Inverse Transport Problems and Applications

II. Optical Tomography and Clear Layers

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Outline for the three lectures

I. Inverse problems in integral geometry

Radon transform and attenuated Radon transform

Ray transforms in hyperbolic geometry

II. Forward and Inverse problems in highly scattering media

Photon scattering in tissues within diffusion approximation

Inverse problems in Optical tomography

III. Inverse transport problems

Singular expansion of albedo operator

Perturbations about "scattering-free" problems

Unsolved practical inverse problems.

Outline for Lecture II

1. Optical tomography

Transport equations and examples of applications

2. Macroscopic modeling of clear layers

Diffusion approximation of transport

Macroscopic modeling of clear layers

3. Reconstruction via the Factorization method

Reconstruction of clear layer and enclosed coefficients

Shape derivative plus level set methods

Mathematical Problems in Optical Tomography

Optical Tomography consists in reconstructing absorption and scattering properties of human tissues by probing them with Near-Infra-Red photons (wavelength of roughly 1μm).

What needs to be done:

- Modeling of forward problem using equations that are easy to solve: photons strongly interact with underlying tissues.

- Devising reconstruction algorithms to image tissue properties from boundary measurements of photon intensities.

$\Big($• Address relevant questions and no more: severely ill-posed problem.$\Big)$

Transport equations in Optical Tomography

The photon density $u(\mathbf{x}, \mathbf{\Omega}; \nu)$ solves the following transport (linear Boltzmann) equation

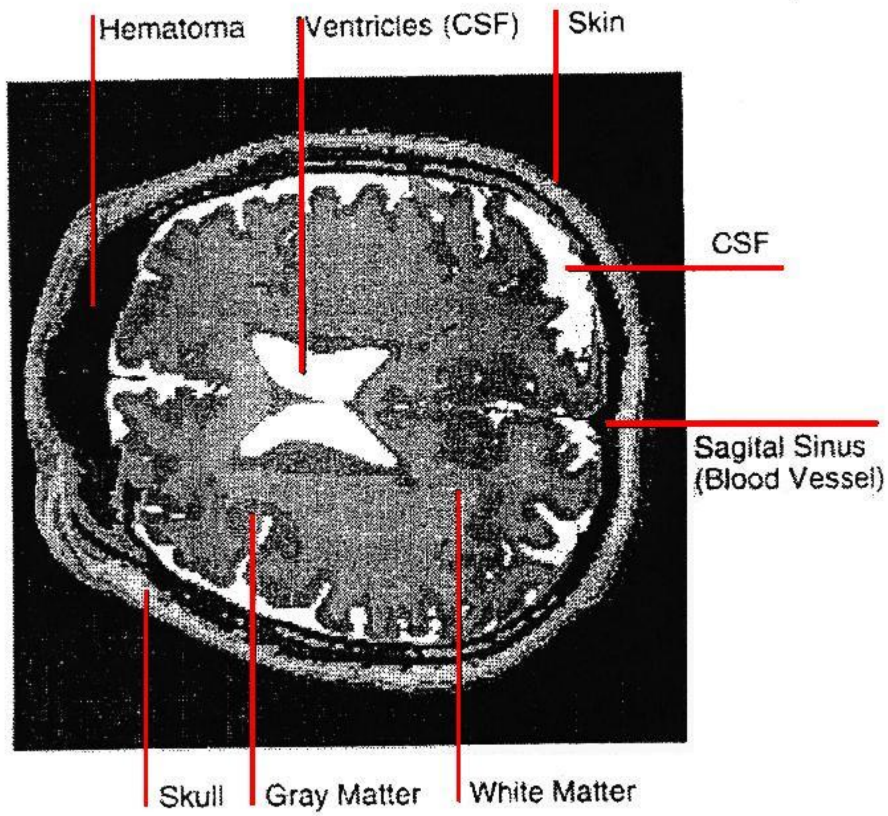
$$\frac{i\nu}{c}u + \mathbf{\Omega} \cdot \nabla u + \sigma_t(\mathbf{x})u = \sigma_s(\mathbf{x}) \int_{S^2} p(\mathbf{\Omega} \cdot \mathbf{\Omega}')u(\mathbf{x}, \mathbf{\Omega}'; \nu)d\mu(\mathbf{\Omega}'),$$

where ν is the (known) modulation of the illumination source, $p(\mu)$ is the phase function of the scattering process (often assumed to be known), $\sigma_t(\mathbf{x})$ is the total absorption coefficient and $\sigma_s(\mathbf{x})$ the scattering coefficient. The last three terms model photon interactions with the underlying medium (tissues).

The inverse problem in OT consists of reconstructing $\sigma_s(\mathbf{x})$ and $\sigma_t(\mathbf{x})$ (and possibly $p(\mu)$) from boundary measurements.

The transport equation is often replaced by its diffusion approximation.

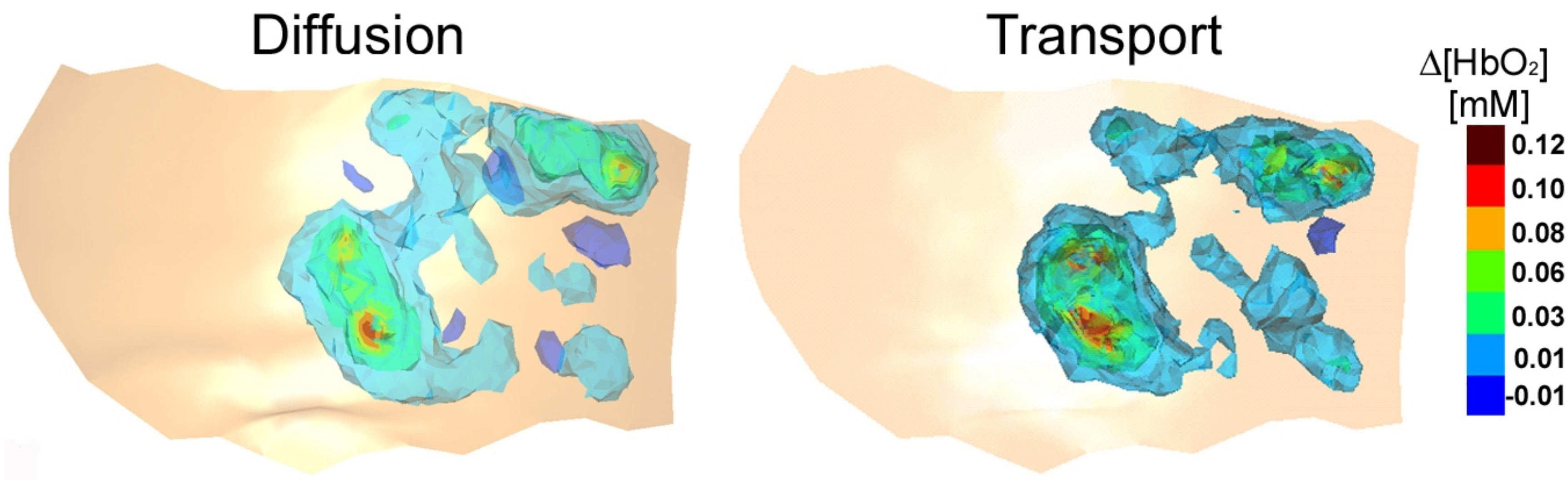
Applications in Near-Infra-Red Spectroscopy



Segmented MRI data for a human brain.

