# KRUSKAL'S UNIQUENESS INQUALITY IS SHARP

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Abstract. Kruskal proved that a tensor in $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ of rank $r$ has a unique decomposition as a sum of $r$ pure tensors if a certain inequality is satisfied. We will show the uniqueness fails if the inequality is weakened. We give 2 different constructions for counterexamples.

In this paper, $\mathbb{F}$ will denote a field. Suppose that $V$ is an $\mathbb{F}$-vector space. For a subset $A \subseteq V$, we define its Kruskal rank $k(A)$ as the largest integer such that $A$ has at least $k(A)$ elements and every subset of $A$ with $k(A)$ elements is linearly independent.

Kruskal proved in [1, 2] the following theorem for $\mathbb{F} = \mathbb{C}$ and $m = 3$:

**Theorem 1** (Kruskal's theorem)**.** *Suppose $V_1, V_2, \ldots, V_m$ are $\mathbb{F}$-vector spaces,*

$$z = \sum_{i=1}^{r} v_{1,i} \otimes v_{2,i} \otimes \cdots \otimes v_{m,i} \in V_1 \otimes V_2 \otimes \cdots \otimes V_m, \tag{1}$$

*and $k_1 + k_2 + \cdots + k_m \geq 2r + m - 1$, where $k_j := k(\{v_{j,1}, v_{j,2}, \ldots, v_{j,r}\})$ for $j = 1, 2, \ldots, m$. Then the decomposition (1) is unique in the following sense: If*

$$z = \sum_{i=1}^{q} w_{1,i} \otimes w_{2,i} \otimes \cdots \otimes w_{m,i}$$

*and $q \leq r$, then we have $q = r$ and*

$$\{v_{1,i} \otimes v_{2,i} \otimes \cdots \otimes v_{m,i} \mid 1 \leq i \leq r\} = \{w_{1,i} \otimes w_{2,i} \otimes \cdots \otimes w_{m,i} \mid 1 \leq i \leq r\}.$$

It was shown in [5] that the case $m > 3$ easily follows from the case $m = 3$. Many easier and shorter proof of this result have been given, see for example [6, 4, 3]. Kruskal's theorem is usually formulated for $\mathbb{F} = \mathbb{C}$, but for the proofs in [3, 4] are valid in arbitrary fields.

We will show that Kruskal's theorem is sharp: The theorem is no longer true if $k_1 + k_2 + \cdots + k_m = 2r + m - 2$.

**Theorem 2.** *Suppose that $\mathbb{F}$ is a field with more than $s$ elements, or that $\mathbb{F}$ is a finite field of characteristic $\geq s$. If $k_1, k_2, \ldots, k_m$ are positive integers with $k_1 + k_2 + \cdots + k_m = s + m - 2$, then there exist $\mathbb{F}$-vector spaces $V_1, V_2, \ldots, V_m$, a positive integer $q$ and vectors $\{v_{i,j}\}$ such that $q \leq s$,*

$$0 = \sum_{i=1}^{q} v_{1,i} \otimes v_{2,i} \otimes \cdots \otimes v_{m,i} \tag{2}$$

*and*

$$k(\{v_{j,1}, v_{j,2}, \ldots, v_{j,q}\}) \geq k_j$$

*for all $j$.*

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**Corollary 3.** *Suppose that $r \geq 2$ and that $\mathbb{F}$ is a field with more than $2r$ elements. Then Theorem 1 is no longer true if we replace the condition $k_1 + \cdots + k_m \geq 2r + m - 1$ by $k_1 + \cdots + k_m \geq 2r + m - 2$.*

*Proof.* We can apply Theorem 2 for $s = 2r$. By Theorem 1, we get $2r + m - 2 = k_1 + k_2 + \cdots + k_m \leq 2q + m - 2$, so $r \leq q \leq 2r$. Then we have

$$\sum_{i=1}^{r} v_{1,i} \otimes v_{2,i} \otimes \cdots \otimes v_{m,i} = \sum_{i=r+1}^{q} (-v_{1,i}) \otimes v_{2,i} \otimes \cdots \otimes v_{m,i}.$$

Since $k_1 + \cdots + k_m > m$, we have $k_j \geq 2$ for some $j$. The vectors $v_{j,1}, v_{j,2}, \ldots, v_{j,q}$ are pairwise linearly independent. It follows that the tensors

$$v_{1,i} \otimes v_{2,i} \otimes \cdots \otimes v_{m,i}, \quad 1 \leq i \leq q$$

are pairwise linearly independent. □

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{F}$. Then we define a Vandermonde matrix

$$V_k(\lambda_1, \lambda_2, \ldots, \lambda_s) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_s^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_s^{k-1} \end{pmatrix}$$

