Fast inversion of the Attenuated Radon Transform (AtRT) with partial measurements

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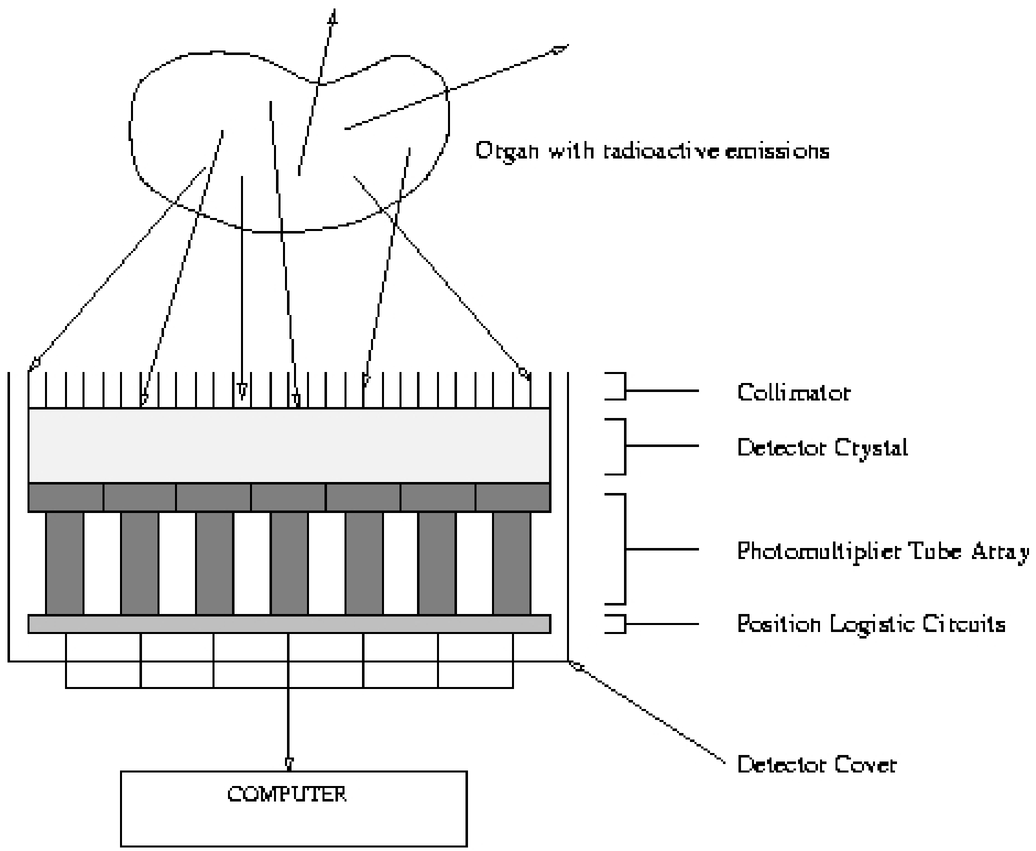
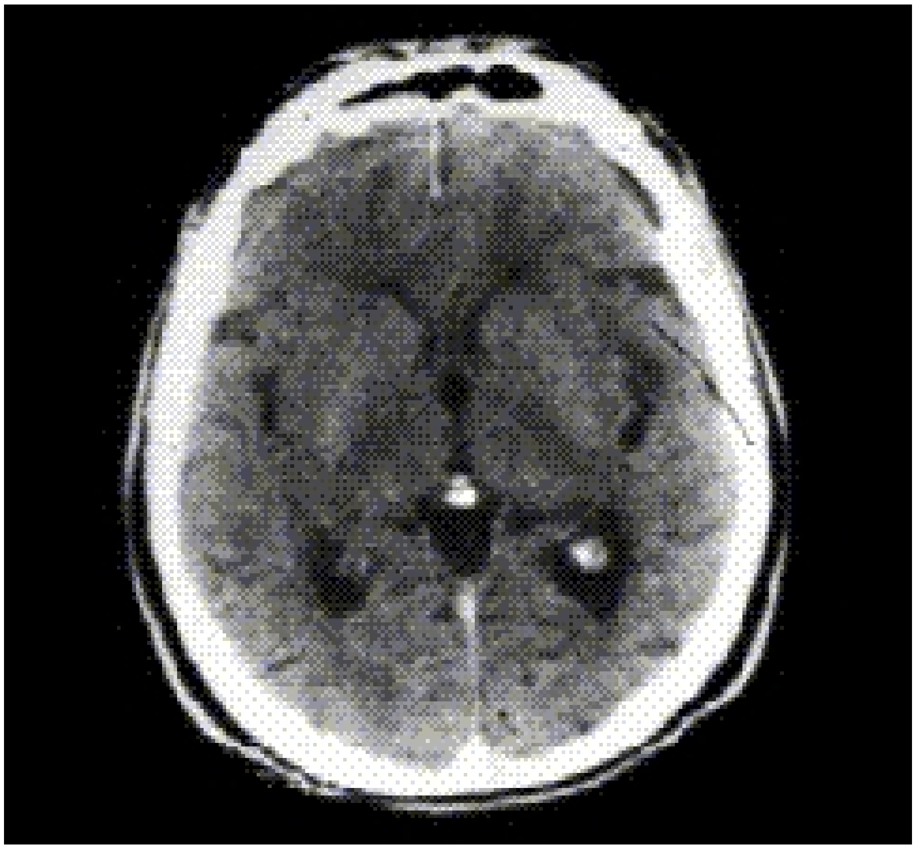
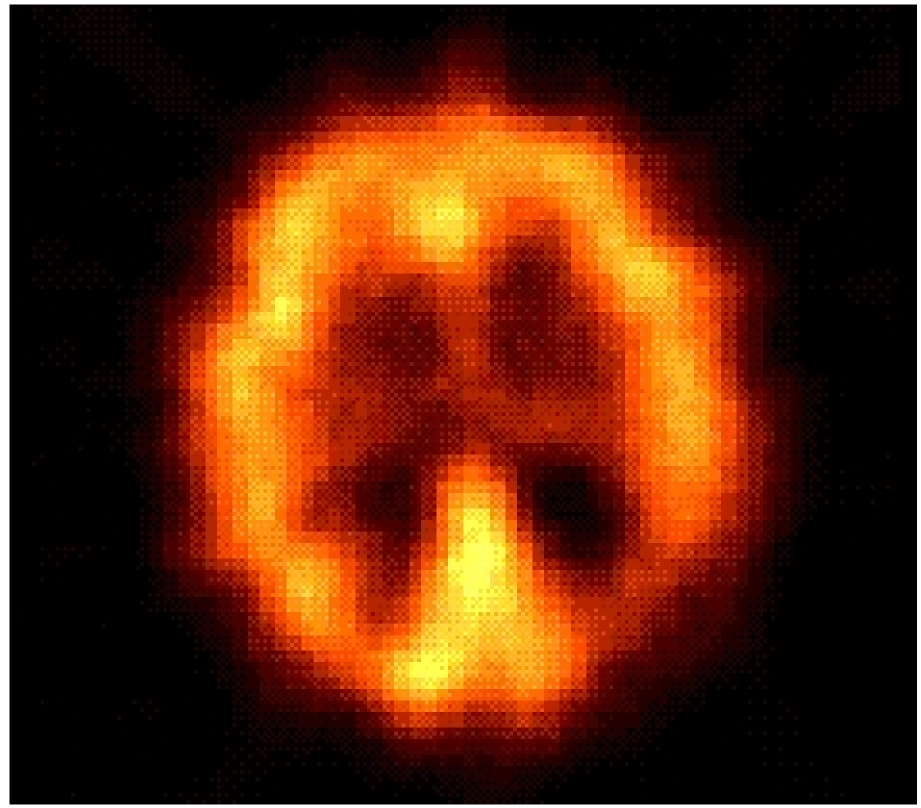
Data Acquisition in SPECT
(Single Photon Emission Computed Tomography)



CT and SPECT measurements in brain





Mathematical modeling

The transport equation with anisotropic source term is given by

$$\boldsymbol{\theta} \cdot \nabla \psi(\mathbf{x}, \theta) + a(\mathbf{x})\psi(\mathbf{x}, \theta) = f(\mathbf{x}, \theta) = \sum_{k=-N}^{N} f_k(\mathbf{x}) e^{ik\theta}, \quad \mathbf{x} \in \mathbb{R}^2, \ \boldsymbol{\theta} \in S^1.$$

We identify $\boldsymbol{\theta} = (\cos\theta, \sin\theta) \in S^1$ and $\theta \in (0, 2\pi)$. We assume that $f_{-k} = \overline{f}_k$ and $f_k(\mathbf{x})$ is compactly supported. The boundary conditions are such that for all $\mathbf{x} \in \mathbb{R}^2$,

$$\lim_{s \to +\infty} \psi(\mathbf{x} - s\boldsymbol{\theta}, \theta) = 0.$$

The absorption coefficient $a(\mathbf{x})$ is smooth and decays sufficiently fast at infinity. The above transport solution admits a unique solution and we can define the *symmetrized* beam transform as

$$D_\theta a(\mathbf{x}) = \frac{1}{2} \int_0^\infty [a(\mathbf{x} - t\boldsymbol{\theta}) - a(\mathbf{x} + t\boldsymbol{\theta})] dt.$$

Mathematical modeling (II)

The symmetrized beam transform satisfies $\boldsymbol{\theta}\cdot\nabla D_\theta a(\mathbf{x}) = a(\mathbf{x})$ so that the transport solution is given by

$$e^{D_\theta a(\mathbf{x})}\psi(\mathbf{x},\theta) = \int_0^\infty (e^{D_\theta a} f)(\mathbf{x}-t\boldsymbol{\theta},\theta)dt.$$