Imaging of human brains.

Applications in Near-Infra-Red Spectroscopy



Imaging of human brains (from A.H. Hielscher, biomedical Engineering, Columbia).

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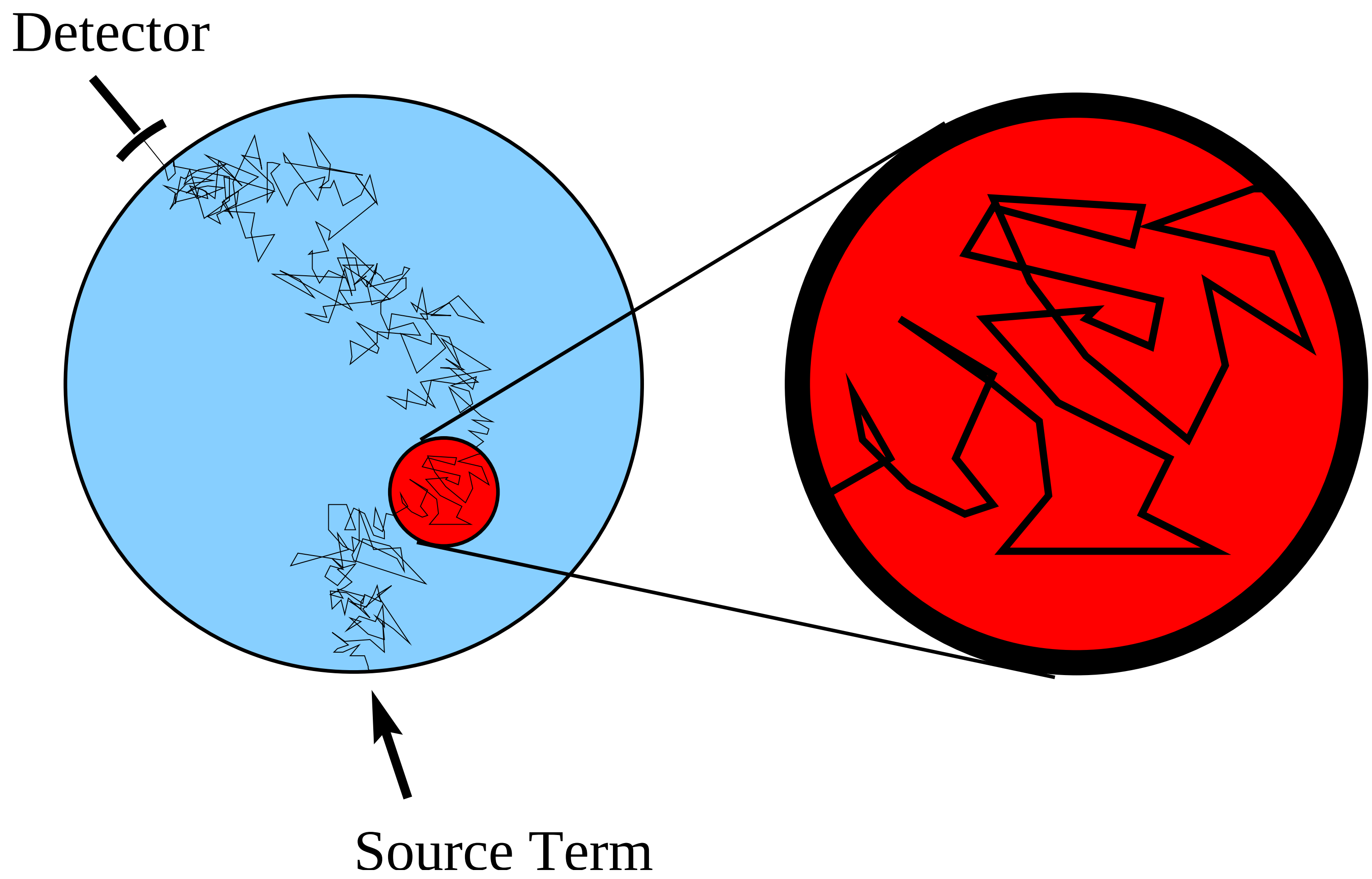
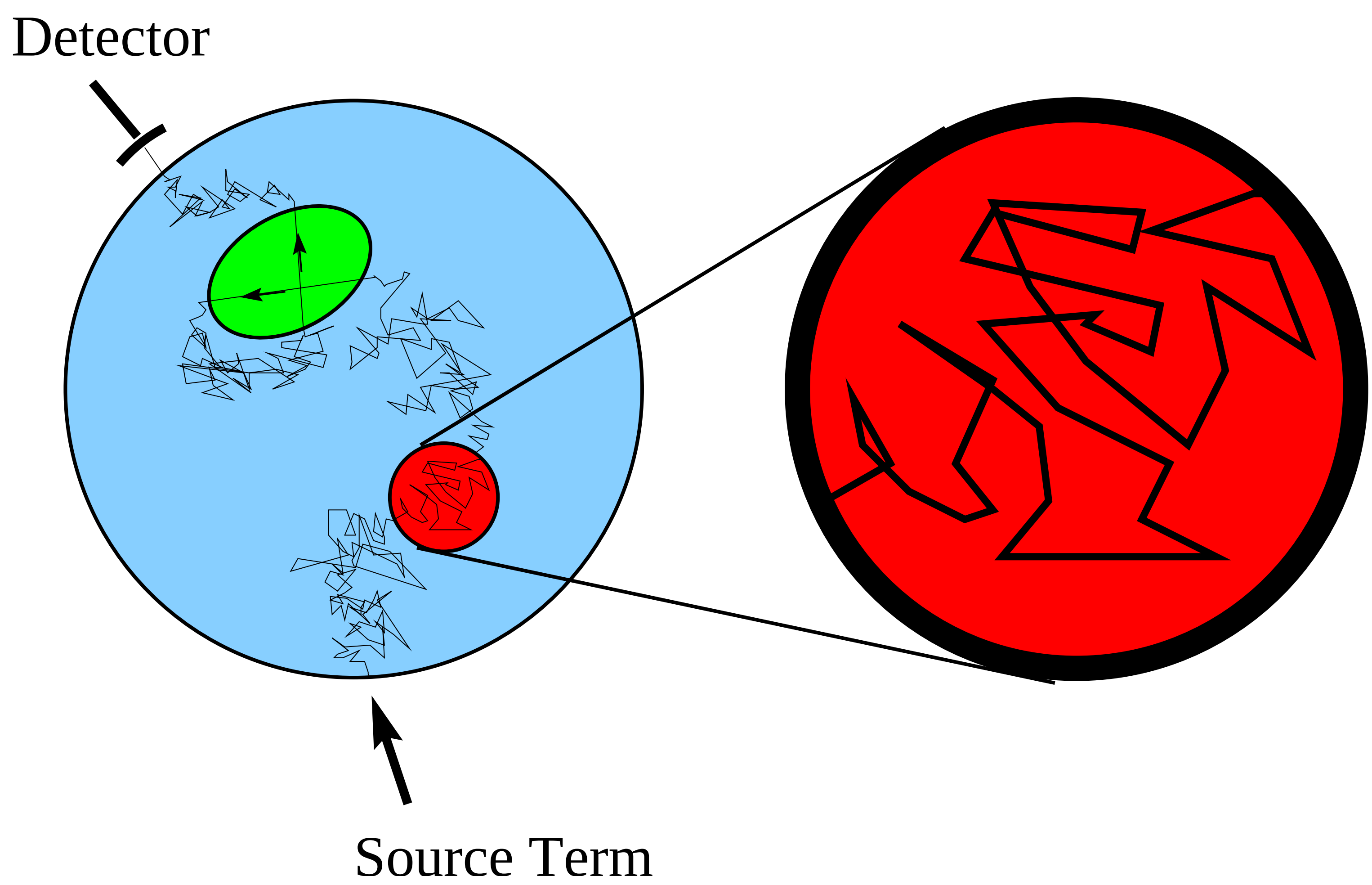
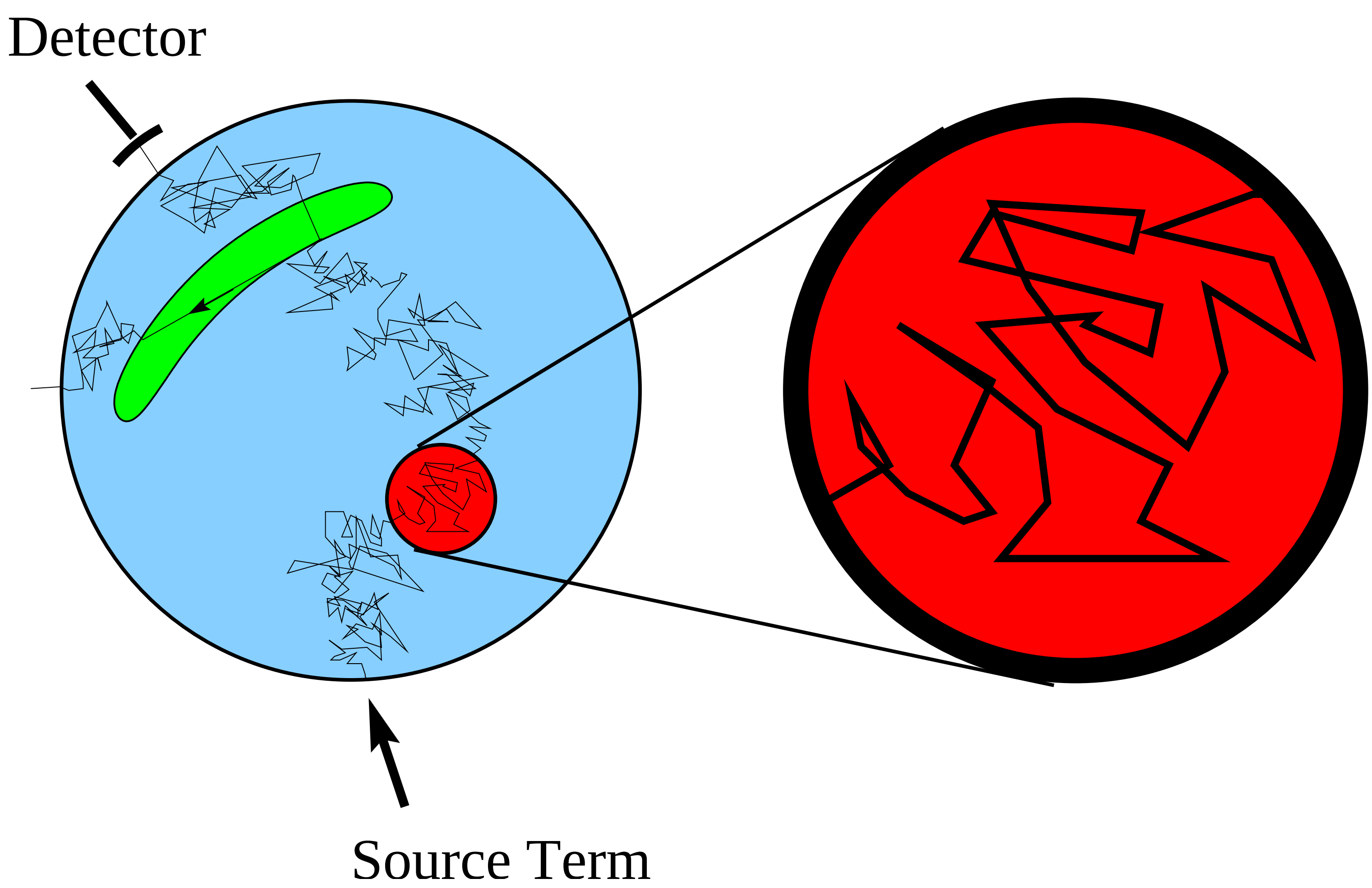
Typical path of a detected photon in a DIFFUSIVE REGION



