On Waring's Problem for Systems of Skew-Symmetric Forms

(joint work with Jia Wan)

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August 2, 2013

Waring's Problem

- $m \in \mathbb{N} \cup \{0\}$ and $n, k \in \mathbb{N}$.
- $W = (n+1)$-dimensional vector space over $\mathbb{C}$.
- $\bigwedge^{k+1} W = (k+1)^{\text{st}}$ exterior power of W.
- A skew-symmetric tensor $\mathbf{w}$ of $\bigwedge^{k+1} W$ is called decomposable if $\exists w_0, \ldots, w_k \in W$ such that $\mathbf{w} = w_0 \wedge \cdots \wedge w_k$.

Problem.

Find the smallest $s(m, n, k)$ such that $m+1$ generic skew-symmetric tensors $\mathbf{w}_0, \ldots, \mathbf{w}_m \in \bigwedge^{k+1} W$ are expressible as linear combinations of the same $s(m, n, k)$ decomposable skew-symmetric tensors.

Geometric Interpretation of Waring's Problem

- $V = (m+1)$-dimensional vector space over $\mathbb{C}$.
- $\mathbb{P}^m$ = projective space of lines in V through the origin.
- X = projective variety in $\mathbb{P}^\ell$.
- The s^{th} **secant variety** of X is defined to be the Zariski closure of the union of secant $(s-1)$-planes to X:

$$\sigma_s(X) = \overline{\bigcup_{p_1,\cdots,p_s \in X} \langle p_1, \ldots, p_s \rangle}.$$

- For simplicity, we write $\mathbb{P}^m \times X$ for the image of the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^\ell$ restricted to $\mathbb{P}^m \times X$.

Geometric Interpretation of Waring's Problem (Cont'd)

- $\mathbb{G}(k,n) \subseteq \mathbb{P}^{\binom{n+1}{k+1}-1}$ = Grassmannian of $(k+1)$-planes of W.

Problem. Find the smallest $s(m,n,k)$ such that

$$\sigma_{s(m,n,k)}(\mathbb{P}^m \times \mathbb{G}(k,n)) = \mathbb{P}^{(m+1)\binom{n+1}{k+1}-1}.$$

Note.

$$(*)\ \dim \sigma_s(\mathbb{P}^m \times \mathbb{G}(k,n)) \leq \min\left\{ s[m + \dim \mathbb{G}(k,n) + 1],\ (m+1)\binom{n+1}{k+1} \right\} - 1$$

It is expected that $s(m,n,k) = \left\lceil (m+1)\binom{n+1}{k+1} / [m + (k+1)(n-k) + 1] \right\rceil$.

Question. Does equality (*) always hold?

Answer. No.

Known Defective Cases ($m = 0$)

- Since $\mathbb{G}(k,n) \simeq \mathbb{G}(n-k-1,n)$, we may consider the problem of classifying defective $\mathbb{G}(k,n)$ only $k \leq (n-1)/2$.
- $\mathbb{G}(1,n)$ is defective for most of n.
- Let $k \geq 2$. The following are the only known defective cases:

		Actual codimension	Expected codimension
(i)	$\sigma_3(\mathbb{G}(2,6))$	1	0
(ii)	$\sigma_3(\mathbb{G}(3,7))$	20	19
(iii)	$\sigma_4(\mathbb{G}(3,7))$	6	2
(iv)	$\sigma_4(\mathbb{G}(2,8))$	10	8

Conjecture (Bauer-Drisma-de Graaf).

If $k \geq 2$, then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension except for (i), (ii), (ii), and (iv).

What happens if $m \geq 1$?

- $X \subseteq \mathbb{P}^\ell$ = non-degenerate subvariety of dimension d.
- (m, ℓ, d) is called **unbalanced** if $m > \ell - d + 1$

Theorem 1 (B-L, B-B-C-C).

Let (m, ℓ, d) be unbalanced. Then $\sigma_s(\mathbb{P}^m \times X)$ is defective iff $\ell - d + 1 < s \leq \min\{m, \ell\}$.

Corollary 2.

$\binom{n+1}{k+1} - (k+1)(n-k) < m \Rightarrow \sigma_s(\mathbb{P}^m \times \mathbb{G}(k, n))$ is defective for

$$\binom{n+1}{k+1} - (k+1)(n-k) < s \leq \min\left\{m, \binom{n+1}{k+1} - 1\right\}.$$

Main Result

Theorem 3 (A-Wan, 2013).