Upon defining $\boldsymbol{\theta}^\perp = (-\sin\theta, \cos\theta)$ and $\mathbf{x} = s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}$, we find that

$$\begin{aligned}\lim_{t\to+\infty} e^{D_\theta a(s\boldsymbol{\theta}^\perp+t\boldsymbol{\theta})}\psi(s\boldsymbol{\theta}^\perp+t\boldsymbol{\theta},\theta) &= \int_{\mathbb{R}} (e^{D_\theta a} f)(s\boldsymbol{\theta}^\perp+t\boldsymbol{\theta},\theta)dt \\ \lim_{t\to+\infty} \psi(s\boldsymbol{\theta}^\perp+t\boldsymbol{\theta},\theta) &= e^{-(P_\theta a)(s)/2}(R_{a,\theta} f)(s),\end{aligned}$$

where P_θ is the Radon transform and $R_{a,\theta}$ the Attenuated Radon Transform (AtRT) defined by:

$$\begin{aligned} P_\theta f(s) &= \int_{\mathbb{R}} f(s\boldsymbol{\theta}^\perp+t\boldsymbol{\theta},\theta)dt = \int_{\mathbb{R}^2} f(\mathbf{x},\theta)\delta(\mathbf{x}\cdot\boldsymbol{\theta}^\perp - s)d\mathbf{x} \\ (R_{a,\theta} f)(s) &= (P_\theta(e^{D_\theta a} f))(s). \end{aligned}$$

Inverse Problem in SPECT

The inverse problem consists then in answering the following questions:

1. Knowing the AtRT $R_{a,\theta}f(s)$ for $\boldsymbol{\theta} \in S^1$ and $s \in \mathbb{R}$, what can we reconstruct in $f(\mathbf{x}, \theta)$?
2. Assuming $f(\mathbf{x}, \theta) = f_0(\mathbf{x}) + 2\cos\theta f_1(\mathbf{x})$, can we obtain explicit formulas for the source term?
3. Can we reconstruct $f(\mathbf{x}, \theta) = f_0(\mathbf{x})$ from half of the measurements?
4. Do we have a reliable numerical technique to obtain fast reconstructions?

Part I: Reconstruction from full measurements
The Novikov formula revisited

We recast the inversion as a Riemann Hilbert (RH) problem. Let us define $T = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$, $D^+ = \{\lambda \in \mathbb{C}, |\lambda| < 1\}$, and $D^- = \{\lambda \in \mathbb{C}, |\lambda| > 1\}$. Let $\varphi(t)$ be a smooth function defined on T. Then there is a unique function $\phi(\lambda)$ such that

- $\phi(\lambda)$ is analytic on D^+ and D^-
- $\lambda\phi(\lambda)$ is bounded at infinity
- $\varphi(t) = \lim_{0<\varepsilon\to 0}(\phi((1-\varepsilon)t) - \phi((1+\varepsilon)t)) \equiv \phi^+(t) - \phi^-(t)$.

Moreover $\phi(\lambda)$ is given by the Cauchy formula

$$\phi(\lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(t)}{t-\lambda}dt, \qquad \lambda \in \mathbb{C}\backslash T.$$

RH for AtRT, a road map

1. Extend the transport equation to the complex plane (complex-valued directions of propagation $\boldsymbol{\theta} \to e^{i\theta} = \lambda \in \mathbb{C}$). Replace the transport solution $\psi(\mathbf{x}, \lambda)$ by $\phi(\mathbf{x}, \lambda)$ which is analytic on D^+ and D^- and $O(\lambda^{-1})$ at infinity by subtracting a finite number of analytic terms on $\mathbb{C}\backslash\{0\}$.

2. Verify that the jump of $\phi(\mathbf{x}, \lambda)$ at $\lambda \in T$ is a function of the measured data $R_{a,\theta} f(s)$.

3. Read off the constraints on the source terms $f_k(\mathbf{x})$ from the Taylor expansion of $\phi(\mathbf{x}, \lambda)$ at $\lambda = 0$.

4. In simplified settings, reconstruct the $f_k(\mathbf{x})$ from the constraints.

Step 1: RH setting

Define

$$\lambda = e^{i\theta}, \qquad z = x + iy \text{ with } \mathbf{x} = (x, y), \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

The transport equation is then recast as

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}} + a(z)\Big)\psi(z,\lambda) = f(z,\lambda).$$

We now consider the above equation for arbitrary complex values of λ. $\psi(z,\lambda)$ is analytic on $\lambda \in \mathbb{C}\backslash(T \cup \{0\})$ and is given by

$$\psi(z,\lambda) = e^{-h(z,\lambda)}\int_{\mathbb{C}} G(z-\zeta,\lambda)e^{h(\zeta,\lambda)}f(\zeta,\lambda)dm(\zeta),$$

where $h(z,\lambda) = \int_{\mathbb{C}} G(z-\zeta,\lambda)a(\zeta)dm(\zeta)$ and

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}}\Big)G(z,\lambda) = \delta(z), \qquad \text{so that} \qquad G(z,\lambda) = \frac{\text{sign}(|\lambda|-1)}{\pi(\lambda\bar{z} - \lambda^{-1}z)}.$$

The source term is given by $f(z,\lambda)=\sum_{k=-N}^{N} f_k(z)\lambda^k$. On D^+ we have

$$G(z,\lambda)=\frac{1}{\pi z}\sum_{m=0}^{\infty}\left(\frac{\overline{z}}{z}\right)^m \lambda^{2m+1}, \qquad \text{and} \qquad \psi(\cdot,\lambda)=\sum_{m=1}^{\infty}(\mathcal{H}_m f(\cdot,\lambda))\lambda^m,$$

where the operators $\mathcal{H}_m$ are explicitly computable with

$$\mathcal{H}_1=\left(\frac{\partial}{\partial\overline{z}}\right)^{-1}, \quad \mathcal{H}_2=-\mathcal{H}_1 a\mathcal{H}_1, \quad \frac{\partial}{\partial\overline{z}}\mathcal{H}_{k+2}+a\mathcal{H}_{k+1}+\frac{\partial}{\partial z}\mathcal{H}_k=0.$$

Using a similar expression on D^-, we find that

$$\phi(z,\lambda)=\psi(z,\lambda)-\sum_{n=-\infty}^{-1}\lambda^n\sum_{m=1}^{\infty}(\mathcal{H}_m f_{n-m})(z)-\sum_{n=-\infty}^{0}\lambda^{-n}\sum_{m=1}^{\infty}(\overline{\mathcal{H}_m} f_{m-n})(z),$$

satisfies the hypotheses of the RH problem: it is analytic on $D^+\cup D^-$ and of order $O(\lambda^{-1})$ at infinity. Its jump across T is the same as that of ψ since the difference $\psi-\phi$ is analytic in $\mathbb{C}\backslash\{0\}$. On D^+ it is given by

$$\phi(z,\lambda)=\sum_{n=0}^{\infty}\lambda^n\sum_{m=1}^{\infty}(\mathcal{H}_m f_{n-m}-\overline{\mathcal{H}_m} f_{n+m})(z).$$

Step 2: jump conditions

Writing $\lambda = re^{i\theta}$ and sending $r-1$ to ± 0, we obtain

$$G_\pm(\mathbf{x},\theta) = \frac{\pm 1}{2\pi i(\boldsymbol{\theta}^\perp\cdot\mathbf{x} \mp i0\text{sign}(\boldsymbol{\theta}\cdot\mathbf{x}))},$$
$$h_\pm(\mathbf{x},\theta) = \pm\frac{1}{2i}(HP_\theta a)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp) + (D_\theta a)(\mathbf{x}), \qquad Hu(t) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{u(s)}{t-s}ds.$$

Here H is the Hilbert transform. We thus obtain that ψ converges on both sides of T parameterized by $\theta \in (0,2\pi)$ to

$$\psi_\pm(\mathbf{x},\theta) = e^{-D_\theta a}e^{\frac{\pm 1}{2i}(HP_\theta a)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)}\frac{\mp 1}{2i}H\Big(e^{\frac{\pm 1}{2i}(HP_\theta a)(s)}P_\theta(e^{D_\theta a}f)\Big)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)$$
$$+e^{-D_\theta a}D_\theta(e^{D_\theta a}f)(\mathbf{x}).$$

Notice that $(\psi_+ - \psi_-)$ is a function of the measurements $R_{a,\theta}f(s) = P_\theta(e^{D_\theta a}f)(s)$ whereas $\psi_\pm$ individually are not.

Jump conditions (ii)

Let us define

$$\varphi(\mathbf{x},\theta) = (\psi^+ - \psi^-)(\mathbf{x},\theta).$$

It depends on the measured data and is given by

$$i\varphi(\mathbf{x},\theta) = [R^*_{-a,\theta}(2H_a)R_{a,\theta}f](\mathbf{x}) = [R^*_{-a,\theta}(2H_a)g(s,\theta)](\mathbf{x}),$$

where

$$R^*_{a,\theta}g(\mathbf{x}) = e^{D_\theta a(\mathbf{x})}g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp), \qquad H_a = \frac{1}{2}(C_cHC_c + C_sHC_s)$$

$$C_cg(s,\theta) = g(s,\theta)\cos(\frac{HRa(s,\theta)}{2}), \quad C_sg(s,\theta) = g(s,\theta)\sin(\frac{HRa(s,\theta)}{2}).$$

Here $R^*_{a,\theta}$ is the adjoint operator to $R_{a,\theta}$. We note that $i\varphi(\mathbf{x},\theta)$ is *real-valued* and that $\boldsymbol{\theta}\cdot\nabla\varphi + a\varphi = 0$.