Same typical path in the presence of a CLEAR INCLUSION



Same typical path in the presence of a CLEAR LAYER



Modeling the Forward Problem:

We want macroscopic equations that model photon propagation *both* in the diffusive and non-diffusive domains.

Outline:

1. Brief recall on the derivation of diffusion equations

2. Generalized equations in the presence of Clear Layers

3. Numerical simulation of transport and diffusion models

Transport Equation and Scaling

The phase-space linear transport equation is given by

$$\frac{1}{\varepsilon} v \cdot \nabla u_\varepsilon(x,v) + \frac{1}{\varepsilon^2} Q(u_\varepsilon)(x,v) + \sigma_a(x) u_\varepsilon(x,v) = 0 \qquad \text{in } \Omega \times V,$$

$$u_\varepsilon(x,v) = g(x,v) \qquad \text{on } \Gamma_- = \{(x,v) \in \partial\Omega \times V \text{ s.t. } v \cdot \nu(x) < 0\}.$$

$u_\varepsilon(x,v)$ is the particle density at $x \in \Omega \subset \mathbb{R}^3$ with direction $v \in V = S^2$.
The scattering operator Q is defined by

$$Q(u)(x,v) = \sigma_s(x)\Big(u(x,v) - \int_V u(x,v') d\mu(v') \Big).$$

The mean free path ε measures the mean distance between successive interactions of the particles with the background medium.
The diffusion limit occurs when $\varepsilon \to 0$.

