Higher Secants of Sato’s Grassmannian

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Grassmannians: functoriality and duality

V finite-dimensional vector space
$\mathbf{Gr}_p(V) := \{v_1 \wedge \cdots \wedge v_p \mid v_i \in V\} \subseteq \bigwedge^p V$
cone over Grassmannian

rank-one alternating tensors

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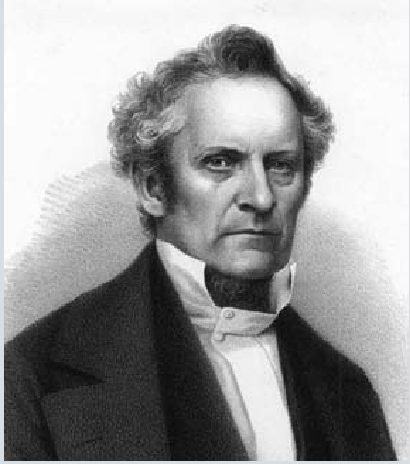
2. if $\dim V =: n + p$
$\rightsquigarrow$ natural map $\bigwedge^p V \to (\bigwedge^n V)^* \to \bigwedge^n (V^*)$
maps $\mathbf{Gr}_p(V) \to \mathbf{Gr}_{n-p}(V^*)$

Plücker varieties



Definition
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$$\mathbf{X}_p : \{\text{vector spaces } V\} \to \{\text{varieties in } \textstyle\bigwedge^p V\}$$

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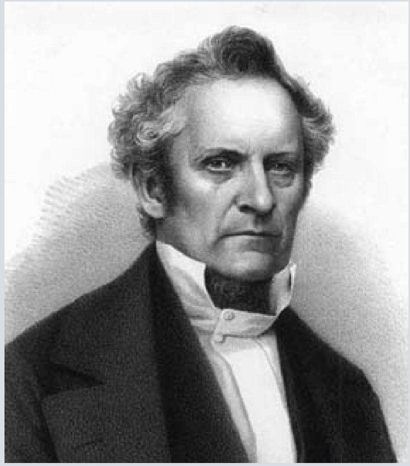

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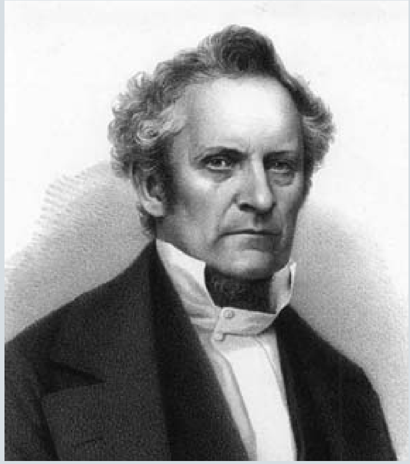

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Constructions
$\mathbf{X}, \mathbf{Y}$ Plücker varieties $\rightsquigarrow$ so are
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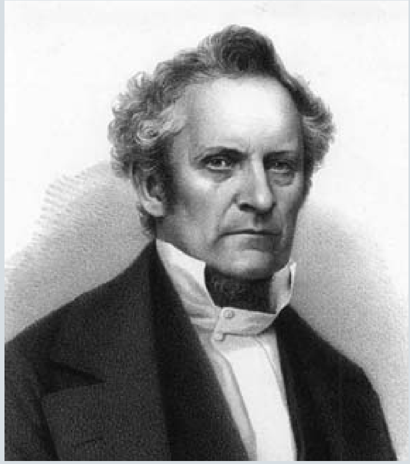

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skew analogue of Snowden's Δ-varieties

Results, with Eggermont

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Theorems apply, in particular, to
$k\mathbf{Gr}$ = {alternating tensors of *alternating rank* $\leq k$}.

The infinite wedge

$$V_\infty := \langle \ldots, x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3, \ldots \rangle$$
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$$\begin{array}{cc} \bigwedge^p V_{np} & \bigwedge^{p+1} V_{n,p+1} \\ \downarrow & \\ \bigwedge^p V_{n+1,p} & \end{array}$$

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Definition
$\bigwedge^{\infty/2} V_\infty := \lim_{\rightarrow} \bigwedge^p V_{n,p}$ *the infinite wedge* (charge-0 part);
basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, $I = \{i_1 < i_2 < \dots\}$, $i_k = k$ for $k \gg 0$

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On $\bigwedge^{\infty/2} V_\infty$ *acts* $\mathrm{GL}_\infty := \bigcup_{n,p} \mathrm{GL}(V_{n,p})$.

