Report on the recent progress on the study of the secant defectivity of Segre-Veronese varieties

(joint work with M. C. Brambilla)

Hirotachi Abo

Department of Mathematics
University of Idaho

abo@uidaho.edu
http://www.uidaho.edu/~abo

October 6, 2011

Notation

- $V = (n+1)$-dimensional vector space over $\mathbb{C}$.
- $\mathbb{P}V (= \mathbb{P}^n) =$ projective space of V.
- $[\mathbf{v}] \in \mathbb{P}V =$ equivalence class containing $\mathbf{v} \in V$.
- $S^d V = d^{\text{th}}$ symmetric power of V.
- $\langle A \rangle =$ linear span of $A \subseteq \mathbb{P}V$.

Secant varieties

- X = projective variety in $\mathbb{P}V$.
- Let $p_1, \ldots, p_s$ be generic points of X. Then $\langle p_1, \ldots, p_s \rangle$ is called a secant $(s-1)$-plane to X.
- The s^{th} secant variety of X is defined to be the Zariski closure of the union of secant $(s-1)$-planes to X:

$$\sigma_s(X) = \overline{\bigcup_{p_1,\cdots,p_s \in X} \langle p_1, \ldots, p_s \rangle}.$$

Secant dimension and secant defectivity

- A simple parameter count implies that the following inequality holds:

$$\dim \sigma_s(X) \leq \min \{s \cdot (\dim X + 1) - 1, \ \dim \mathbb{P}V\}.$$

- If Equality holds, we say X has the expected dimension.
- Not all the secant varieties have the expected dimension.
- If $\sigma_s(X)$ does not have the expected dimension, then $\sigma_s(X)$ is said to be defective.
- X is said to be defective if $\sigma_s(X)$ is defective for some s.

Why do we care about the secant defectivity?

The dimensions of higher secant varieties of parameter spaces of tensors have something to do with:

- Tensor rank, bordar rank, and typical tensor rank.
- Uniqueness of tensor decomposition.
- Equations.

Secant varieties of other classically studied varieties

- Let $v_d : \mathbb{P}V \to \mathbb{P}S^dV$ be the d^{th} Veronese map, i.e., v_d is the map given by $v_d([\mathbf{v}]) = [\mathbf{v}^d]$.
- Theorem (Alexander-Hirschowitz, 1995)

  $\sigma_s[v_d(\mathbb{P}V)]$ *is non-defective except for the following cases:*

$\dim \mathbb{P}V$	d	s
≥ 2	2	$2 \leq s \leq n$
2	4	5
3	4	9
4	3	7
4	4	14

Secant defectivity for Segre varieties and Grassmann varieties

- There are corresponding conjecturally complete lists of defective secant varieties for Segre varieties (A-Ottaviani-Peterson, 2009) and Grassmann varieties (Bauer-Draisma-de Graaf, 2007).
- There is no general conjecture on defective secant varieties for the Segre-Veronese case known yet.

Secant varieties of Segre-Veronese varieties

- $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$.
- $V_i = (n_i + 1)$-dimensional vector space.
- $\mathrm{Seg} : \prod_{i=1}^{k} \mathbb{P}V_i \to \mathbb{P}\left(\bigotimes_{i=1}^{k} V_i\right) =$ Segre map, i.e., the map given by $\mathrm{Seg}([\mathbf{v}_1], \ldots, [\mathbf{v}_k]) = [\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k]$.
- $X_{\mathbf{n},\mathbf{d}} := \mathrm{Seg}\left(\prod_{i=1}^{k} v_{d_i}(\mathbb{P}V_i)\right) \hookrightarrow \mathbb{P}\left(\bigotimes_{i=1}^{k} S^{d_i} V_i\right)$ is called a Segre-Veronese variety.

Conjecturally complete list of defective two factor cases

n	**d**	s
(m, n) with $m \geq 2$	$(d, 1)$	$\binom{m+d}{d} - m < s < \min\left\{\binom{m+d}{d}n + 1\right\}$
$(2, 2k+1)$	$(1, 2)$	$3k + 2$
$(4, 3)$	$(1, 2)$	6
$(1, 2)$	$(1, 3)$	5
$(1, n)$	$(2, 2)$	$n + 2 \leq s \leq 2n + 1$
$(2, 2)$	$(2, 2)$	$7,\ 8$
$(2, n)$	$(2, 2)$	$\left\lfloor \frac{3n^2+9n+5}{n+3} \right\rfloor \leq s \leq 3n + 2$
$(3, 3)$	$(2, 2)$	$14,\ 15$
$(3, 4)$	$(2, 2)$	19
$(n, 1)$	$(2, 2k)$	$kn + k + 1 \leq s \leq kn + k + n$

This conjecture is based on:

- already existing results (by many people including E. Carlini, T. Geramita, J. Draisma, and G. Ottaviani).
- computational experiments;
- the theorems we proved.