Volume Diffusion Equation

Asymptotic Expansion: $u_\varepsilon(x,v) = u_0(x) + \varepsilon u_1(x,v) + \varepsilon^2 u_2(x,v)\dots$

Equating like powers of ε in the transport equation yields

$$\begin{aligned}
\text{Order } \varepsilon^{-2}: && Q(u_0) &= 0\\
\text{Order } \varepsilon^{-1}: && v\cdot\nabla u_0 + Q(u_1) &= 0\\
\text{Order } \varepsilon^{0}: && v\cdot\nabla u_1 + Q(u_2) + \sigma_a u_0 &= 0.
\end{aligned}$$

Krein-Rutman theory:

$$\begin{aligned}
&\text{Order } \varepsilon^{-2}: && u_0(x,v) = u_0(x)\\
&\text{Order } \varepsilon^{-1}: && u_1(x,v) = -\frac{1}{\sigma_s(x)} v\cdot\nabla u_0(x),\\
&\text{Order } \varepsilon^{0}: && \boxed{-\text{div} D(x)\cdot\nabla u_0(x) + \sigma_a(x)u_0(x) = 0 \quad \text{in } \Omega}
\end{aligned}$$

where the diffusion coefficient is given by $\boxed{D(x) = \dfrac{1}{3\sigma_s(x)}}$

Diffusion Equations with Boundary Conditions

The volume asymptotic expansion does not hold in the vicinity of boundaries. After boundary layer analysis we obtain

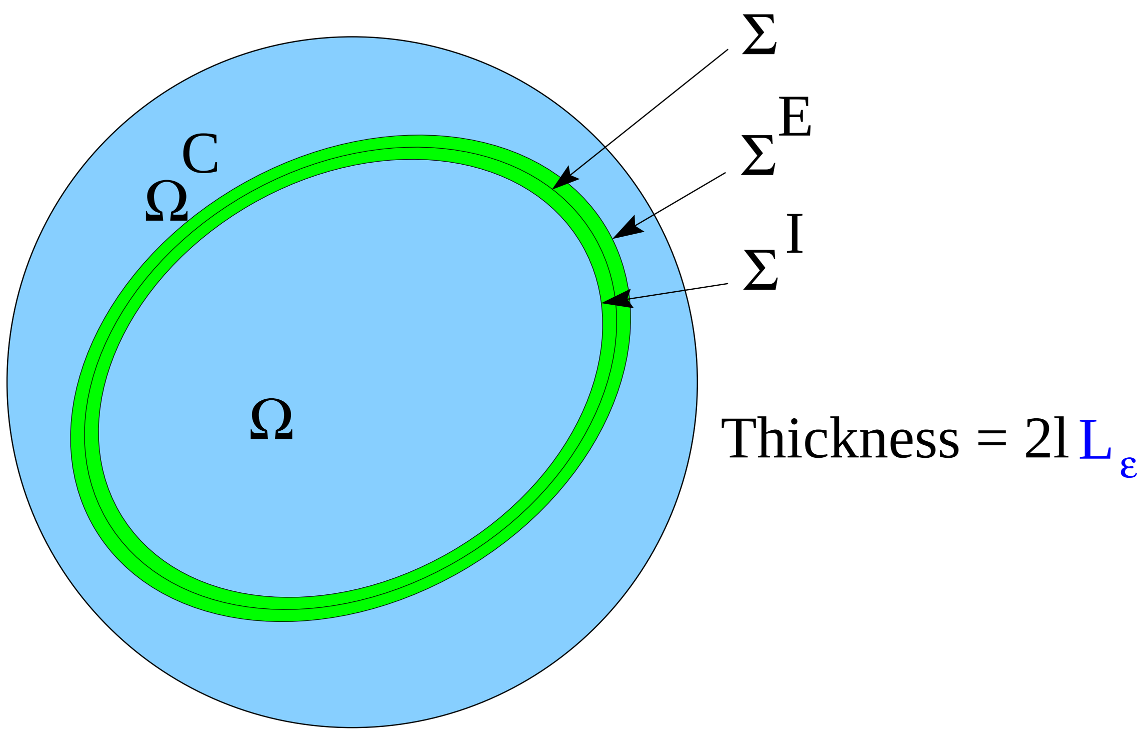
$$-\text{div} D(x) \cdot \nabla u_0(x) + \sigma_a(x) u_0(x) = 0 \quad \text{in} \quad \Omega$$
$$u_0(x) = \Lambda(g(x,v)) \quad \text{on} \quad \partial\Omega.$$

Λ is a linear form on $L^\infty(V_-)$.

We obtain in any reasonable sense that

$$\boxed{u_\varepsilon(x,v) = u_0(x) + O(\varepsilon).}$$

Generalization to an Extended Object of small thickness (Clear Layer)



Geometry of the Clear Layer Ω^C of boundary $\begin{cases} \Sigma^E = \Sigma + lL_\varepsilon \nu(x), \\ \Sigma^I = \Sigma - lL_\varepsilon \nu(x), \end{cases}$
where $\nu(x)$ is the outgoing normal to Σ at $x \in \Sigma$.

Local Generalized Diffusion Model

Assuming $L_\varepsilon^2|\ln L_\varepsilon| \sim \varepsilon$, in the limit $\varepsilon \to 0$ we obtain in a joint work with **Kui Ren** the following generalized diffusion model

$$-\nabla \cdot D(x)\boldsymbol{\nabla} U(x) + \sigma_a(x)U(x) = 0 \qquad \text{in } \Omega\backslash\Sigma$$

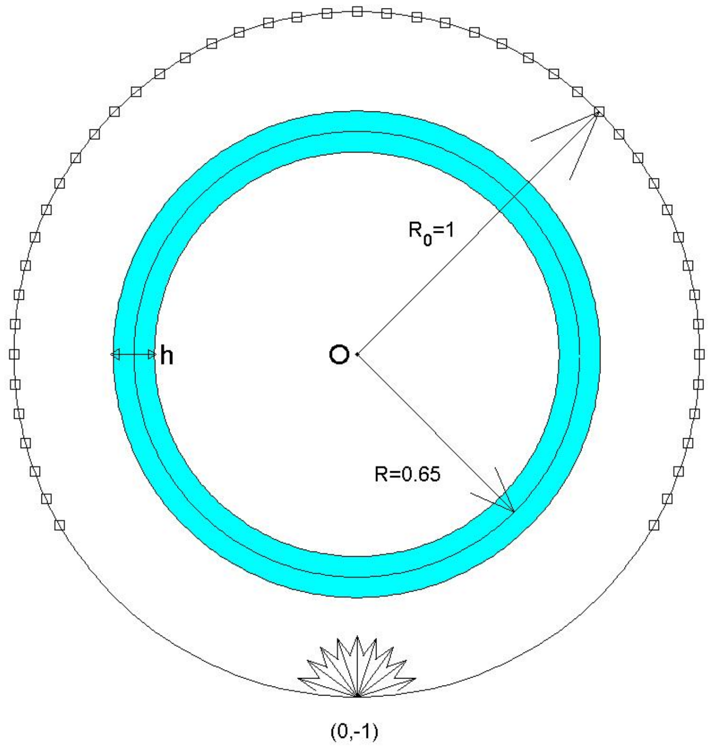
$$U(x) + 3L_3\varepsilon D(x)\nu(x)\cdot\nabla U(x) = \Lambda(g(x,v)) \qquad \text{on } \partial\Omega$$