The limit of a Plücker variety

Dual diagram

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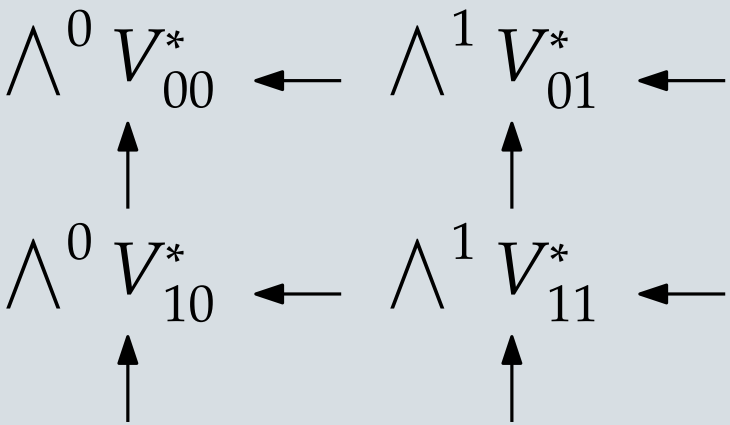

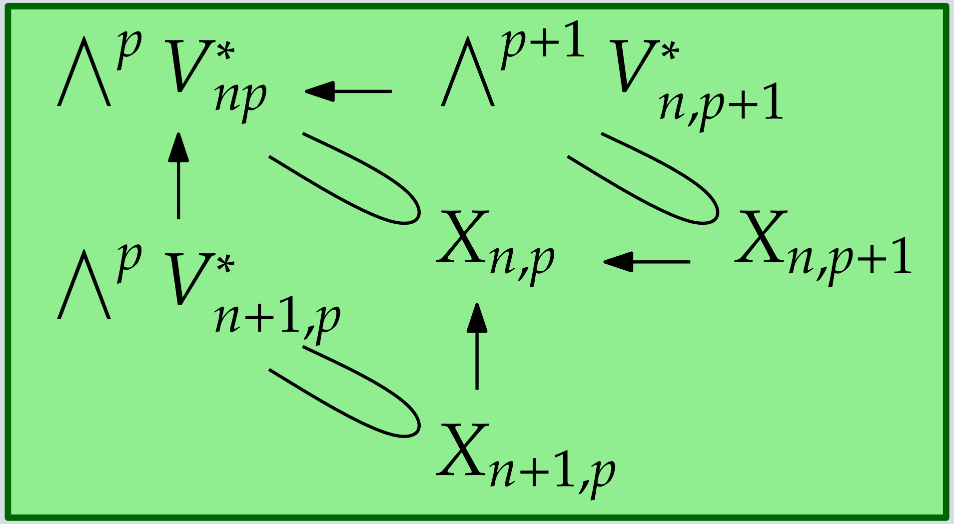

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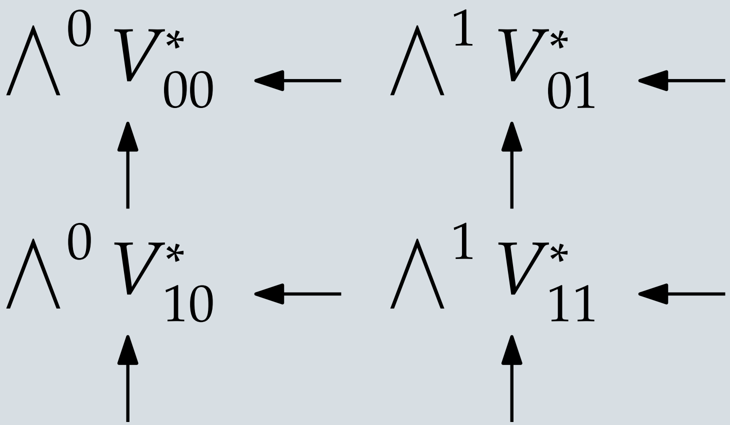