$\sigma_s(\mathbb{P}^m \times \mathbb{G}(k,n))$ *is defective if* $(m,n,k.s)$ *is one of the following:*

	(m,n,k,s)	Actual codimension	Expected codimension
(i)	$(2, 4\ell+2, 1, 3\ell+2)$	1	0
(ii)	$(2,5,1,4)$	3	1
(iii)	$(2,7,1,5)$	10	9
(iv)	$(1,5,2,3)$	8	7
(v)	$(2,5,2,5)$	1	0

Table 1: Previously unknown defective cases

What happens if $m \geq 1$? (Cont'd)

It is an immediate consequence that if (m, n, k) is one of the following, then $s(m, n, k)$ is strictly larger than expected:

- (m, n, k) with $\binom{n+1}{k+1} - (k+1)(n-k) < m$,
- $(2, 4\ell + 2, 3\ell + 2)$ with $\ell \in \mathbb{N}$,
- $(2, 5, 2)$.

What happens if $m \geq 1$? (Cont'd)

Question.

Are there any quadruples (m, n, k, s) (other than the ones listed in Corollary 2 and Table 2) such that $\sigma_s(\mathbb{P}^m \times \mathbb{G}(k, n))$ are defective?

Main Result Revisited

Theorem 3 (A-Wan, 2013).

$\sigma_s(\mathbb{P}^m \times \mathbb{G}(k,n))$ *is defective if* $(m, n, k.s)$ *is one of the following:*

	(m,n,k,s)	Actual codimension	Expected codimension
(i)	$(2, 4\ell+2, 1, 3\ell+2)$	1	0
(ii)	$(2,5,1,4)$	3	1
(iii)	$(2,7,1,5)$	10	9
(iv)	$(1,5,2,3)$	8	7
(v)	$(2,5,2,5)$	1	0

Table 2: Previously unknown defective cases

Proof for (i)

- $\sigma_{3\ell+2}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell+2))$ is expected to fill out $\mathbb{P}^{3\binom{4\ell+3}{2}-1}$.
- The existence of a point of $\mathbb{P}^{3\binom{4\ell+3}{2}-1}$, which does not lie on $\sigma_{3\ell+2}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell+2))$, shows that $\sigma_{3\ell+2}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell+2))$ does not have the expected dimension.

A Theorem of Chiantini and Ciliberto

- $X \subseteq \mathbb{P}^\ell$ = non-degenerate, non-singular subvariety.
- $p_1, \dots, p_s \in X$ = generic points.

Theorem 4 (Chiantini-Ciliberto).

If $\exists$ non-singular suvbariety C of X through $p_1, \dots, p_s$, then

$$\dim \sigma_s(X) \leq s(\dim X - \dim C) + \dim \sigma_s(C).$$

In particular, if

$$s(\dim X - \dim C) + \dim \sigma_s(C) < \min\{s(\dim X + 1) - 1, \ell\},$$

then $\sigma_s(X)$ is defective.

Example

- $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ = the d^{th} Veronese embedding.
- $n = d = s = 2$.
- $\forall p_1, p_2 \in \nu_2(\mathbb{P}^2)$ $\exists$ a unique conic $C \in \nu_2(\mathbb{P}^2)$ passing through p_1 and p_2.
- We have

$$\begin{aligned}\dim \sigma_2(\nu_2(\mathbb{P}^2)) &\leq 2[\dim \nu_2(\mathbb{P}^2) - \dim C] + \dim \sigma_2(C) \\ &= 2(2-1)+2 \\ &= 4 < 5.\end{aligned}$$

- By Theorem 4, $\nu_2(\mathbb{P}^2)$ is defective.

Proofs for (ii)-(v)

The defectivity of each of (ii)-(v) is verified by the existence of the following subvarieties:

	(m,n,k,s)	C
(ii)	$(2,5,1,4)$	$\mathbb{P}^2 \times \nu_2(\mathbb{P}^2)$
(iii)	$(2,7,1,5)$	$\mathbb{P}^2 \times \nu_2(\mathbb{P}^3)$
(iv)	$(1,5,2,3)$	$\mathbb{P}^2 \times \nu_3(\mathbb{P}^3)$
(v)	$(2,5,2,5)$	$\nu_8(\mathbb{P}^2)$

For example, if $(m,n,k,s)=(2,5,2,5)$, then

$$\begin{aligned}
\dim\sigma_5(\mathbb{P}^2\times\mathbb{G}(2,5)) &\leq 5[\dim(\mathbb{P}^2\times\mathbb{G}(2,5))-\dim\nu_8(\mathbb{P}^2)]+\dim\sigma_5(\nu_8(\mathbb{P}^2))\\
&\leq 5[2+3(5-2)-1]+8\\
&\leq 58 \ < \ 59.
\end{aligned}$$

Problem

- An application of Terracini's lemma shows that

$$\dim \sigma_5(\mathbb{P}^2 \times \mathbb{G}(2,5)) \geq 58,$$

which implies that $\sigma_5(\mathbb{P}^2 \times \mathbb{G}(2,5)) \subset \mathbb{P}^{59}$ is a hypersurface.

Problem. Find the equation for $\sigma_5(\mathbb{P}^5 \times \mathbb{G}(2,5))$.

Thank you very much for your attention!