$$[U](x) = 0 \qquad \text{on } \Sigma$$

$$\boxed{[\nu \cdot D\nabla U](x) = -\nabla_\perp d^c \nabla_\perp U.}$$

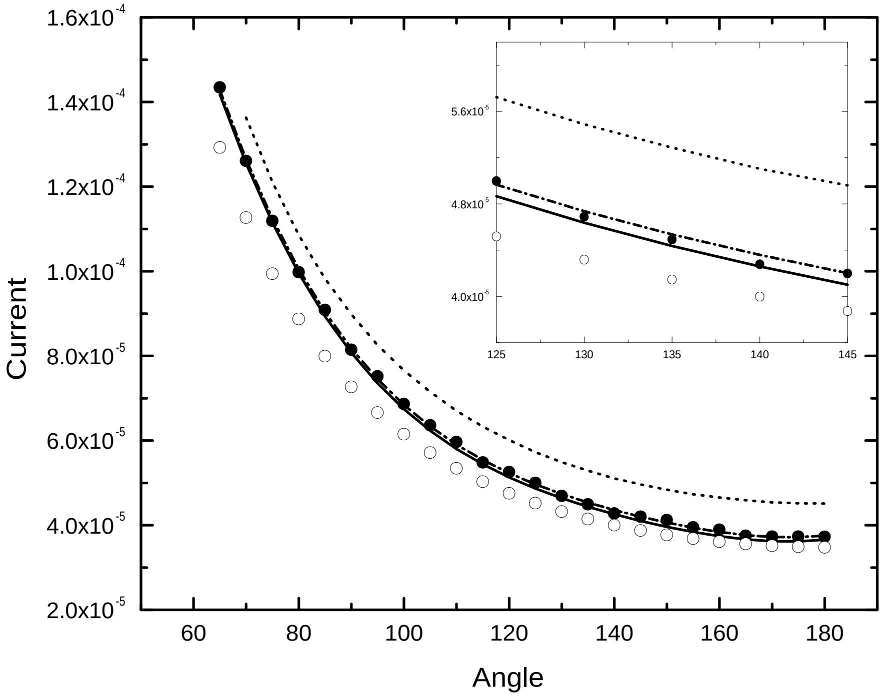
The clear layer is modeled as a tangential (supported on Σ) diffusion process with coefficient $d^c(\mathbf{x})$. The approximation (w.r.t. transport solution) is of order $\sqrt{\varepsilon}$ when Σ has positive curvature and can be as bad as $|\ln\varepsilon|^{-1}$ for straight clear layers.

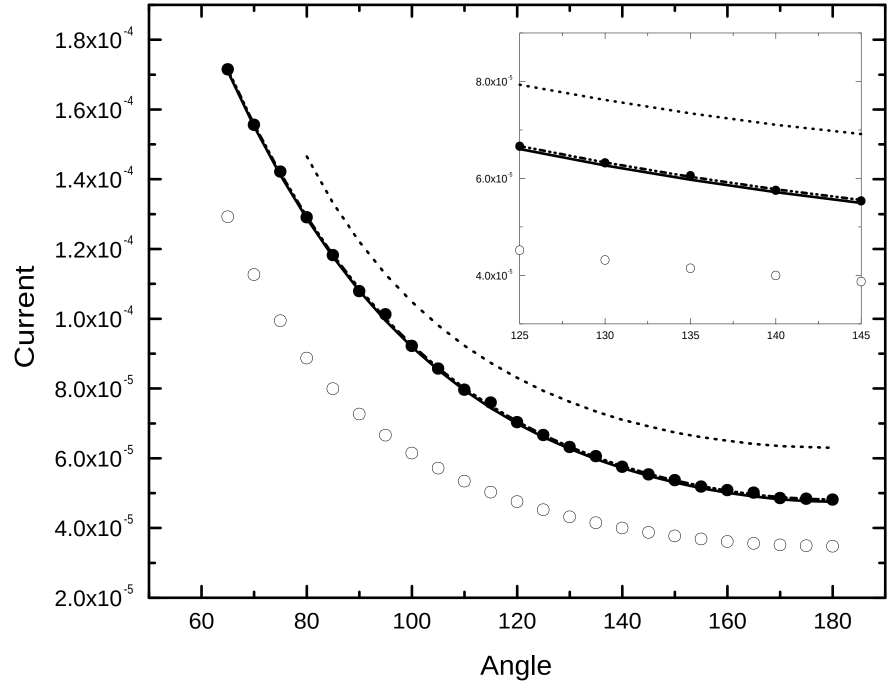
Numerical simulations



Geometry of domain with circular/spherical clear layer.

Two-dimensional Numerical simulation





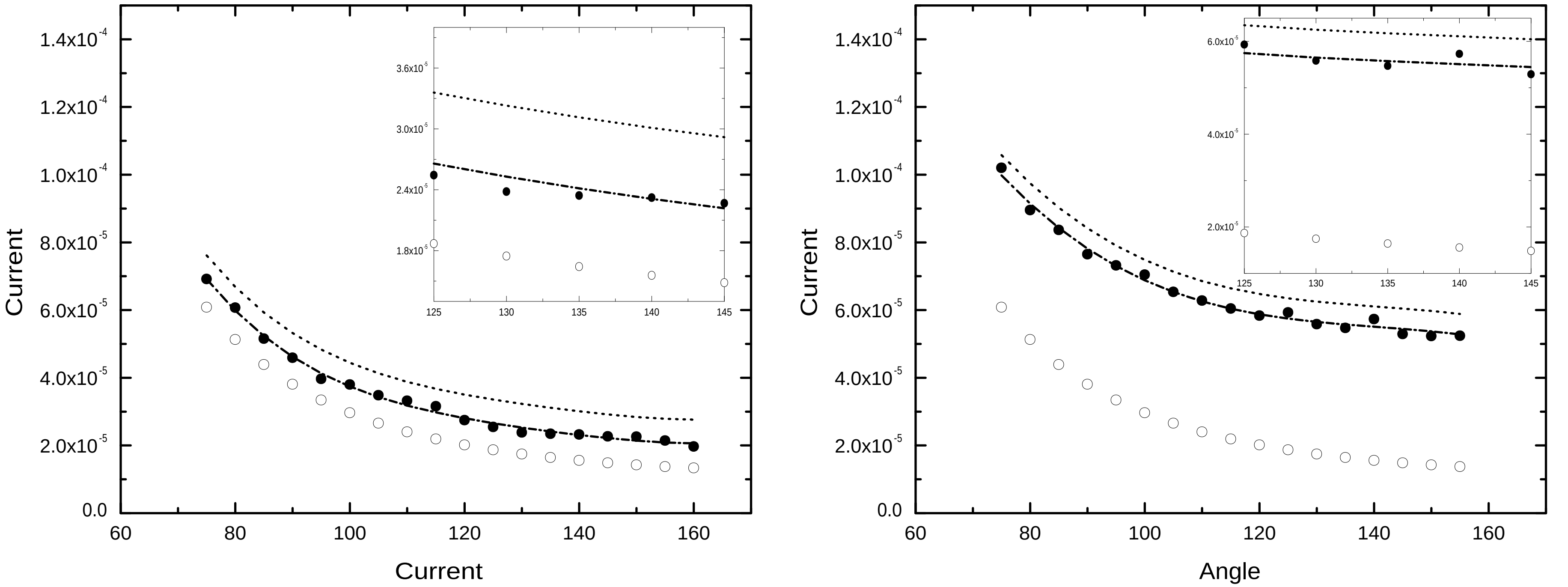
Outgoing current for clear layers of 2 and 5 mean free paths.