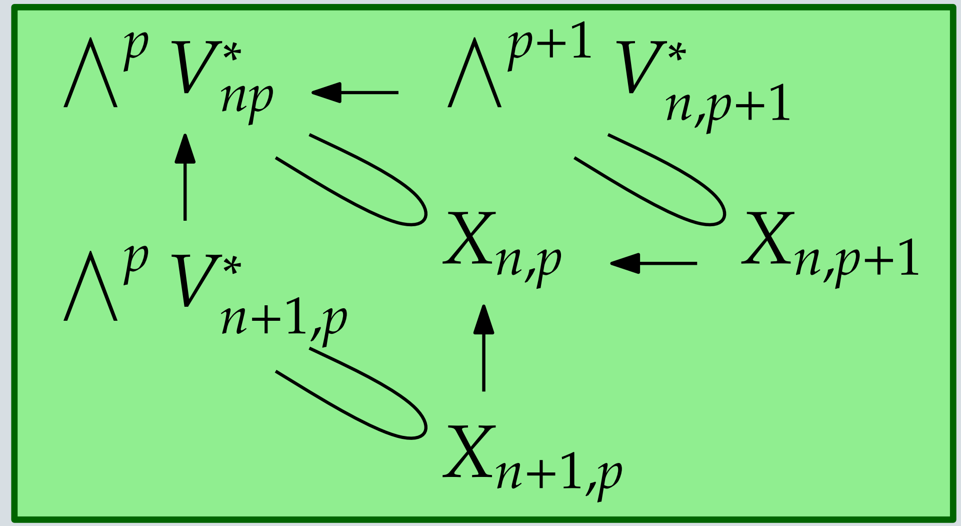

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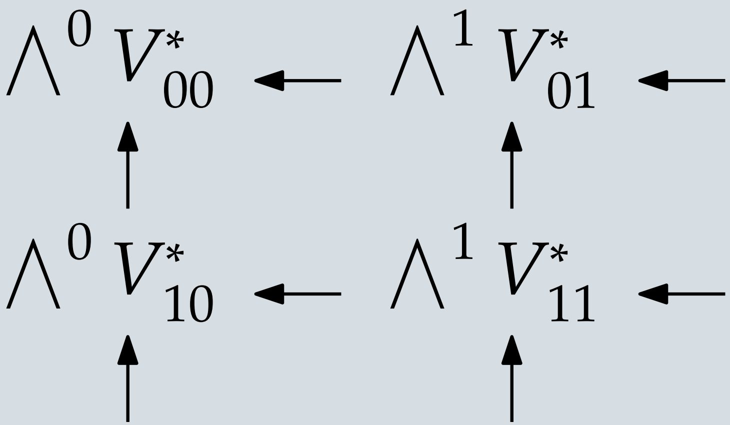

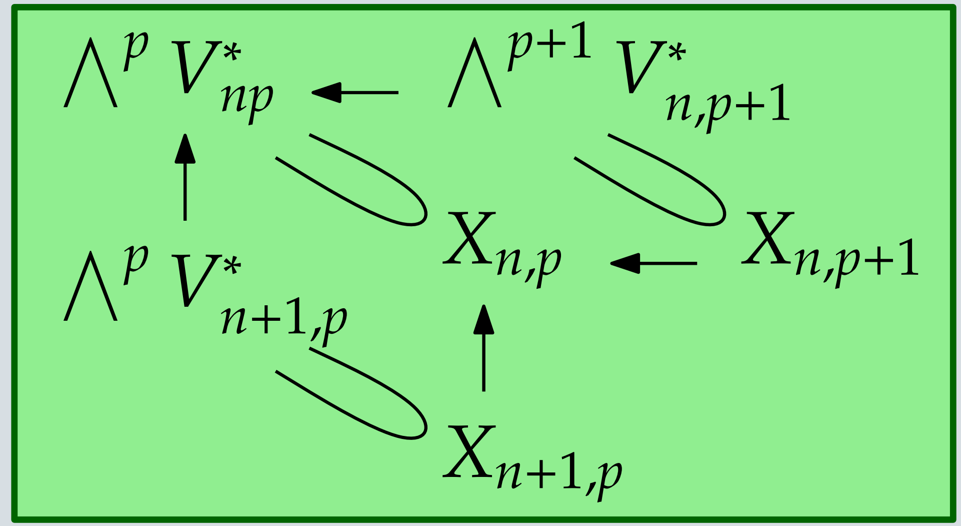

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Theorem (implies other theorems)
If $\mathbf{X}$ bounded $\rightsquigarrow \mathbf{X}_\infty$ cut out by finitely many GL_∞-orbits of equations.