Step 3: constraints on source terms

The function ϕ is sectionally analytic, of order $O(\lambda^{-1})$ at infinity and such that

$$\varphi(z,\theta) = \phi^{+}(z,\theta) - \phi^{-}(z,\theta) \qquad \text{on } T.$$

So ϕ is the unique solution to the RH problem given by

$$\phi(z,\lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)}{t-\lambda}dt = \sum_{n=0}^{\infty} \lambda^n \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)dt}{t^{n+1}}$$

on D^{+} so that

$$\boxed{\sum_{m=1}^{\infty} (\mathcal{H}_m f_{n-m} - \overline{\mathcal{H}_m} f_{n+m})(z) = \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)dt}{t^{n+1}} \equiv \varphi_n(z), \qquad n \geq 0.}$$

Because $\frac{\partial}{\partial \bar{z}}\varphi_n + a\varphi_{n+1} + \frac{\partial}{\partial z}\varphi_{n+2} = 0$, there are actually only two independent constraints for $n=0$ and $n=1$. This *characterizes the redundancy* of order 2 of the AtRT.

Step 4: reconstruction in simplified setting.

Assume that $N = 1$ so that $f(\mathbf{x}, \lambda) = f_0(\mathbf{x}) + \lambda f_1(\mathbf{x}) + \lambda^{-1} f_{-1}(\mathbf{x})$. Then

$$\begin{aligned}\mathcal{H}_1 f_{-1}(z) - \overline{\mathcal{H}_1} f_1(z) &= \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)dt}{t} = \varphi_0(z)\\ \mathcal{H}_2 f_{-1}(z) + \mathcal{H}_1 f_0(z) &= \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)dt}{t^2} = \varphi_1(z).\end{aligned}$$

Define $\boldsymbol{\omega} = (\cos\omega, \sin\omega) \in S^1$ and impose for $\rho_1(z)$ real-valued:

$$f_1(z) = e^{i\omega}\rho_1(z), \quad f_{-1}(z) = e^{-i\omega}\rho_1(z),$$
so that $f_1(z)e^{i\theta} + f_{-1}(z)e^{-i\theta} = 2\cos(\theta+\omega)\rho_1(z)$.

Since $\mathcal{H}_1$ is multiplication by $2/(i\xi_z)$ in the Fourier domain, we obtain

$$\begin{aligned} f_1(\mathbf{x}) &= \frac{1}{4} D_{\boldsymbol{\omega}_s} \Delta(i\varphi_0)(\mathbf{x}), \qquad \boldsymbol{\omega}_s = (\sin\omega, \cos\omega),\\ f_0(\mathbf{x}) &= \frac{1}{4\pi}\int_0^{2\pi} \boldsymbol{\theta}^\perp \cdot \nabla(i\varphi)(\mathbf{x},\theta)d\theta + \frac{1}{2} D_{\boldsymbol{\omega}_s} \boldsymbol{\omega}_s^\perp \cdot \nabla(i\varphi_0)(\mathbf{x}).\end{aligned}$$

When $\varphi_0 \equiv 0$ this is the classical Novikov formula.

Step 4 bis: Application to Doppler tomography.

In Doppler tomography, the source term of interest is of the form

$$f(\mathbf{x},\theta) = \mathbf{F}(\mathbf{x})\cdot\boldsymbol{\theta} \qquad \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x})).$$

So we define the source term $f_1(\mathbf{x}) = \frac{1}{2}(F_1(\mathbf{x}) - iF_2(\mathbf{x}))$ and $f_k(\mathbf{x}) \equiv 0$ for $|k| \neq 1$. The constraint $n = 0$ gives

$$\nabla \times \mathbf{F}(\mathbf{x}) = \frac{\partial F_2(\mathbf{x})}{\partial x} - \frac{\partial F_1(\mathbf{x})}{\partial y} = \frac{1}{2}\Delta(i\varphi_0)(\mathbf{x}).$$

The constraint $n = 1$ gives $\mathcal{H}_2 f_{-1}(z) = \varphi_1(z)$ so that

$$\frac{1}{2}\Big(F_1(z) + iF_2(z)\Big) = -\frac{\partial}{\partial\bar{z}}\frac{1}{a(z)}\frac{\partial\varphi_1(z)}{\partial\bar{z}}.$$

This *explicit* reconstruction formula is valid on the support of $a(\mathbf{x})$ and has *no equivalent* when $a \equiv 0$.

Redundancy and compatibility conditions.

When $\varphi_0(\mathbf{x}) \equiv 0$ (compatibility condition), we obtain $f_1(\mathbf{x}) = 0$ and the data can be obtained as the AtRT of a source term $f(\mathbf{x}, \theta) = f(\mathbf{x})$.

In general, we can reconstruct two functions from the AtRT measurements; say $f_0(\mathbf{x})$ and $f_1(\mathbf{x})$ at $\boldsymbol{\omega}$ fixed.

The reconstruction is optimal in the following sense. Consider some data $g(s, \theta)$ and reconstruct f_0 and f_1 as above with $\rho_1(\mathbf{x}) = |f_1(\mathbf{x})|$. Then the AtRT of $f(\mathbf{x}, \theta) = f_0(\mathbf{x}) + 2\cos(\theta + \omega)\rho_1(\mathbf{x})$ is equal to the measured data $g(s, \theta)$ (this relies on the uniqueness to the RH problem).

Part II: Reconstruction from partial measurements

Since we can reconstruct two functions from the AtRT, can we reconstruct one from half of the measurements? The answer is yes under a smallness constraint on the variations of the absorption parameter.

The setting is as follows. We assume that $g(s,\theta)$ is available for all values of $s \in \mathbb{R}$ and for $\theta \in M \subset [0, 2\pi)$. The assumption on M is that $M^c = [0, 2\pi)\backslash M \subset \overline{M + \pi}$; for instance $M = [0, \pi)$ and $M^c = [\pi, 2\pi)$.

We also assume that the source term $f(\mathbf{x})$ is compactly supported in the unit ball B.

The derivation is based on decomposing the Novikov reconstruction formula into skew-symmetric and symmetric components in $\mathcal{L}(L^2(B))$.

Decomposition of the identity operator

Let us define $\dfrac{i\varphi(\mathbf{x},\theta)}{2} = R^*_{-a,\theta} H_a R_{a,\theta} f(\mathbf{x}) \equiv \Phi_{a,\theta} f(\mathbf{x})$ and the operators

$$\begin{aligned} F_\theta &= \boldsymbol{\theta}^\perp \cdot \nabla \Phi_{a,\theta} = F_{1,\theta} + F_{2,\theta} \\ F_{1,\theta} &= R^*_{-a,\theta} \frac{\partial}{\partial s} H_a R_{a,\theta}, \qquad\qquad R^*_{a,\theta} g(\mathbf{x}) = e^{D_\theta a(\mathbf{x})} g(\mathbf{x} \cdot \boldsymbol{\theta}^\perp) \\ F_{2,\theta} &= \Big(\boldsymbol{\theta}^\perp \cdot \nabla R^*_{-a,\theta} - R^*_{-a,\theta} \frac{\partial}{\partial s} \Big) H_a R_{a,\theta}. \end{aligned}$$

The Novikov formula shows formally that

$$2\pi I = \int_0^{2\pi} F_\theta d\theta,$$

which we recast as

$$2\pi I = \int_M F_\theta d\theta + \int_{M^c} F^*_{1,\theta} d\theta + \int_{M^c} (F_{1,\theta} - F^*_{1,\theta}) d\theta + \int_{M^c} F_{2,\theta} d\theta.$$

Decomposition of the identity operator (ii)

The main interest of the decomposition is that

$$F_{1,\theta}^* = R_{a,\theta}^* H_a \frac{\partial}{\partial s} R_{-a,\theta}$$

so that $F_{1,\theta}^*$ on M^c involves

$$R_{-a,\theta} f(s) = R_{a,\theta+\pi} f(-s), \qquad \text{because} \qquad D_{\theta+\pi}(-a)(\mathbf{x}) = D_\theta a(\mathbf{x}),$$

where now $\theta + \pi \in M$ by construction. Thus $F_{1,\theta}^*$ on M^c depends on the measured data. Defining $F_{2,\theta}^s = \frac{1}{2}(F_{2,\theta} + F_{2,\theta}^*)$ and $F_{2,\theta}^a = F_{2,\theta} - F_{2,\theta}^s$, we obtain

$$\begin{aligned} I &= F^d + F^a + F^s, \qquad F^d = \frac{1}{2\pi}\int_M F_\theta d\theta + \frac{1}{2\pi}\int_{M^c} F_{1,\theta}^* d\theta \\ F^a &= \frac{1}{2\pi}\int_{M^c} (F_{1,\theta} - F_{1,\theta}^* + F_{2,\theta}^a) d\theta, \qquad F^s = \frac{1}{2\pi}\int_{M^c} F_{2,\theta}^s d\theta. \end{aligned}$$

Reconstruction from partial measurements

The preceding decomposition allows us to recast the reconstruction problem as

$$f(\mathbf{x}) = d(\mathbf{x}) + F^a f(\mathbf{x}) + F^s f(\mathbf{x}), \qquad d(\mathbf{x}) = F^d f(\mathbf{x}),$$

where F^a is formally skew-symmetric and F^s is formally symmetric.

Theorem 1. The operators F^a and F^s are bounded in $\mathcal{L}(L^2(B))$ and F^s is compact in the same sense with range in $H^{1/2}(B)$.

Theorem 2. Provided that $\rho(F^s) < 1$, we can reconstruct $f(\mathbf{x})$ uniquely from $g(s, \theta)$ for $\theta \in M$. Since F^s is compact we can always reconstruct the singular part of $f(\mathbf{x})$ that is not in the Range of F^s.