Two-dimensional Numerical simulation

h	0.01	0.02	0.03	0.04	0.05	0.06	0.07
d^C_{theory}	0.0124	0.0455	0.0971	0.166	0.253	0.355	0.475
$d^C_{\text{best fit}}$	0.0129	0.0465	0.0983	0.167	0.253	0.356	0.474
E_{GDM} (%)	1.17	1.56	1.43	1.09	0.81	0.56	0.60
E_{BF} (%)	0.73	0.65	0.57	0.49	0.46	0.47	0.46
E_{DI} (%)	3.3	10.2	17.7	24.5	30.2	35.3	39.8

Tangential diffusion coefficients and relative L^2 error between the transport Monte Carlo simulations and the various diffusion models for several thicknesses of the clear layer.

Three-dimensional Numerical simulation



Outgoing current for clear layers of 3 and 6 mean free paths.

Summary of Forward Modeling:

- We have a macroscopic model that captures particle propagation *both* in scattering and non-scattering regions, such as embedded objects and clear layers.
- The generalized diffusion model is computationally only slightly more expensive than the classical diffusion equation (essentially, one term is added in the variational formulation) and much less expensive than the full phase-space transport model.
- The accuracy of the macroscopic equation is sufficient to address the inverse problem where absorption and scattering cross sections are reconstructed from boundary measurements.

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Inverse Problem

- Optical Tomography uses near-infrared photons to image properties of human tissues.

- **Advantages:** Non-invasive (as are all "imaging" techniques); Cheap (that is, for a medical technique); Quite harmless (as light should be); and Good Discrimination properties between healthy and non-healthy tissues.

- **Disadvantages:** photons scatter a lot with underlying medium because they have low energy. This implies that the images have a low spatial resolution, and forward models are computationally expensive.

- Here we focus on reconstructing the clear layer and what it encloses. We also consider the similar problem in Impedance tomography.

Model problem in Impedance Tomography

The potential $u(\mathbf{x})$ solves the following equation:

$$\begin{array}{rcll} \nabla \cdot \gamma \nabla u & = & 0, & \text{in } \Omega \backslash \Sigma \\ [u] & = & 0 & \text{on } \Sigma \\ [\mathbf{n} \cdot \gamma \nabla u] & = & -\nabla_{\perp} \cdot d \nabla_{\perp} u & \text{on } \Sigma \\ \mathbf{n} \cdot \gamma \nabla u & = & g & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} u \; d\sigma & = & 0. & \end{array}$$

Assume that the above hypotheses are satisfied and that $g \in H_0^{-1/2}(\partial\Omega)$. Then the above system admits a unique solution $u \in H^1_{0,\Sigma}(\Omega)$ with trace $u_{|\partial\Omega} \in H_0^{1/2}(\partial\Omega)$. The variational formulation is

$$\int_{\Omega} \gamma \nabla u \cdot \nabla v \; d\mathbf{x} + \int_{\Sigma} d \nabla_{\perp} u \cdot \nabla_{\perp} v \; d\sigma = \int_{\partial\Omega} g v \; d\sigma.$$

Model problem in Optical Tomography

The photon density $u(\mathbf{x};\omega)$ solves the following equation

$$\begin{aligned} i\omega u - \nabla\cdot\gamma\nabla u + au &= 0, && \text{in } \Omega\backslash\Sigma \\ [u] &= 0 && \text{on } \Sigma \\ [\mathbf{n}\cdot\gamma\nabla u] &= -\nabla_\perp\cdot d\nabla_\perp u && \text{on } \Sigma \\ \mathbf{n}\cdot\gamma\nabla u &= g && \text{on } \partial\Omega. \end{aligned}$$

Assume that $a(\mathbf{x})$ is bounded when $\omega \neq 0$ and that $a(\mathbf{x})$ is uniformly bounded from below by a positive constant when $\omega = 0$, and that $g \in H^{-1/2}(\partial\Omega)$. Then the above system admits a unique solution $u \in H^1_\Sigma(\Omega)$ with trace $u_{|\partial\Omega} \in H^{1/2}(\partial\Omega)$. The variational formulation is:

$$\int_\Omega \Big(i\omega + a\Big)uv + \int_\Omega \gamma\nabla u\cdot\nabla v\ d\mathbf{x} + \int_\Sigma d\nabla_\perp u\cdot\nabla_\perp v\ d\sigma = \int_{\partial\Omega} gv\ d\sigma.$$

Assumptions and what we can reconstruct

• Main Assumption: The conductivity tensor γ is known on $\Omega \backslash \overline{D}$ such that $\Sigma = \partial D$.

• We reconstruct the interface $\Sigma = \partial D$ using a factorization method. The method is "constructive".

• Next we find the tangential diffusion tensor $d(\mathbf{x})$ on Σ.

• Finally we reconstruct what we can (using known theories) on γ from the knowledge of the Dirichlet-to-Neumann map.

A few more assumptions (Impedance case)

We define the Neumann-to-Dirichlet operator Λ_Σ as

$$\Lambda_\Sigma \;:\; H_0^{-1/2}(\partial\Omega) \longrightarrow H_0^{1/2}(\partial\Omega), \qquad g \longmapsto u_{|\partial\Omega}.$$

We define the "background" Neumann-to-Dirichlet operator Λ_0 as above with $\gamma(\mathbf{x})$ replaced by a known background $\gamma_0(\mathbf{x})$ and with $d(\mathbf{x})$ replaced by 0.

Our assumptions on the background $\gamma_0(\mathbf{x})$ is that it is the true conductivity tensor on $\Omega\backslash\overline{D}$ and a *lower-bound* to the true conductivity tensor on D:

$$\gamma_0(\mathbf{x}) \leq \gamma(\mathbf{x}) \quad \text{on } D, \qquad \gamma_0(\mathbf{x}) = \gamma(\mathbf{x}) \quad \text{on } \Omega\backslash\overline{D}.$$

The main assumption is thus that we *assume* that everything is known in $\Omega\backslash\overline{D}$.

A typical result

Theorem. Let us assume that the tensor $\gamma(\mathbf{x})$ is of class $C^2(\Omega)$ for $n = 2, 3$, is known on $\Omega\backslash\overline{D}$, and is proportional to identity (i.e., $\gamma(\mathbf{x}) = \frac{1}{n}\mathrm{Tr}(\gamma(\mathbf{x}))I$) on $\overline{D}$.

Then the surface $\Sigma = \partial D$, the tangential diffusion tensor $d(\mathbf{x})$, and the conductivity tensor $\gamma(\mathbf{x})$ are uniquely determined by the Cauchy data $\{u_{|\partial\Omega}, \mathbf{n} \cdot \gamma\nabla u_{|\partial\Omega}\}$ in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$.