Sato's Grassmannian

Definition
$\mathbf{Gr}_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^*$ is *Sato's Grassmannian*
defined by polynomials $\sum_{i \in I} \pm x_{I-i} \cdot x_{J+i} = 0$
where $i_k = k - 1$ for $k \gg 0$ and $j_k = k + 1$ for $k \gg 0$

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But in fact the GL_∞-orbit of
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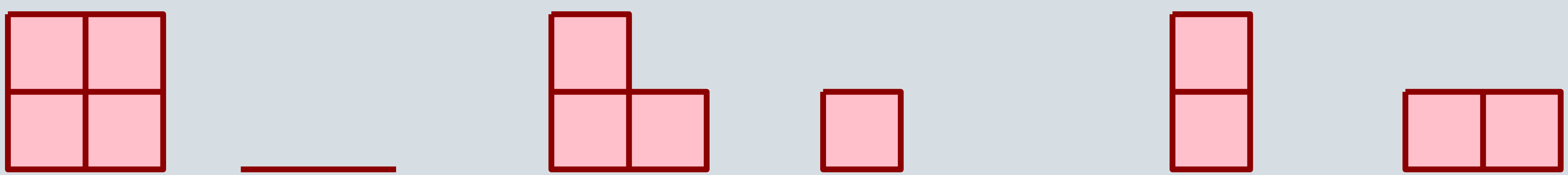

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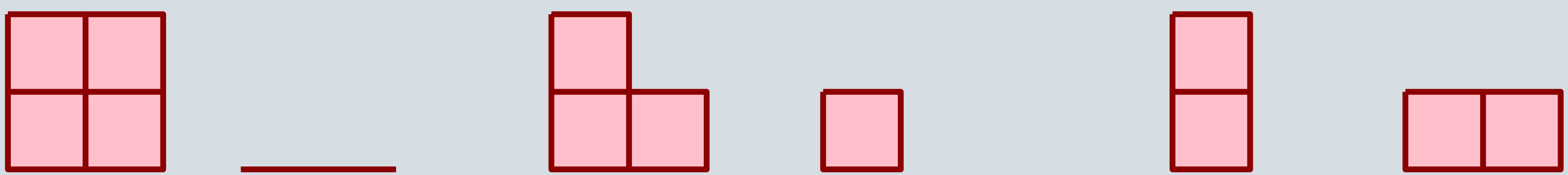

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Our theorems imply that also higher secant varieties of Sato's Grassmannian are defined by finitely many GL_∞-orbits of equations. . . *we just don't know which!*

Poly time

Setting
$\mathbf{X}$ bounded Plücker variety $\rightsquigarrow \exists\, n_0, p_0$ such that GL_∞-orbits of equations of $X_{n_0,p_0} \subseteq \bigwedge^{p_0} V^*_{n_0,p_0}$ define $\mathbf{X}_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^*$.

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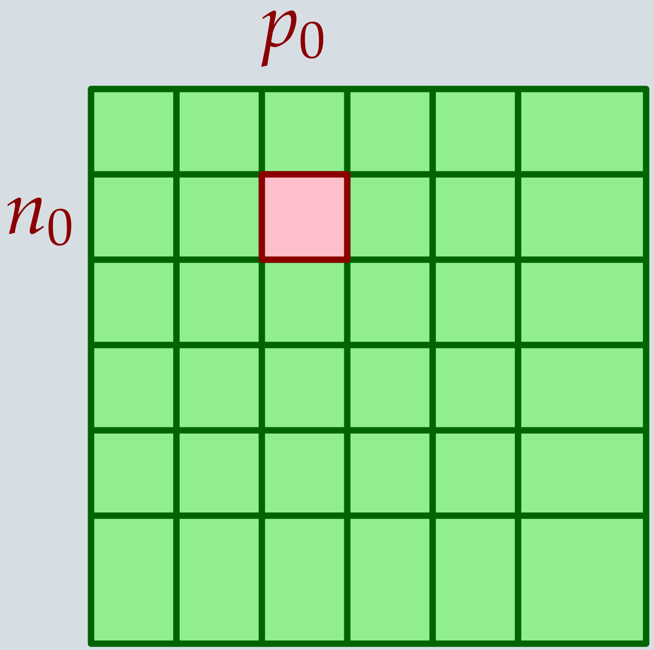
Shape of randomised algorithm

Input: $p, V, T \in \bigwedge^p V$

Output: $T \in X_p(V)$?