Explicit Iterative Reconstruction

The reconstruction is obtained as follows: We have

$$\begin{aligned} f(\mathbf{x}) &= (I - F^s)^{-1/2} h(\mathbf{x}) \\ h(\mathbf{x}) &= (I - F^s)^{-1/2} d(\mathbf{x}) + (I - F^s)^{-1/2} F^a (I - F^s)^{-1/2} h(\mathbf{x}). \end{aligned}$$

Defining the skew-symmetric operator $G^a = (I - F^s)^{-1/2} F^a (I - F^s)^{-1/2}$ and $\gamma = (1 + \|G^a\|_2^2)^{-1}$, we observe that the iterative scheme

$$h^{k+1}(\mathbf{x}) = \gamma (I - F^s)^{-1/2} d(\mathbf{x}) + ((1-\gamma) I + \gamma G^a) h^k(\mathbf{x})$$

converges to $h(\mathbf{x})$ in $L^2(B)$ as $\|(1-\gamma)I + \gamma G^a\|_2 = \dfrac{\|G^a\|_2}{(1 + \|G^a\|_2^2)^{1/2}} < 1$.

The uniquely defined solution of

$$f^a(\mathbf{x}) = d(\mathbf{x}) + F^a f^a(\mathbf{x})$$

is such that $f(\mathbf{x}) - f^a(\mathbf{x}) \in \text{Range}(F^s)$.

Sketch of proof of Theorem 1

We need to consider terms of the form $h(\mathbf{x}) = \frac{1}{2\pi}\int_\alpha^\beta \boldsymbol{\theta}^\perp \cdot \nabla \Phi_{a,\theta} f(\mathbf{x}) d\theta$. For $a \equiv 0$ we use the Fourier slice theorem to show that

$$\widehat{h}(\boldsymbol{\xi}) = \frac{1}{2}\Big(\chi_{(\alpha,\beta)}(\xi_B) + \chi_{(\alpha,\beta)}(\xi_F)\Big)\widehat{f}(\boldsymbol{\xi}).$$

for some angles ξ_B and ξ_F. So $\|h\|_2 \le \|f\|_2$.

In the general case we have terms of the form

$$h(\mathbf{x}) = \frac{1}{2\pi}\int_\alpha^\beta \boldsymbol{\theta}^\perp \cdot \nabla \Big(u(\mathbf{x},\theta) H[v(s,\theta) P_\theta(w(\mathbf{x},\theta))(s)](\mathbf{x}\cdot\boldsymbol{\theta}^\perp)\Big) d\theta,$$

with u and v smooth [and $\equiv 1$ when $a \equiv 0$] and $w(\mathbf{x},\theta) = e^{D_\theta a(\mathbf{x})} f(\mathbf{x})$. The term involving $\Big(\boldsymbol{\theta}^\perp \cdot \nabla u(\mathbf{x},\theta)\Big)$ yields a compact contribution whereas application of the Fourier slice theorem shows that the term involving $u(\mathbf{x},\theta)\boldsymbol{\theta}^\perp \cdot \nabla$ yields a bounded contribution in $\mathcal{L}(L^2(B))$.

Case of constant absorption (ERT)

When $a(\mathbf{x})$ is constant and equal to μ on the unit disk and vanishes elsewhere, we verify that

$$\boldsymbol{\theta}^{\perp}\cdot\nabla(e^{D_\theta a(\mathbf{x})}g(\mathbf{x}\cdot\boldsymbol{\theta}^{\perp},\theta)) = e^{D_\theta a(\mathbf{x})}\Big(\frac{\partial g(s,\theta)}{\partial s}\Big)(\mathbf{x}\cdot\boldsymbol{\theta}^{\perp}),$$

so that $F_{2,\theta} \equiv 0$.

We thus recover a result by Noo and Wagner (IP 2001) that $f(\mathbf{x})$ can uniquely be reconstructed. Furthermore we have that

$$I = \frac{2}{2\pi}\int_M F_\theta d\theta + \frac{1}{2\pi}\int_{M+\pi}(F_\theta - F_{\theta+\pi})d\theta = F^d + F^a,$$

where $d(\mathbf{x}) = F^d f(\mathbf{x})$ is the measured data and F^a is skew-symmetric.

Part III: Fast numerical reconstruction using the slant stack algorithm

Joint work with Philippe Moireau, Ecole Polytechnique.

Let us represent $f(\mathbf{x})$ by an image with $n \times n$ pixels. The objectives are:

- to compute an accurate approximation of $g(s,\theta) = R_{a,\theta} f(s)$
- to compute it fast (with a cost of $O(n^2 \log n)$)
- to invert the AtRT accurately and fast from full or partial measurements.

Slant stack algorithm for the Radon transform

Follows presentation in recent papers by *Averbuch, Coifman, Donoho, Israeli, and Waldén.*

Let us define $\Theta_1 = [\frac{-\pi}{4}, \frac{\pi}{4})$, $\Theta_2 = [\frac{\pi}{4}, \frac{3\pi}{4})$, $\Theta_3 = [\frac{3\pi}{4}, \frac{5\pi}{4})$, $\Theta_4 = [\frac{5\pi}{4}, \frac{7\pi}{4})$, *and the slant stack transform*

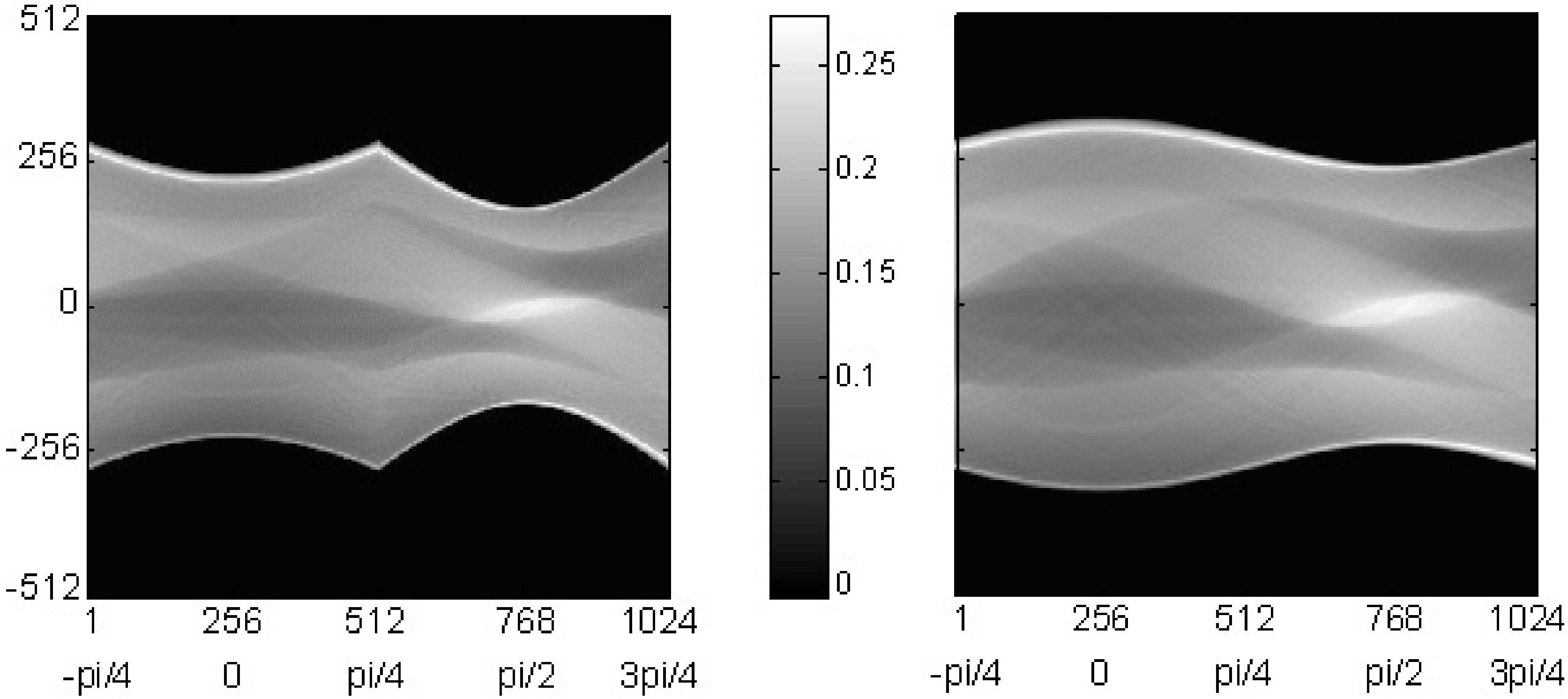
$$Sf(t,\theta) = \begin{cases} S_1 f(t,\theta) = \int_{\mathbb{R}} f(x, x\tan\theta + t)\frac{dx}{\cos\theta}, & \theta \in \Theta_1 \\ S_2 f(t,\theta) = \int_{\mathbb{R}} f(y\cot\theta - t, y)\frac{dy}{\sin\theta}, & \theta \in \Theta_2. \end{cases}$$

We have the reconstruction formula

$$f(\mathbf{x}) = \frac{1}{2\pi}\Big(S_1^*(\frac{\partial}{\partial y}HSf)(\mathbf{x}) + S_2^*(\frac{\partial}{\partial x}HSf)(\mathbf{x})\Big).$$

Differentiations in x and y are Cartesian-friendly. The operators S_k, S_k^*, and H are local in the Fourier domain.

Comparison of Slant Stack and Radon Transform



$Sf(\theta; t)$ [Lineogram] *versus* $Rf(\theta; t)$ [Sinogram].