Moreover the method to recover Σ is constructive and based on a suitable factorization of $\Lambda_0 - \Lambda_\Sigma$.

The Factorization method

The idea is to reconstruct the support of objects *without* knowing what is inside.

Originally proposed by Colton and Kirsch and analyzed to detect obstacles in the scattering context by Kirsch [IP1998].

Analyzed in impedance tomography for objects with different impedance than the background by Brühl [SIMA01].

Factorization method idea

The idea is to factor the difference of NtD operators as

$$\Lambda_0 - \Lambda_\Sigma = L^* F L,$$

where L and L^* are defined on $\Omega \backslash \overline{D}$ and where F can be decomposed as $B^* B$ with B^* surjective. This implies that

$$\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{1/2}) = \mathcal{R}(L^*).$$

We then construct functions $\mathbf{y} \mapsto g_{\mathbf{y}}(\cdot)$ from the measured data that solve the source-less diffusion equation in $\Omega \backslash \overline{D}$ and are in the Range of L *if and only* if $\mathbf{y} \in D$.

This allows us to constructively image the interface Σ from the boundary measurements.

Details of the factorization

Let us define v and v^* as the solutions to:

$$\begin{aligned} \nabla \cdot \gamma \nabla v &= 0, && \text{in } \Omega \backslash \overline{D} \\ \mathbf{n} \cdot \gamma \nabla v &= 0 && \text{on } \Sigma \\ \mathbf{n} \cdot \gamma \nabla v &= \phi && \text{on } \partial\Omega \\ \int_{\partial\Omega} v \; d\sigma &= 0, \end{aligned} \qquad \begin{aligned} \nabla \cdot \gamma \nabla v^* &= 0, && \text{in } \Omega \backslash \overline{D} \\ \mathbf{n} \cdot \gamma \nabla v^* &= -\phi && \text{on } \Sigma \\ \mathbf{n} \cdot \gamma \nabla v^* &= 0 && \text{on } \partial\Omega \\ \int_{\Sigma} v^* \; d\sigma &= 0. \end{aligned}$$

L maps $\phi \in H_0^{-1/2}(\partial\Omega)$ to $v_{|\Sigma} \in H_0^{1/2}(\Sigma)$ and its adjoint operator L^* maps $\phi \in H_0^{-1/2}(\Sigma)$ to $v^*_{|\partial\Omega} \in H_0^{1/2}(\partial\Omega)$. We have

$$(u, L^* v)_{\partial\Omega} \equiv \int_{\partial\Omega} u L^* v \; d\sigma = \int_{\Sigma} v L u \; d\sigma \equiv (Lu, v)_{\Sigma}.$$

Construction of F

Let us define w as the solution to

$$\begin{aligned} \nabla\cdot\gamma\nabla w &= 0, && \text{in } \Omega\backslash\Sigma \\ [w] &= \phi, && \text{on } \Sigma \\ [\mathbf{n}\cdot\gamma\nabla w] &= -\nabla_\perp d\nabla_\perp w^- && \text{on } \Sigma \\ \mathbf{n}\cdot\gamma\nabla w &= 0 && \text{on } \partial\Omega \\ \int_{\partial\Omega} w\ d\sigma &= 0. \end{aligned}$$

The operator F_Σ maps $\phi\in H_0^{1/2}(\Sigma)$ to $\mathbf{n}\cdot\gamma\nabla w^+\in H_0^{-1/2}(\Sigma)$.
The variational formulation to the above problem is

$$\int_\Omega \gamma\nabla w\cdot\nabla w\ d\mathbf{x}+\int_\Sigma d\nabla_\perp w^-\cdot\nabla_\perp w^-\ d\sigma=\int_\Sigma \mathbf{n}\cdot\gamma\nabla w^+\ \phi\ d\sigma=\int_\Sigma F_\Sigma\phi\ \phi\ d\sigma.$$

F_0 is defined similarly with $\gamma(\mathbf{x})$ replaced by $\gamma_0(\mathbf{x})$ and $d(\mathbf{x})\equiv 0$.
The operator F is defined as $F=F_0-F_\Sigma$. It is symmetric and we have $\Lambda_0-\Lambda_\Sigma=L^*FL$.

Coercivity of F

After a few integrations by parts we obtain with $\delta w = w_\Sigma - w_0$:

$$\int_\Sigma F\phi\ \phi = \int_\Omega \gamma \nabla \delta w \cdot \nabla \delta w + \int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_0 + \int_\Sigma d \nabla_\perp w_\Sigma^- \cdot \nabla_\perp w_\Sigma^- .$$

We can then show that F is coercive in the sense that

$$(F\phi, \phi) \geq \alpha \|\phi\|^2_{H_0^{1/2}(\Sigma)},$$

for some $\alpha > 0$.

What makes the factorization useful (I)

In the case of a jump of the diffusion coefficient across the interface Σ, one can show that F is an isomorphism so that it can be written $F = B^*B$ with B also an isomorphism.

In the clear layer case, F may not be an isomorphism. It can still be decomposed. Let $\mathcal{I}$ be the canonical isomorphism between $H_0^{-1/2}(\Sigma)$ and $H_0^{1/2}(\Sigma)$ and define

$$\mathcal{I} = \mathcal{J}^*\mathcal{J}, \qquad \mathcal{J} : \ H_0^{-1/2}(\Sigma) \ \to \ L_0^2(\Sigma), \quad \mathcal{J}^* : \ L_0^2(\Sigma) \ \to \ H_0^{1/2}(\Sigma).$$

We can thus recast the coercivity of F as

$$(F\phi, \phi) = (F\mathcal{J}^*u, \mathcal{J}^*u) = (\mathcal{J}F\mathcal{J}^*u, u) \geq \alpha\|\phi\|^2_{H_0^{1/2}(\Sigma)} = \alpha\|u\|^2_{L_0^2(\Sigma)}.$$

What makes the factorization useful (II)

Since

$$(\mathcal{J} F \mathcal{J}^* u, u) \geq \alpha \|u\|^2_{L^2_0(\Sigma)},$$

$\mathcal{J} F \mathcal{J}^*$ is symmetric and positive definite as an operator on $L^2_0(\Sigma)$. So we can decompose the operator as

$$\mathcal{J} F \mathcal{J}^* = C^* C, \qquad \text{with } C,\ C^* \text{ positive operators from } L^2_0(\Sigma) \text{ to } L^2_0(\Sigma).$$

So we have the decomposition

$$\boxed{F = B^* B}, \qquad B = \mathcal{J}^{-1} C^* \text{ maps } H^{1/2}_0(\Sigma) \text{ to } L^2_0(\Sigma).$$

Since F is coercive, we deduce that $\|B\phi\|_{L^2_0(\Sigma)} \geq C \|\phi\|_{H^{1/2}_0(\Sigma)}$.

This implies that B^* is surjective.