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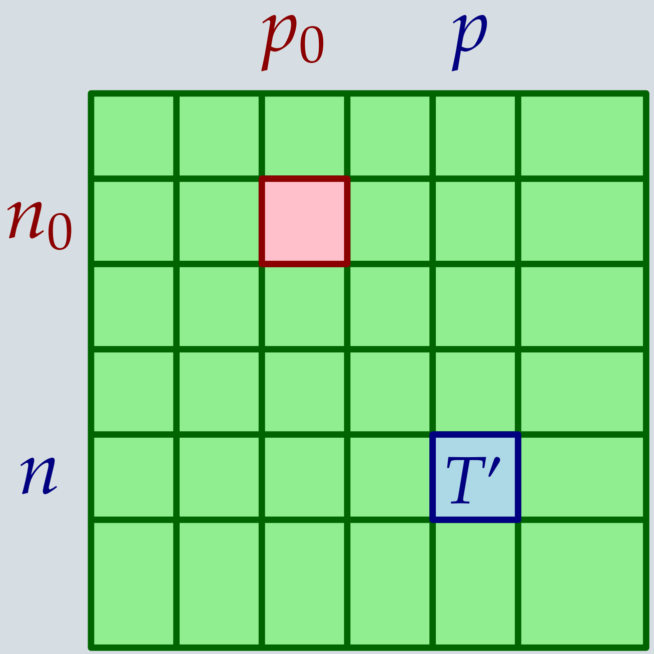

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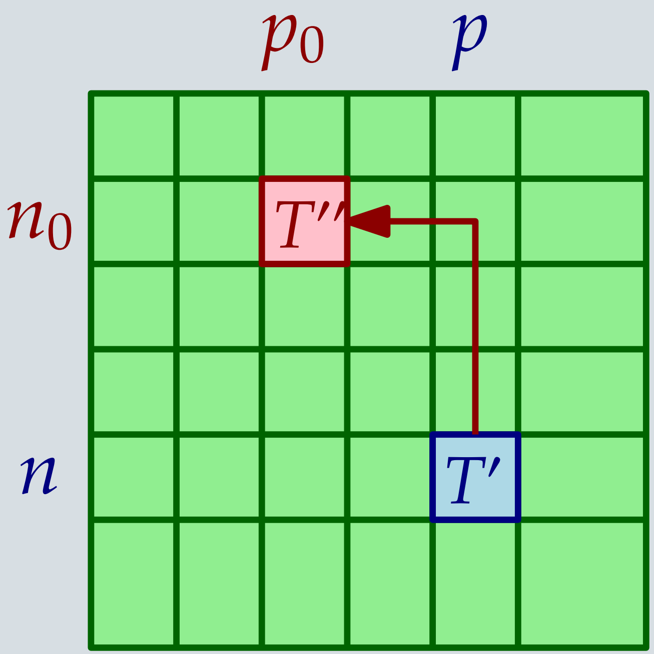
4. set $T'' :=$ image of T' in $V^*_{n_0,p_0}$
5. return $T'' \in X_{n_0,p_0}$?



Wrapping up

Pfaffians

$$\Upsilon^{k,l} := \{t \in (\textstyle\bigwedge^{\infty/2} V_\infty)^* \mid \forall g \in \mathrm{GL}_\infty :$$
$$\text{image of } gt \text{ in } \textstyle\bigwedge^2 V_{2,2l} \text{ has rank} \leq 2l \text{ and}$$
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$\rightsquigarrow$ *defined by orbits of two Pfaffians*

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Problem

How to make things work ideal-theoretically?

Landsberg-Ottaviani's *skew flattenings*?