1^* \otimes \bigwedge^2 B_1^* \\ \mathrm{Sym}^0 B_2 \end{bmatrix} \xleftarrow{\partial_2} \begin{bmatrix} S(-4) \\ \bigwedge^4 A \\ \mathrm{Div}^2 B_1^* \otimes \bigwedge^2 B_1^* \\ \mathrm{Div}^0 B_2^* \otimes \bigwedge^2 B_2^* \end{bmatrix} \longleftarrow 0$$

- Betti diagram: $\beta(F(\phi, w)_\bullet) = \begin{pmatrix} 3 & - & - \\ - & 6 & - \\ - & - & 3 \end{pmatrix}$.
- Fixing bases, $\partial_1$ and $\partial_2$ can be written in terms of a distinguished flattening of $\phi$.

# $(4 \times 2 \times 2)$-example

$$\phi^\flat : A^* \to B_1^* \otimes B_2^* \quad \text{as} \quad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} x_{1,(1,1)} & x_{2,(1,1)} & x_{3,(1,1)} & x_{4,(1,1)} \\ x_{1,(1,2)} & x_{2,(1,2)} & x_{3,(1,2)} & x_{4,(1,2)} \\ x_{1,(2,1)} & x_{2,(2,1)} & x_{3,(2,1)} & x_{4,(2,1)} \\ x_{1,(2,2)} & x_{2,(2,2)} & x_{3,(2,2)} & x_{4,(2,2)} \end{pmatrix} \end{array}.$$

Then

$$\partial_1^T = \begin{array}{c} \\ f_{\{1,2\},\{1,2\},\varnothing} \\ f_{\{1,3\},\{1,2\},\varnothing} \\ f_{\{1,4\},\{1,2\},\varnothing} \\ f_{\{2,3\},\{1,2\},\varnothing} \\ f_{\{2,4\},\{1,2\},\varnothing} \\ f_{\{3,4\},\{1,2\},\varnothing} \end{array} \begin{array}{c} \begin{matrix} g_{\varnothing,\varnothing,(2,0)} & g_{\varnothing,\varnothing,(1,1)} & g_{\varnothing,\varnothing,(0,2)} \end{matrix} \\ \begin{pmatrix} \Delta_{ac;12} & \Delta_{ad;12} + \Delta_{bc;12} & \Delta_{bd;12} \\ \Delta_{ac;13} & \Delta_{ad;13} + \Delta_{bc;13} & \Delta_{bd;13} \\ \Delta_{ac;14} & \Delta_{ad;14} + \Delta_{bc;14} & \Delta_{bd;14} \\ \Delta_{ac;23} & \Delta_{ad;23} + \Delta_{bc;23} & \Delta_{bd;23} \\ \Delta_{ac;24} & \Delta_{ad;24} + \Delta_{bc;24} & \Delta_{bd;24} \\ \Delta_{ac;34} & \Delta_{ad;34} + \Delta_{bc;34} & \Delta_{bd;34} \end{pmatrix} \end{array}.$$

# The support of $M(\phi, w)$

- Let $Y(\phi)$ denote the support of the module $M(\phi, w)$.

- $Y(\phi)$ is independent of $w$.
- $Y(\phi)$ is an integral subvariety of $\mathbb{A}^{a \times b_1 \times \cdots \times b_n}$ of codimension $a - \sum_i (b_i - 1)$.
- $Y(\phi)$ is $G$-equivariant.

**Theorem [BEKS]:**

If $a' = 1 + \sum_i (b_i - 1)$, then $Y(\phi)$ is set-theoretically defined by

$$\langle \text{hyperdeterminant of } \phi' \mid \phi' \text{ is } (a' \times b_1 \times \cdots \times b_n)\text{-subtensor of } \phi \rangle.$$

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# Ties to algebraic statistics

- In recent work, Sturmfels and Zwiernik express $2 \times 2 \times \cdots \times 2$-hyperdeterminants in terms of cumulants. The hypersurface cut out by such an equation is a context-specific independence (CSI) split model for binary random variables.

## Questions:

- Does $Y(\phi)$ have a statistical interpretation?
- Can statistical tools be applied to understand the defining equations of $Y(\phi)$?