Factorization: The Range Characterization

From the above calculations we obtain that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L = L^* B^* (L^* B^*)^* = A^* A.$$

Since the Range of $(A^*A)^{1/2}$ is the Range of A^*, we deduce:

$$\boxed{\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{1/2}) = \mathcal{R}(L^* B^*) = \mathcal{R}(L^*)}$$

since B^* is surjective. Now consider the solution of

$$\begin{aligned} \nabla \cdot \gamma_0 \nabla N(\cdot; \mathbf{y}) &= \delta(\cdot - \mathbf{y}), && \text{in } \Omega \\ \mathbf{n} \cdot \gamma_0 \nabla N(\cdot; \mathbf{y}) &= 0 && \text{on } \partial\Omega \quad \int_\Sigma N(\cdot; \mathbf{y}) \, d\sigma = 0. \end{aligned}$$

Then $\mathbf{n} \cdot \gamma \nabla N(\mathbf{x}; \mathbf{y})_{|\Sigma} \in H_0^{-1/2}(\Sigma)$ and $N(\mathbf{x}; \mathbf{y}) \in \mathcal{R}(L^*)$ *if and only if* $\mathbf{y} \in D$. Notice that this requires that $\gamma(\mathbf{x}) = \gamma_0(\mathbf{x})$ be known on $\Omega \backslash \overline{D}$.

How do we get the rest?

Now that Σ is known, we have on $\mathcal{R}(\Lambda_0 - \Lambda_\Sigma) \subset \mathcal{R}(L^*)$:

$$(L^*)^{-1}(\Lambda_0 - \Lambda_\Sigma) = FL.$$

L is dense in $H_0^{1/2}(\Sigma)$ since $\overline{\mathcal{R}(L)} = \mathcal{N}(L^*)^\perp = \{0\}^\perp = H_0^{1/2}(\Sigma)$ so we have access to the full mapping F in $\mathcal{L}(H_0^{1/2}(\Sigma), H_0^{-1/2}(\Sigma))$ and $F_\Sigma = F + F_0$.

The Range of $G_\Sigma = F_\Sigma L$ is dense since G_Σ^* is injective. This provides knowledge of the full Cauchy data:

$$\Big\{ w^-_{|\Sigma} \in H_0^{1/2}(\Sigma); \qquad \mathbf{n} \cdot \gamma \nabla w^+_{|\Sigma} \in H_0^{-1/2}(\Sigma) \Big\},$$

whence of the Dirichlet to Neumann operator

$$\Lambda_D = -\nabla_\perp \cdot d \nabla_\perp + \tilde{\Lambda}_D,$$

where $\tilde{\Lambda}_D$ is the Dirichlet to Neumann map of the domain D.

Reconstruction of $d(\mathbf{x})$.

Recall that

$$\Lambda_D = -\nabla_\perp \cdot d\nabla_\perp + \tilde{\Lambda}_D.$$

The second contribution $\tilde{\Lambda}_D$ is a bounded operator from $H_0^1(\Sigma)$ to $L_0^2(\Sigma)$. Let Σ be given locally by $x^n = 0$ in the coordinates $(\mathbf{x}', x^n)$. Since $\tilde{\Lambda}_D$ differentiates only once, it is clear that

$$\boldsymbol{\omega}' \cdot d(\mathbf{x}')\boldsymbol{\omega}' = \lim_{s\to\infty} \frac{-1}{s^2} e^{-is\boldsymbol{\omega}'\cdot\mathbf{x}'} \Lambda_D e^{is\boldsymbol{\omega}'\cdot\mathbf{x}'}, \qquad \text{for all} \qquad \boldsymbol{\omega}' \in S^{n-2}.$$

This fully characterizes the symmetric tensor $d(\mathbf{x}')$.

Reconstruction of $\gamma(\mathbf{x})$.

Once $d(\mathbf{x})$ is known, we have access to the Dirichlet to Neumann map $\tilde{\Lambda}_D$ of the domain D.

We then use known results to show that $\gamma(\mathbf{x})$ can uniquely be reconstructed if it is a sufficiently smooth (depending on space dimension) scalar-valued conductivity.
For anisotropic tensors in dimension $n = 2$ we have that γ_1 and γ_2 in $C^{2,\alpha}(D)$, $0 < \alpha < 1$, with boundary ∂D of class $C^{3,\alpha}$ with same data Λ_γ are such that there exists a $C^{3,\alpha}(D)$ diffeomorphism Φ with $\Phi_{|\partial\Omega} = I_{\partial\Omega}$, the identity operator on $\partial\Omega$, and

$$\gamma_2(x) = \frac{(D\Phi)^T \gamma_1 (D\Phi)}{|D\Phi|} \circ \Phi^{-1}(x).$$

In dimension $n \geq 3$ the same results holds provided that γ_1, γ_2, and ∂D (then Φ) are real-analytic.

Shape sensitivity analysis.

In a joint work with **Kui Ren**, we have developed a shape-sensitivity-based method to reconstruct the singular interface (clear layer) Σ. This is recast as a regularized nonlinear least square problem:

$$\mathcal{F}_\alpha(\Sigma) := \frac{1}{2}\left\|u - u_m^\delta\right\|^2_{L^2(\Gamma)} + \alpha \int_\Sigma d\sigma(\mathbf{x}) \to \min_{\Sigma\in\Pi}.$$

Here and below, $\Gamma = \delta\Omega$. For a smooth vector field $\mathbf{V}(\mathbf{x})$ we define

$$\mathbf{F}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{V}(\mathbf{x}), \qquad \Sigma_t = \mathbf{F}_t(\Sigma),$$

and show that

$$d\mathcal{F}_\alpha(\Sigma) = (u - u_m^\delta, u')_{(\Gamma)} + \alpha(\kappa(\mathbf{x}), V_n)_{(\Sigma)},$$

where u' is the shape derivative of the current estimate $u = u(\Sigma)$.

Level-set based numerical simulation.

We show that $d\mathcal{F}_\alpha(\Sigma) \leq 0$ when $\mathbf{V}$ is chosen such that

$$V_n = d\kappa \nabla_\perp u \cdot \nabla_\perp w + \mathbf{n} \cdot \nabla u^+ \mathbf{n} \cdot D \nabla w^+ - \mathbf{n} \cdot \nabla u^- \mathbf{n} \cdot D \nabla w^- - \alpha\kappa,$$

where w solves an adjoint equation.

Combined with a level-set approach, it allows us to construct velocity fields and to numerically move a guessed interface (assuming that $d(\mathbf{x})$ is *known*) so as to lower the discrepancy with the measured data.
The method was first proposed by F. Santosa to image the interface between two areas with known (and different) diffusion coefficients.

Simulations show how **noise** in the data **degrades** the reconstruction.

Ellipse 0% noise | Ellipse 5% noise | Star 0% noise | Star 2% noise

Conclusions.

Assuming the conductivity is known between $\partial\Omega$ and Σ and a lower bound is known on Ω, we can reconstruct the singular interface Σ, the tangential diffusion tensor $d(\mathbf{x})$, and a scalar-valued conductivity on the region enclosed by Σ.

The method to obtain Σ is constructive.

This means that we can image Σ and through Σ. This a positive result. The extension to Optical Tomography is straightforward.

Numerical simulations (based on shape derivatives and the level set method) show reconstructions of the interface from boundary measurements.

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