The following is a well-known property of the Vandermonde matrix:

**Lemma 4.** *If $\lambda_1, \lambda_2, \ldots, \lambda_s$ are distinct, then every $k$ columns of $V_k(\lambda_1, \lambda_2, \ldots, \lambda_s)$ are linearly independent.*

**Proposition 5.** *Suppose that $\mathbb{F}$ is a field with more than $s$ elements. Choose $\lambda \in \mathbb{F} \setminus \{0\}$ whose multiplicative order is at least $s$. Define*

$$v_{j,i} := \begin{pmatrix} 1 \\ \lambda^i \\ \lambda^{2i} \\ \vdots \\ \lambda^{(k_j-1)i} \end{pmatrix}$$

*for $1 \leq j \leq m$ and $0 \leq i < s$. Define $p(x) \in \mathbb{F}[x]$, $p_0, \ldots, p_{s-1} \in \mathbb{F}$ by*

$$p(x) = (x-1)(x-\lambda)\cdots(x-\lambda^{s-2}) = p_0 + p_1 x + \cdots + p_{s-1}x^{s-1}.$$

*Then we have*

$$\sum_{i=0}^{s-1} p_i (v_{1,i} \otimes v_{2,i} \otimes \cdots \otimes v_{m,i}) = 0, \tag{3}$$

*and*

$$k(\{v_{j,0}, v_{j,1}, \ldots, v_{j,s-1}\}) = k_j$$

*for $j = 1, 2, \ldots, m$.*

*Proof.* For every $k$ choose the basis

$$e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_{k-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

in $\mathbb{F}^k$ and $f_0, f_1, \dots, f_{k-1}$ be a dual basis. Suppose that $0 \le t_j \le k_j - 1$ for all $j$. An element $f_{t_1} \otimes f_{t_2} \otimes \cdots \otimes f_{t_m}$ can be viewed as linear function on $\mathbb{F}^k \otimes \mathbb{F}^k \otimes \cdots \mathbb{F}^k$ via

$$f_{t_1} \otimes f_{t_2} \otimes \cdots \otimes f_{t_m}(e_{u_1} \otimes e_{u_2} \otimes \cdots \otimes e_{u_m}) = f_{t_1}(e_{u_1}) f_{t_2}(e_{u_2}) \cdots f_{t_m}(e_{u_m}) =$$
$$= \begin{cases} 1 & \text{if } t_i = u_i \text{ for } i = 1, 2, \dots, m; \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$(f_{t_1} \otimes f_{t_2} \otimes \cdots \otimes f_{t_m}) \Big( \sum_{i=0}^{s-1} p_i (v_{1,i} \otimes v_{2,i} \otimes \cdots \otimes v_{m,i}) \Big) = \sum_{i=0}^{s-1} p_i \lambda^{(t_1 + t_2 + \cdots + t_m)i} = p(\lambda^{t_1 + t_2 + \cdots + t_m}) = 0$$

because $t_1 + t_2 + \cdots + t_m \le k_1 + k_2 + \cdots + k_m - m \le s - 2$ and $\lambda^{t_1 + t_2 + \cdots + t_m}$ is a root of $p(x)$.

The vectors $v_{j,0}, v_{j,1}, \dots, v_{j,s-1}$ are the columns of $V_{k_j}(1, \lambda, \lambda^2, \dots, \lambda^{s-1})$. Since $1, \lambda, \dots, \lambda^{s-1}$ are distinct, we have

$$k(\{v_{j,0}, v_{j,1}, \dots, v_{j,s-1}\}) = k_j.$$

□

We will need the following well-known combinatorial identity:

**Lemma 6.** *If* $0 \le k \le n - 1$*, then we have*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^k = 0$$

*Proof.* Define a $\mathbb{Q}$-linear operator $S : \mathbb{Q}[x] \to \mathbb{Q}[x]$ by $S(p(x)) = p(x+1)$ and let $I : \mathbb{Q}[x] \to \mathbb{Q}[x]$ be the identity operator. If $p(x)$ is a polynomial of degree $k$, then $(S - I)(p(x))$ is a polynomial of degree $\le k - 1$. In particular, we have

$$0 = (S - I)^n p(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} p(x + i) = 0.$$

Now the lemma follows from the case $p(x) = x^k$, after substituting $x = 0$. □

**Proposition 7.** *Suppose that* $\mathbb{F}$ *is a field of characteristic* 0 *or characteristic at least* $s$*. Define*

$$v_{j,i} = \begin{pmatrix} 1 \\ i \\ i^2 \\ \vdots \\ i^{k_j - 1} \end{pmatrix}$$

*for* $0 \le i \le s-1$ *and* $1 \le j \le m$. *Then we have*

$$\sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}v_{1,i}\otimes v_{2,i}\otimes\cdots\otimes v_{m,i}=0,$$

*and*

$$k(\{v_{j,1},v_{j,2},\dots,v_{j,s}\})=k_j$$

*for* $j=1,2,\dots,m$.