Discrete slant stack

Set $\theta \in \Theta_1$ and $m = 2n$. Let F be a $n \times n$ image. Define $F^1 = E^1 F$, where E^1 zero pads the image into a $n \times m$ image. Set $\mathcal{T}_n = \{-\frac{n}{2}, -\frac{n}{2} + 1, \cdots, \frac{n}{2} - 1\}$ and define the interpolation

$$F_u^1(y) = \sum_{v \in \mathcal{T}_m} F_{u,v}^1 D_m(y - v), \quad D_m(t) = \frac{\sin m\pi t}{m \sin t} \text{ (Dirichlet kernel).}$$

Define the semi-discrete slant stack transform as

$$S_n F_t(\theta) = \frac{1}{\cos\theta} \frac{1}{n} \sum_{u \in \mathcal{T}_n} F_u^1(u \tan\theta + t), \quad \theta \in \Theta_1, t \in \mathcal{T}_m.$$

Choose the directions of integration such that

$$\Theta_1^n = \{\theta_l = \arctan\frac{2l}{n}, \quad l \in \mathcal{T}_n\},$$

and define the discrete slant stack transform as $S_n F_{t,l} = S_n F_t(\theta_l)$.

Fast calculation

For the specific choice of angles Θ_1^n, we have

$$S_n F_{t,l} = \sum_{k \in \mathcal{T}_m} e^{i\frac{2\pi}{m}(k+\frac{1}{2})t} \widehat{S_n F}_{k,l}, \qquad t \in \mathcal{T}_m, \quad l \in \mathcal{T}_n,$$

where

$$\widehat{S_n F}_{k,l} = \sqrt{1 + \left(\frac{2l}{n}\right)^2} \hat{F}^1\Big(-\frac{2\pi}{m}(k+\frac{1}{2})\frac{2l}{n}, \frac{2\pi}{m}(k+\frac{1}{2})\Big).$$

Define

$$\tilde{F}_u^1\Big(\frac{2\pi}{m}(k+\frac{1}{2})\Big) = \frac{1}{m} \sum_{v \in \mathcal{T}_m} e^{-i\frac{2\pi}{m}(k+\frac{1}{2})} F_{u,v}^1.$$

Then with $(\mathcal{F}_\alpha V)_l = \sum_{u \in \mathcal{T}_n} V_u e^{-i\frac{2\pi}{n}\alpha l u}$ the fractional FT,

$$\Big(\mathcal{F}_{\frac{-2(k+1/2)}{m}} \tilde{F}_u^1\Big(\frac{2\pi}{m}(k+\frac{1}{2})\Big)\Big)_l = \hat{F}^1\Big(-\frac{2\pi}{m}(k+\frac{1}{2})\frac{2l}{n}, \frac{2\pi}{m}(k+\frac{1}{2})\Big).$$

Implementation of the algorithm

1. We zero-pad the $n \times n$ image F to obtain the $n \times 2n$ image F^1,

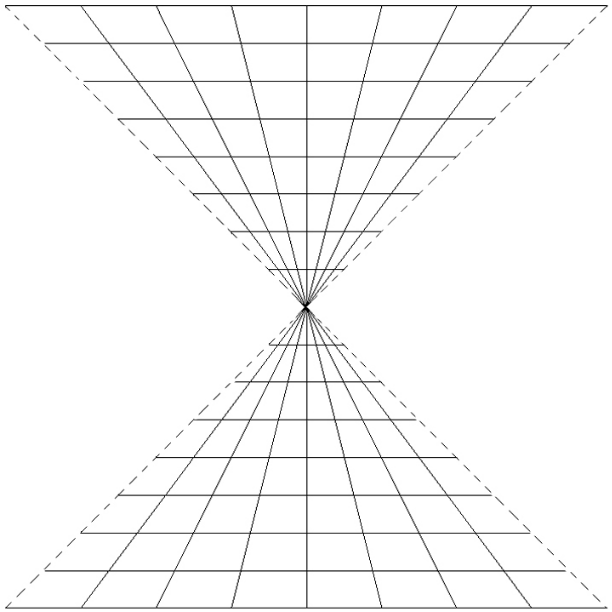
2. We compute a Discrete Fourier Transform (DFT) on the columns,

3. We compute a fractional DFT on the rows,

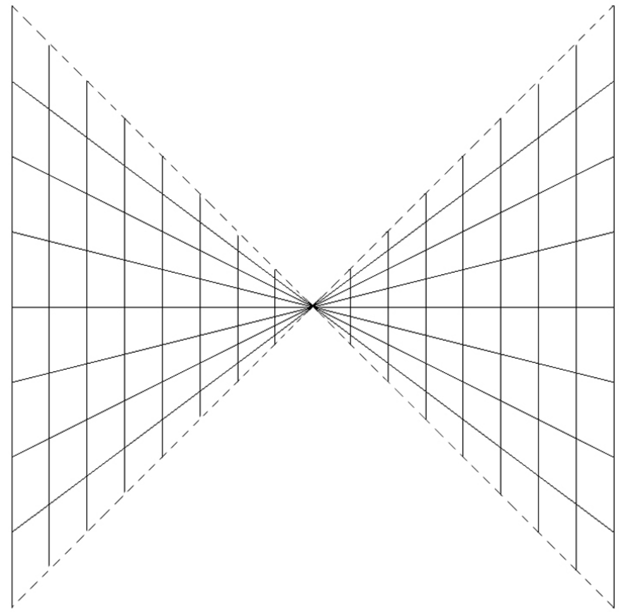
4. We compute an inverse DFT (IDFT) on the columns.

Each of these operations can be performed in $O(n^2 \log n)$ operations. Moreover the discrete transform converges to the exact transform with spectral accuracy.

Discrete Fourier slice theorem



D_1^n



D_2^n

The discrete FT of the discrete slant stack involves the Fourier transform of the image at the above discrete points. Left: angles $\theta \in \Theta_1^n$. Right: angles $\theta \in \Theta_2^n$.

Adjoint transform, inversion, accuracy

Let S_n be the fast slant stack operator. The discretization of the Riesz operator I_n^{-1} is local in Fourier and the "back-projection" operator S_n^* can also be estimated in $O(n^2 \log n)$ operations. The exact reconstruction $S^* I^{-1} S = Id$ is now replaced by

$$Id_n \approx G_n = S_n^* I_n^{-1} S_n.$$

The matrix G_n is symmetric. Moreover its eigenvalues are all positive (this was proved for many small values of n). So we can write

$$Id_n = G_n^{-1} G_n = [G_n^{-1} S_n^* I_n^{-1}] S_n, \quad \text{i.e. ,} \quad S_n^{-1} = G_n^{-1} S_n^* I_n^{-1}.$$

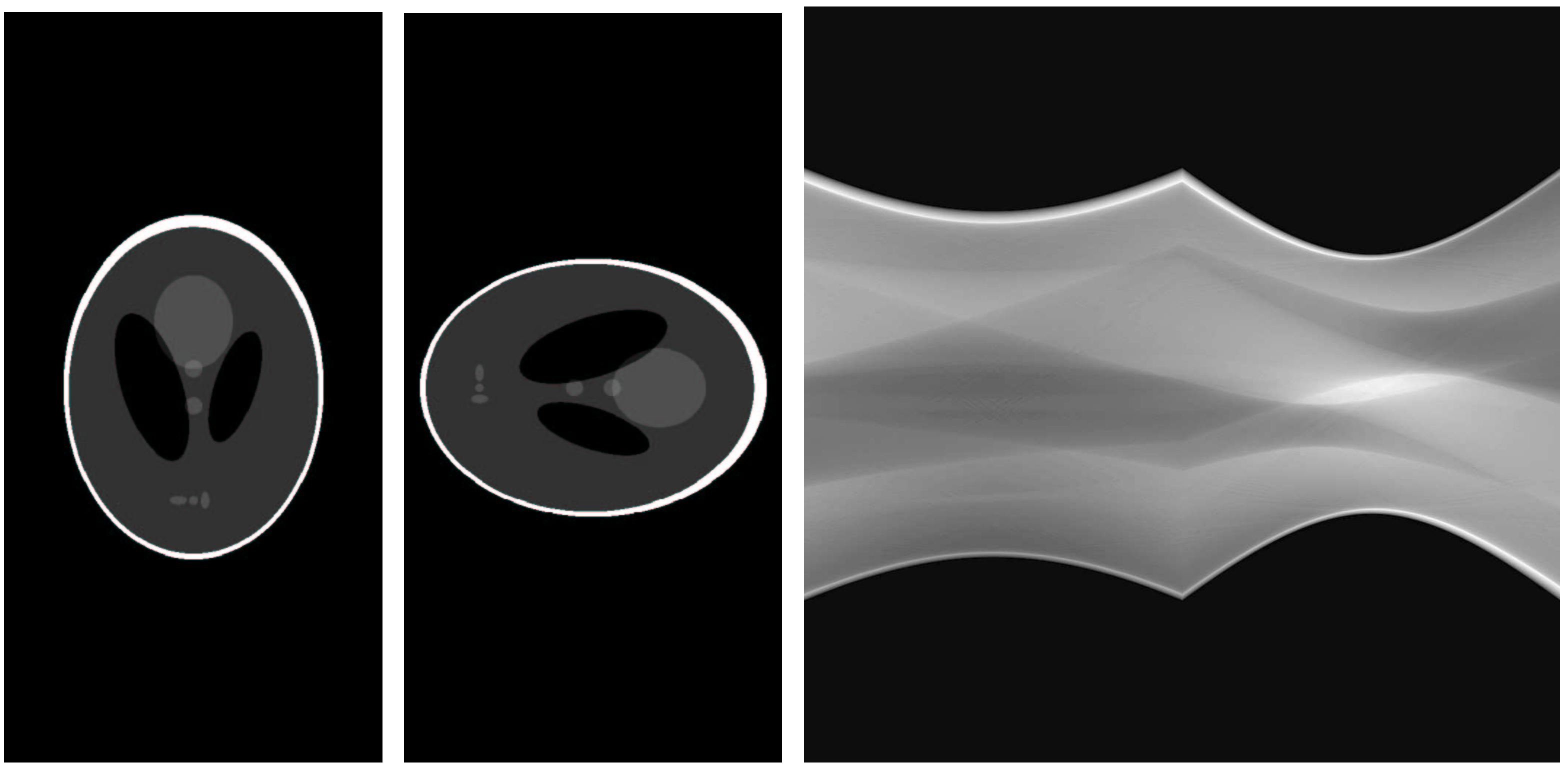
The discrete transform can be inverted exactly, for instance iteratively by Conjugate Gradient (CG).

