Ranks of Real Symmetric Tensors

Greg Blekherman

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Algebraic Geometry of Tensor Decompositions

Real Symmetric Tensor Decompositions

Let f be a form of degree d in $\mathbb{R}[x_1, \ldots, x_n]$. We would like to decompose f as

$$f = \sum_{i=1}^{r} c_i \ell_i^d,$$

where ℓ_i are **real** linear forms. The minimal r for which such decomposition exists is the rank of f.

Illustrative example: $2x^3 - 6xy^2 = (x + \sqrt{-1}y)^3 + (x - \sqrt{-1}y)^3$.

Over $\mathbb{C}$ a generic form has unique rank, given by the Alexander-Hirschowitz theorem.

Call rank r **typical** for forms in $\mathbb{R}[x_1, \ldots, x_n]_d$ if the set of forms of ranks r includes an open subset of $\mathbb{R}[x_1, \ldots, x_n]_d$. For real forms there can be many typical ranks.

Binary Forms

Theorem(Comon-Ottaviani, Causa-Re, Reznick): Let $f \in \mathbb{R}[x, y]_d$ be a form with distinct roots. Then f has rank d if and only if all roots of f are real.

Conjecture (Comon-Ottaviani): All ranks r with $\lfloor \frac{d+2}{2} \rfloor \leq r \leq d$ are typical for forms in $\mathbb{R}[x, y]_d$. Now a Theorem (B.).

Enter Apolarity

To a monomial $x^\alpha = x_1^{a_1} \dots x_n^{a_n}$ associate a differential operator

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x_1^{a_1} \dots \partial x_n^{a_n}}.$$

To a polynomial $f = \sum c_\alpha x^\alpha$ associate a differential operator

$$\partial f = \sum c_\alpha \frac{\partial}{\partial x^\alpha}.$$

For $f \in k[x_1, \dots, x_n]$ define **the apolar ideal** $f^\perp$ of f by

$$f^\perp = \{g \in k[x_1, \dots, x_n] \mid \partial g(f) = 0\}.$$

Apolar inner product:

$$\langle f, g \rangle = \partial f(g).$$

The Key Lemma

Apolarity Lemma: Let f be a form of degree d. Then f can be written as $f = \sum_{i=1}^{r} \ell_i^d$ with $\ell_i = c_{i1}x_1 + \cdots + c_{in}x_n$ if and only if the vanishing ideal of the points $c_i = (c_{i1}, \ldots, c_{in})$, $1 \leq i \leq r$ is contained in the apolar ideal $f^\perp$.

Restatement about rank: A form f has rank at most r if and only if the apolar ideal $f^\perp$ contains the vanishing ideal of some r points.

For binary forms: A form f has rank at most r if and only if $f^\perp$ contains a form of degree r that factors completely into distinct factors (over the appropriate field!).

Idea of Proof: Let f be a form of rank r. We need that for any small perturbation g of f the apolar ideal $g^\perp$ contains a form of degree r with all real roots and no forms of degree $r-1$ with all real roots.

Nonnegative Decompositions

Two Questions: (1) Given a real form f of even degree $2d$ is is possible to write f as a *positive combination* of $2d$-th powers of linear forms:

$$f = \sum \alpha_i \ell_i^{2d}, \quad \alpha_i \geq 0.$$

(2) Given a linear functional $\mathcal{L} : \mathbb{R}[x]_{n,2d} \to \mathbb{R}$ is it possible to write $\mathcal{L}$ as integration with respect to a measure μ:

$$\mathcal{L}(f) = \int_{\mathbb{R}^n} f \, d\mu.$$

The 2nd question is the **homogeneous truncated moment problem**. But the questions are equivalent via apolarity! Let $\mathbf{v} = (v_1 x_1 + \cdots + v_n x_n)^{2d}$ and define linear operator $\mathcal{L}_{\mathbf{v}}$ by

$$\mathcal{L}_{\mathbf{v}}(f) = \partial\mathbf{v}(f) = f(v).$$

Some Convexity

Sums of $2d$-th powers of linear forms are a convex cone in $\mathbb{R}[x]_{n,2d}$. By the above this cone is *dual* to the cone of nonnegative forms $P_{n,2d}$ in $\mathbb{R}[x]_{n,2d}$.

The dual cone $\Sigma^*_{n,2d}$ to the cone of sums of squares $\Sigma_{n,2d}$ consists of forms with *positive semidefinite* middle catalecticant matrix; this is the matrix of the quadratic form $Q_f : \mathbb{R}[x]_{n,d} \to \mathbb{R}$ given by:

$$Q_f(p) = \partial f(p^2).$$

Corollary: Suppose $P_{n,2d} = \Sigma_{n,2d}$ and f has a positive semidefinite middle catalecticant matrix. Then f is a sum of $2d$-th powers and nonnegative rank of f is equal to the rank of the middle catalecticant matrix.

By Hilbert's Theorem this only happens for (1) Binary forms (2) Quadratic Forms (3) Ternary Quartics.

A Generalization

Can this situation be repeated? YES!

Theorem: (B.) Let $f \in \mathbb{R}[x]_{n,2d}$ and suppose that

$$\operatorname{rank} Q_f \leq 3d - 3 \quad \text{for} \quad d \geq 3 \quad \text{or} \quad \operatorname{rank} Q_f \leq 5 \quad \text{for} \quad d = 2.$$

Then f is a sum of $2d$-th powers and $\operatorname{rank} f = \operatorname{rank} Q_f$. These bounds are tight.

Proposition: (B.) Let $f \in \mathbb{R}[x]_{3,6}$ be a sum of 6-th powers. Then $\operatorname{rank} f \leq 11$ and this bound is sharp.

Can construct example where nonnegative rank is larger than the real rank.

Completely Speculative

Let $f \in \mathbb{R}[x]_{n,2d}$. Suppose that we demand that the decomposition is not real but *conjugate invariant*, i.e. if ℓ^{2d} occurs in the decomposition of f, then so does $\bar{\ell}^{2d}$.

What can be said about the conjugate rank?

Open Questions

- What happens for more variables?

  Describe all typical ranks for ternary quartics. Can show that 7 is a typical rank and any ternary quartic has rank at most 8.

- Conjugate invariant rank.

- Describe all cases (in terms of degree and number of variables) where nonnegative rank is equal to the real rank.

THANK YOU!