*Proof.* Suppose that $0 \le t_j \le k_j - 1$ for all $j$. Then we have

$$(f_{t_1}\otimes f_{t_2}\otimes\cdots\otimes f_{t_m})\Big(\sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}(v_{1,i}\otimes v_{2,i}\otimes\cdots\otimes v_{m,i})\Big)=\sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}i^{t_1+t_2+\cdots+t_m}=0$$

because of Lemma 6 and the inequality

$$t_1+t_2+\cdots+t_m\le(k_1-1)+(k_2-1)+\cdots+(k_m-1)\le s-2.$$

The vectors $v_{j,0},v_{j,1},\dots,v_{j,s-1}$ are the columns of the matrix $V_{k_j}(0,1,2,\dots,s-1)$. Since $0,1,\dots,s-1\in\mathbb{F}$ are distinct, we have

$$k(\{v_{j,0},v_{j,1},\dots,v_{j,s-1}\})=k_j.$$

□

*Proof of Theorem 2.* The theorem follows from Proposition 5 and Proposition 7. Note that in Proposition 5 we can replace $v_{1,i}$ by $p_i v_{1,i}$ so that (3) becomes (2) after relabeling the $v_{j,i}$'s. For some $i$ we may have $p_i=0$. In that case $q$ will be strictly less than $s$. Clearly we have $p_{s-1}=1$, so $q\ge 1$. In fact, $q$ is more than $\max\{k_1,\dots,k_m\}$ because otherwise, the vectors on the right-hand side of (2) would be linearly independent. □

**Example 8.** The construction in Proposition 7 for $\mathbb{F}=\mathbb{C}$, $r=3$, $s=2r=6$, $m=3$, $k_1=k_2=2$, $k_3=3$ yields:

$$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\end{pmatrix}-5\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\\1\end{pmatrix}+10\begin{pmatrix}1\\2\end{pmatrix}\otimes\begin{pmatrix}1\\2\end{pmatrix}\otimes\begin{pmatrix}1\\2\\4\end{pmatrix}=$$
$$10\begin{pmatrix}1\\3\end{pmatrix}\otimes\begin{pmatrix}1\\3\end{pmatrix}\otimes\begin{pmatrix}1\\3\\9\end{pmatrix}-5\begin{pmatrix}1\\4\end{pmatrix}\otimes\begin{pmatrix}1\\4\end{pmatrix}\otimes\begin{pmatrix}1\\4\\16\end{pmatrix}+\begin{pmatrix}1\\5\end{pmatrix}\otimes\begin{pmatrix}1\\5\end{pmatrix}\otimes\begin{pmatrix}1\\5\\25\end{pmatrix}.$$

**Example 9.** We use the construction in Proposition 5 for $\mathbb{F}=\mathbb{R}$, $r=3$, $s=2r=6$, $m=3$, $k_1=k_2=2$, $k_3=3$, $\lambda=2$. We expand

$$(x-1)(x-2)(x-4)(x-8)(x-16)=-1024+1984x-1240x^2+310x^3-31x^4+x^5.$$

We have

$$1024\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\\1\end{pmatrix}+1240\begin{pmatrix}1\\4\end{pmatrix}\otimes\begin{pmatrix}1\\4\end{pmatrix}\otimes\begin{pmatrix}1\\4\\16\end{pmatrix}+31\begin{pmatrix}1\\16\end{pmatrix}\otimes\begin{pmatrix}1\\16\end{pmatrix}\otimes\begin{pmatrix}1\\16\\256\end{pmatrix}=$$
$$1984\begin{pmatrix}1\\2\end{pmatrix}\otimes\begin{pmatrix}1\\2\end{pmatrix}\otimes\begin{pmatrix}1\\2\\4\end{pmatrix}+310\begin{pmatrix}1\\8\end{pmatrix}\otimes\begin{pmatrix}1\\8\end{pmatrix}\otimes\begin{pmatrix}1\\8\\64\end{pmatrix}+\begin{pmatrix}1\\32\end{pmatrix}\otimes\begin{pmatrix}1\\32\end{pmatrix}\otimes\begin{pmatrix}1\\32\\1024\end{pmatrix}.$$

Note that in this example the tensors have nonnegative entries when they are viewed as multi-arrays. Whenever one chooses $\lambda > 0$ in Proposition 5 one obtains counterexamples with nonnegative entries, because exactly half of the coefficients of $p(x)$ are positive and half of them are negative when $s = 2r$ is even.

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