Méthode d'Horace différentielle

- $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$ with $d_1 \geq 3$.
- $\mathbf{n}' = (n_1 - 1, n_2, \ldots, n_k)$, $\mathbf{d}' = (d_1 - 1, d_2, \ldots, d_k)$, $\mathbf{d}'' = (d_1 - 2, d_2, \ldots, d_k)$.
- For a given positive integer s, let s' and ϵ be the quotient and remainder when dividing the following integer by $\sum_{i=1}^{k} n_i$:

$$s\left(1 + \sum_{i=1}^{k} n_i\right) - \binom{n_1 + d_1 - 1}{d_1 - 1} \prod_{i=2}^{k} \binom{n_i + d_i}{d_i}.$$

Méthode d'Horace différentielle (cont'd)

- Theorem (A-Brambilla, 2009)

  If $\sigma_{s'}(X_{\mathbf{n}',\mathbf{d}})$, $\sigma_{s-s'}(X_{\mathbf{n},\mathbf{d}'})$, and $\sigma_{s-s'-\epsilon}(X_{\mathbf{n},\mathbf{d}''})$ have the expected dimension and if

$$(s-s'-\epsilon)\left(1+\sum_{i=1}^{k} n_i\right) \geq \binom{n_1+d_1-2}{d_1-2}\prod_{i=2}^{k}\binom{n_i+d_i}{a_i},$$

  then $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ also has the expected dimension.

Results

- Theorem (A-Brambilla, 2009)

  Let $n, a \geq 1$, $b \geq 3$, $\mathbf{n} = (n, 1)$ *and* $\mathbf{d} = (a, b)$. *Then* $X_{\mathbf{n},\mathbf{d}}$ *is not defective except for* $(n, a, b) = (n, 2, 2k)$ *with* $k \geq 1$.

- Remark. In 2011, this theorem was extended to $b \geq 1$ by E. Ballico, A. Bernardi and M. V. Catalisano.

- Corollary (A-Brambilla, 2009)

  Suppose that $X_{\mathbf{n},\mathbf{d}}$ *is not defective for every* $\mathbf{n}$ *and for* $\mathbf{d} = (3, 3)$, $(3, 4)$ *and* $(4, 4)$. *Then* $X_{\mathbf{n},\mathbf{d}}$ *is not defective for every* $\mathbf{n}$ *and for every* $\mathbf{d} \geq (3, 3)$.

More realistic conjecture

- Conjecture (A-Brambilla, 2009)

  If $\mathbf{d} \geq (3,3)$, *then there are no defective two-factor Segre-Veronese varieties* $X_{\mathbf{n},\mathbf{d}}$ *for all* $\mathbf{n} \in \mathbb{N}^2$.

- The completion of this conjecture is equivalent to establishment of the non-defectivity of $X_{\mathbf{n},\mathbf{d}}$ for $\mathbf{d} \in \{(3,3),(3,4),(4,4)\}$.

- Problem. *Show the non-defectivity of* $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ *for smaller* $\mathbf{d}'s$. (It is frequent that the Horace theorem cannot be applied directly v to such cases.)

Méthode d'Horace différentielle Revisited

- $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$ with $d_1 \geq 3$.
- $\mathbf{n}' = (n_1 - 1, n_2, \ldots, n_k)$, $\mathbf{d}' = (d_1 - 1, d_2, \ldots, d_k)$, $\mathbf{d}'' = (d_1 - 2, d_2, \ldots, d_k)$.
- Theorem (A-Brambilla, 2009)

  If $\sigma_{s'}(X_{\mathbf{n}',\mathbf{d}})$, $\sigma_{s-s'}(X_{\mathbf{n},\mathbf{d}'})$, and $\sigma_{s-s'-\epsilon}(X_{\mathbf{n},\mathbf{d}''})$ have the expected dimension and if

$$(s - s' - \epsilon)\left(1 + \sum_{i=1}^{k} n_i\right) \geq \binom{n_1 + d_1 - 2}{d_1 - 2} \prod_{i=2}^{k} \binom{n_i + d_i}{a_i},$$

  then $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ also has the expected dimension.

What about Segre-Veronese varieties with 3 or more factors?

- Theorem (A-Brambilla, 2010)

  Let $k \in \{3, 4\}$, *let* $\mathbf{n} = (n_1, \ldots, n_k)$ *and let* $\mathbf{d} = (1, \ldots, 1, 2)$. *Then there exist infinitely many defective secant varieties of* $X_{\mathbf{n},\mathbf{d}}$, *which were previously not known.*

Time for reception!