Spectral properties of G_n and CG iterations

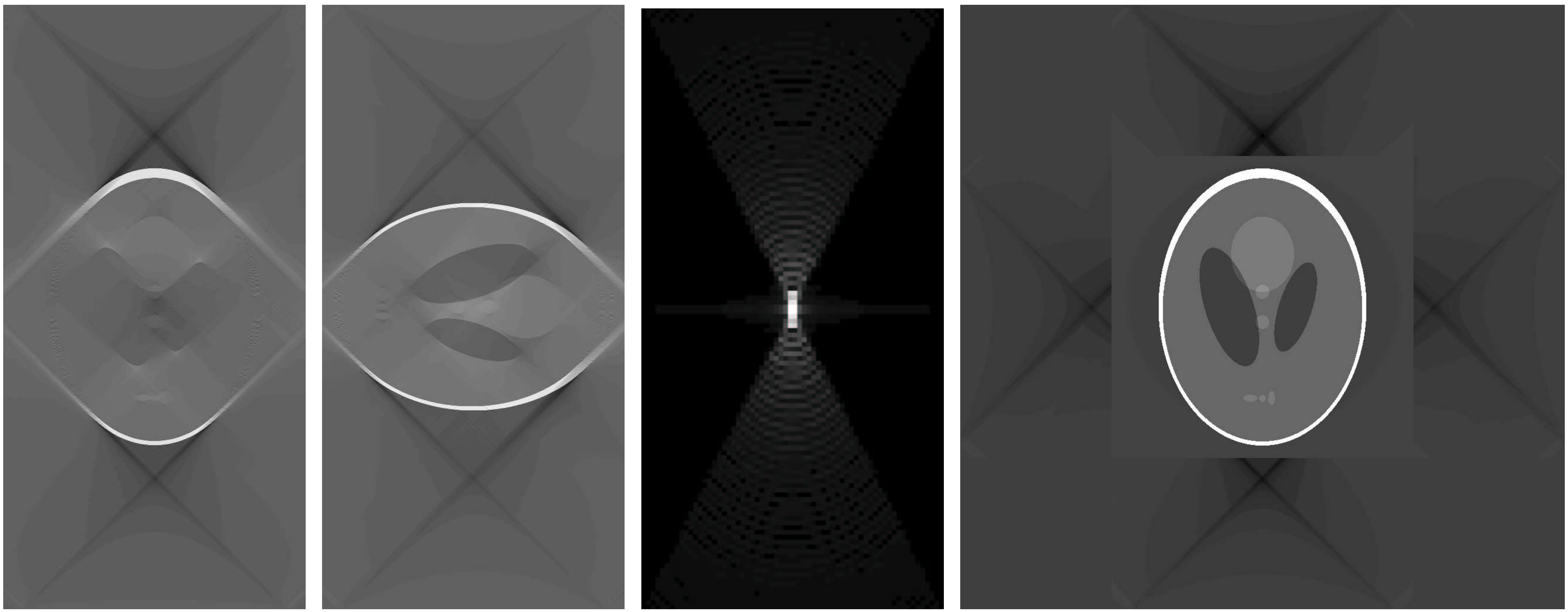
Case (n, ZP, CG)	First	Second	Third	$n-1$	Last
16,0,0	0.88686	0.88686	0.97878	1.0897	1.3531
32,0,0	0.82501	0.82501	0.97795	1.0984	1.4539
32,1,0	0.99566	0.99566	0.99755	1.0204	1.0615
32,0,4	0.99993	$\cdots$	$\cdots$	$\cdots$	1.0001
64,0,0	0.7599	0.7599	0.96266	1.1097	1.534
64,1,0	0.99585	0.99585	0.9969	1.0212	1.0657
64,0,4	0.99983	$\cdots$	$\cdots$	$\cdots$	1.0004
128,0,0	0.69675	0.69675	0.93882	1.1543	1.5977

Spectral data (three smallest and two largest eigenvalues) for different simulations: $n \times n$ number of pixels of image; ZP additional zero padding such that the algorithm zero pads the original image into a $2n \times 2n$ images for $ZP = 1$; CG the number of conjugate gradient iterations to invert G_n.

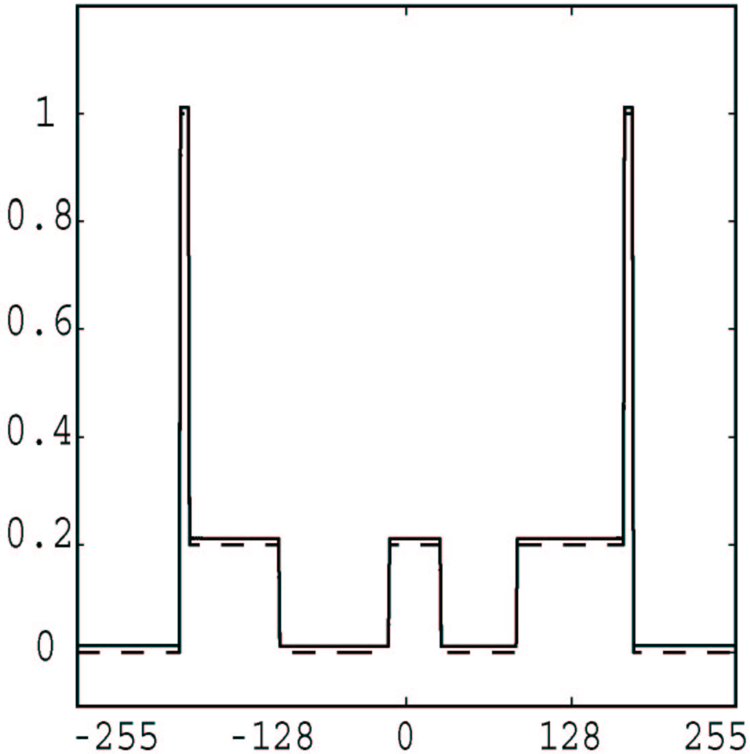
Classical phantom reconstruction



Classical phantom reconstruction (ii)



Classical phantom reconstruction (iii)



Generalization to the AtRT

Recall that the AtRT is given by

$$R_a f(s,\theta) = R[e^{D_\theta a}(\mathbf{x}) f(\mathbf{x})](s,\theta) \equiv R_{a,\theta} f(s).$$

There are two difficulties. (i) We need to compute the coefficients $e^{D_\theta a}(\mathbf{x})$. (ii) We need to compute the Radon transform of a source term $f(\mathbf{x},\theta)$. *None* of these operations can be performed fast. The FFT based on the Fourier slice theorem only works for spatially dependent source terms.

Let us define the Fourier coefficients

$$w_k(\mathbf{x}) = \int_0^{2\pi} e^{D_\theta a(\mathbf{x})} e^{-ik\theta} \frac{d\theta}{2\pi}.$$

We can recast the AtRT as

$$S_a f(t,\theta) = \sum_{k\in\mathbb{Z}} e^{ik\theta} S[w_k(\mathbf{x}) f(\mathbf{x})](t,\theta).$$

Fast AtRT calculation in simplified setting

Let us assume that $e^{D_\theta a(\mathbf{x})}$ can be approximated by $N+1$ predetermined Fourier coefficients. Then each slant stack transform $S[w_k(\mathbf{x})f(\mathbf{x})](s,\theta)$ can be estimated in $O(n^2 \log n)$ calculations. The total complexity of the discrete AtRT

$$S_{aN}f(t,\theta) = \sum_{k=-N/2}^{N/2} e^{ik\theta} S[w_k(\mathbf{x})f(\mathbf{x})](t,\theta),$$

is thus $O(Nn^2 \log n)$.

The calculation of the Fourier coefficients can be performed in $O(n^3 \log n)$ operations using a modification of the slant stack algorithm. The fast algorithm is therefore useful when the AtRT corresponding to *many sources* must be calculated with the same absorption map. This is the case when the γ radiation of isotopes is monitored in time.

Novikov formula and Discrete Reconstruction

The Novikov inversion formula in the slant stack variables reads

$$f(\mathbf{x}) = \frac{1}{4\pi}\Big(\frac{\partial}{\partial y}\int_{\Theta_1\cup\Theta_3} S^*_{-a,\theta} H_a g(\mathbf{x}) d\theta + \frac{\partial}{\partial x}\int_{\Theta_2\cup\Theta_4} S^*_{-a,\theta} H_a g(\mathbf{x}) d\theta\Big).$$

The operator H_a involves multiplications (local in the spatial domain) and the Hilbert transform (local in the Fourier domain). The adjoint operators $S^*_{-a,\theta}$ can also be estimated in $O(Nn^2 \log n)$ operations provided that the Fourier coefficients of $e^{-D_\theta a(\mathbf{x})}$ are precalculated.

Thanks to the Novikov formula, we thus have a fast algorithm to calculate and invert the AtRT *in the case where* the Fourier coefficients of the cone beam transform of $a(\mathbf{x})$ are known.

Accuracy of the method

The Novikov formula $I = S_a^* H_a S_a$ is approximated by

$$Id_n \sim G_{na} = (S^* H)_{na} S_{na}.$$

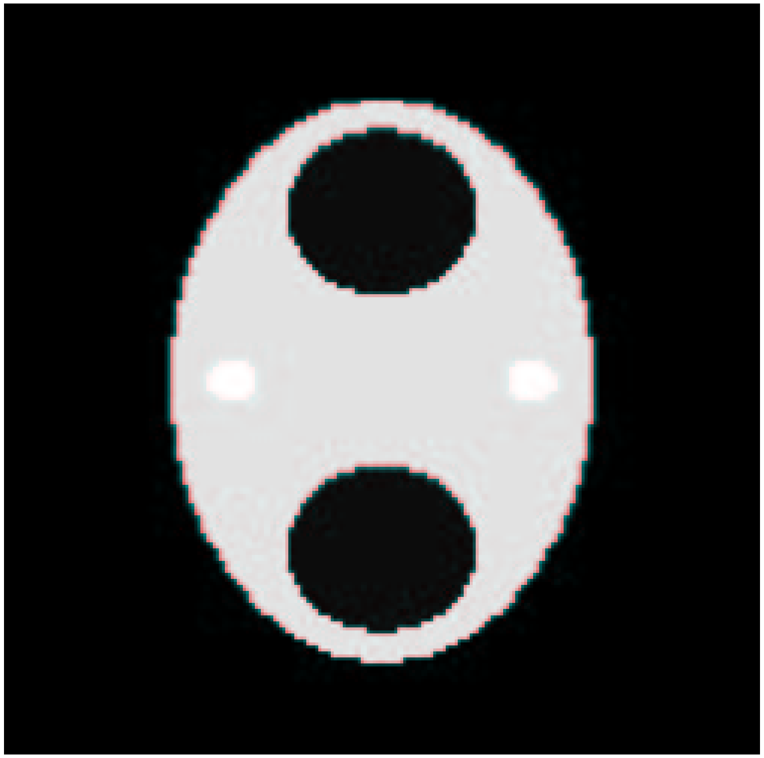
The operator G_{na} need *no longer* be symmetric. To obtain a better approximation of identity we thus consider

$$Id_n \sim (G_{na}^* G_{na})^{-1} G_{na}^* G_{na}.$$

There is a difficulty here: $(G_{na}^* G_{na})^{-1}$ is well defined and bounded when $a \equiv 0$. This is *no longer* the case for $a \neq 0$.

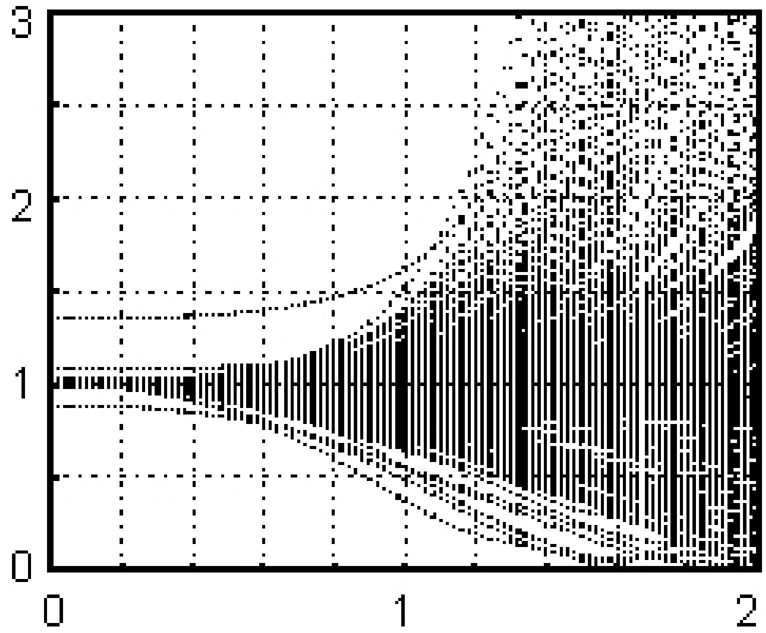
An example of spectral analysis of G_{na}

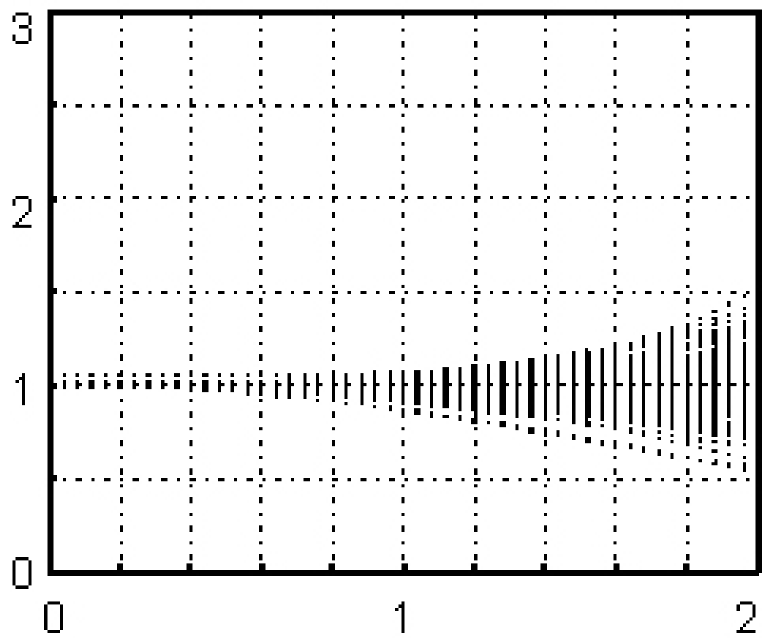
Consider the absorption maps $a(\mathbf{x}; L) = La(\mathbf{x})$ with $a(\mathbf{x})$ given by [$a(\mathbf{x}) = 6.5; 6; 0$ on the white;grey;black parts], and L a multiplicative constant.



An example of spectral analysis of G_{na} (ii)

Singular values of G_{na} as a function of L *without* (left) and *with* (right) zero-padding.





Spectral analysis of G_{na} (iii)

We thus observe that for sufficiently small values of absorption, the discrete AtRT method will provide good reconstructions and $G_{na}^* G_{na}$ is invertible. Conjugate gradient iterations can be used to obtain reconstructions that are as accurate as one wishes.

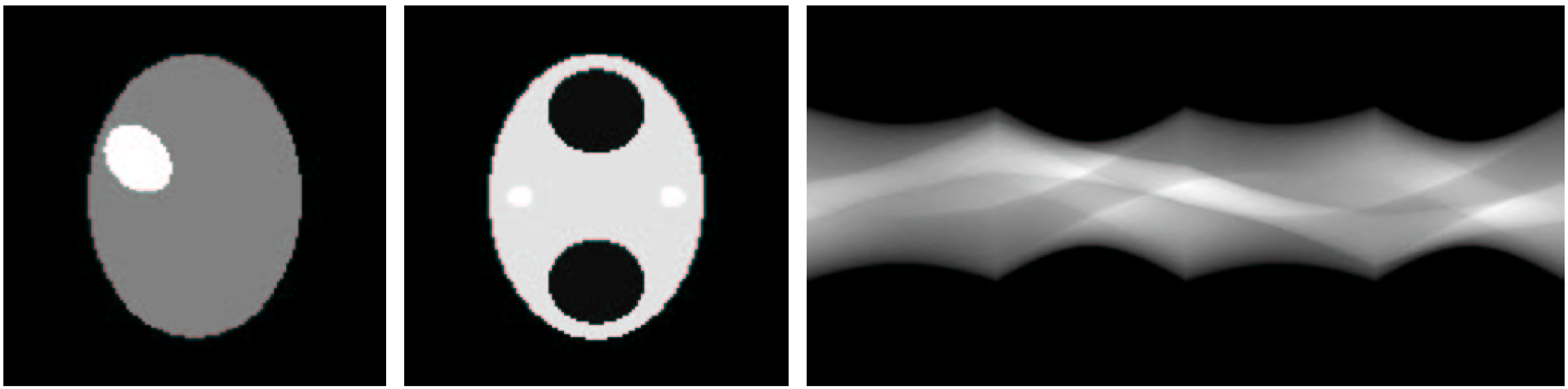
However for larger values of absorption (and how large depends on the image size n), some singular values of $G_{na}^* G_{na}$ become arbitrarily small. Although we do not have any theoretical proof for this, the solution is then to zero-pad the original image into a bigger image, for instance $2n \times 2n$. The spectrum of the AtRT reconstruction G_{na} after zero-padding is then again very close to identity.

Examples of spectral data

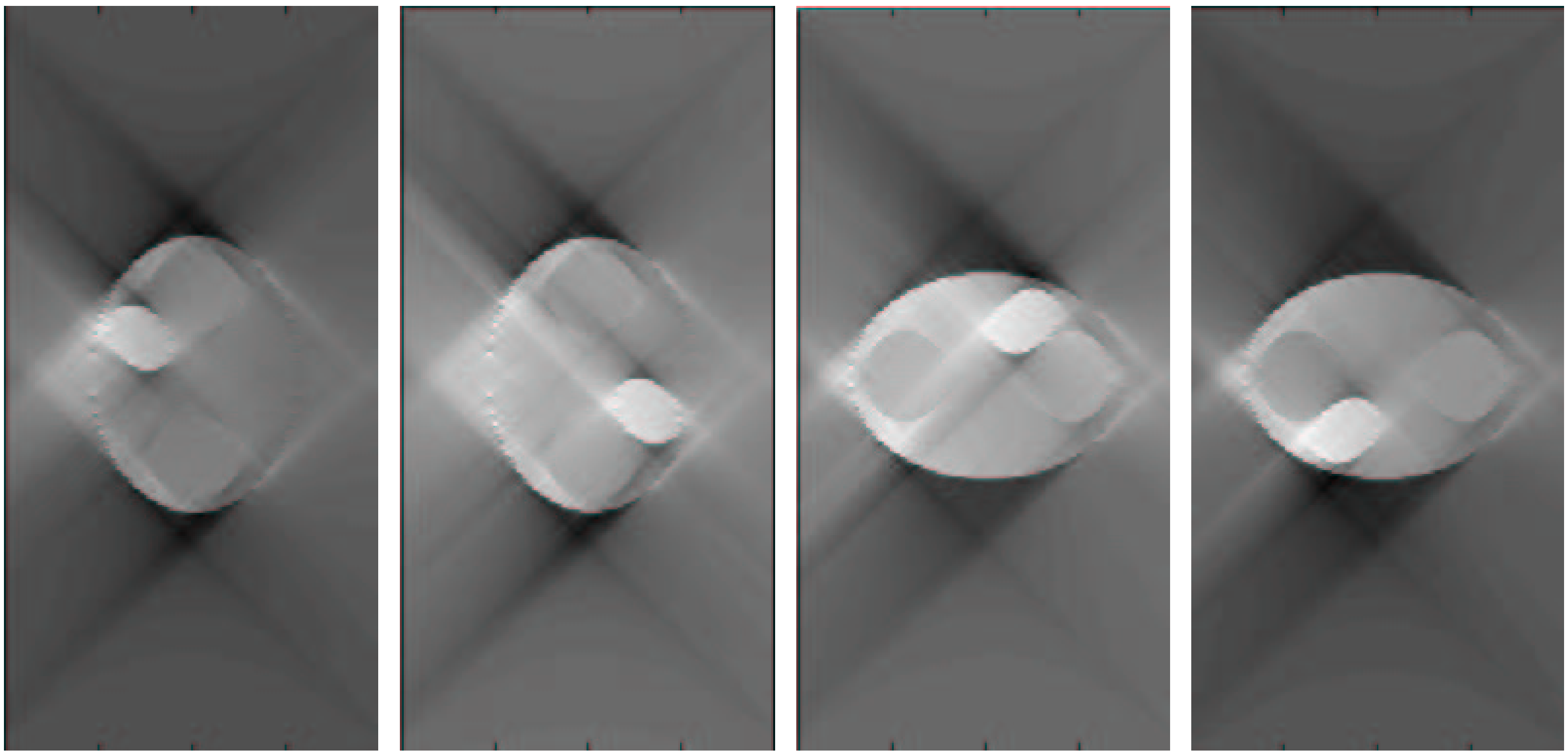
Case (L, n, ZP, CG)	First	Second	Third	$n-1$	Last
0.5,16,0,0	0.8217	0.8240	0.8540	1.0927	1.3806
0.5,16,0,4	0.9997	0.9998	0.9998	1.0001	1.0001
0.5,16,1,0	0.9671	0.9675	0.9770	1.0225	1.0913
1,16,0,0	0.3747	0.4922	0.5514	1.5186	1.6228
1,16,0,4	0.7209	0.8743	0.9142	1.0146	1.0156
1,16,1,0	0.8678	0.8717	0.9090	1.0633	1.0678
1,32,0,0	0.3548	0.3972	0.5340	1.5317	1.9680
1,32,0,4	0.6507	0.7086	0.8877	1.0162	1.0169
1,32,1,0	0.8671	0.8891	0.8970	1.0701	1.0816

Spectral data (three smallest and two largest eigenvalues) for different simulations: λ: multiplicative factor of absorption; $n \times n$ number of pixels of image; ZP additional zero padding such that the algorithm zero pads the original image into a $2n \times 2n$ images for $ZP = 1$; CG the number of conjugate gradient iterations to invert G_{an}.

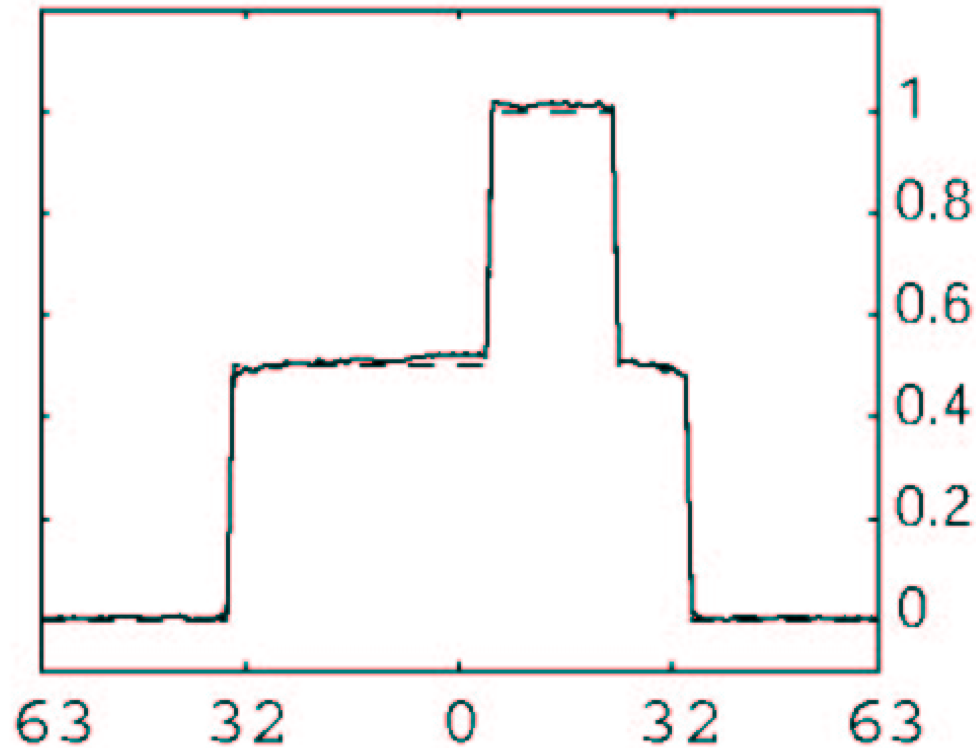
Example of reconstruction with 4 CG iterations (i)



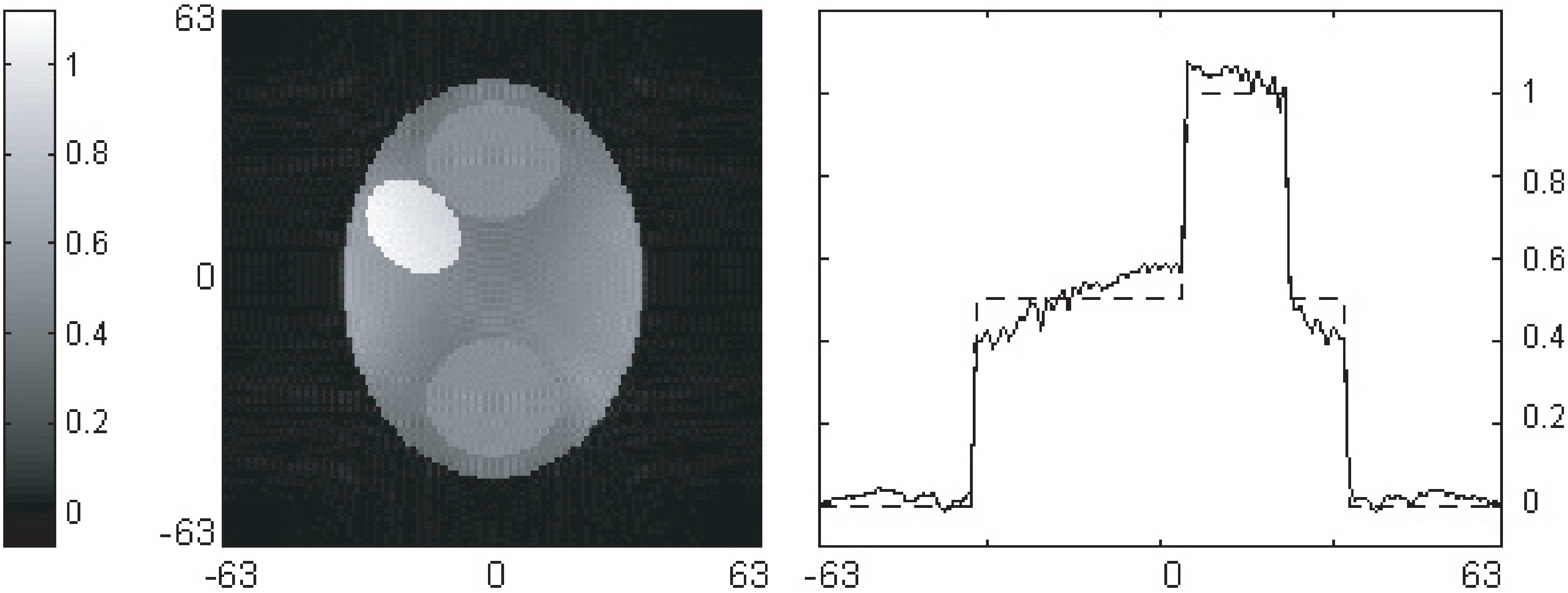
Example of reconstruction with 4 CG iterations (ii)



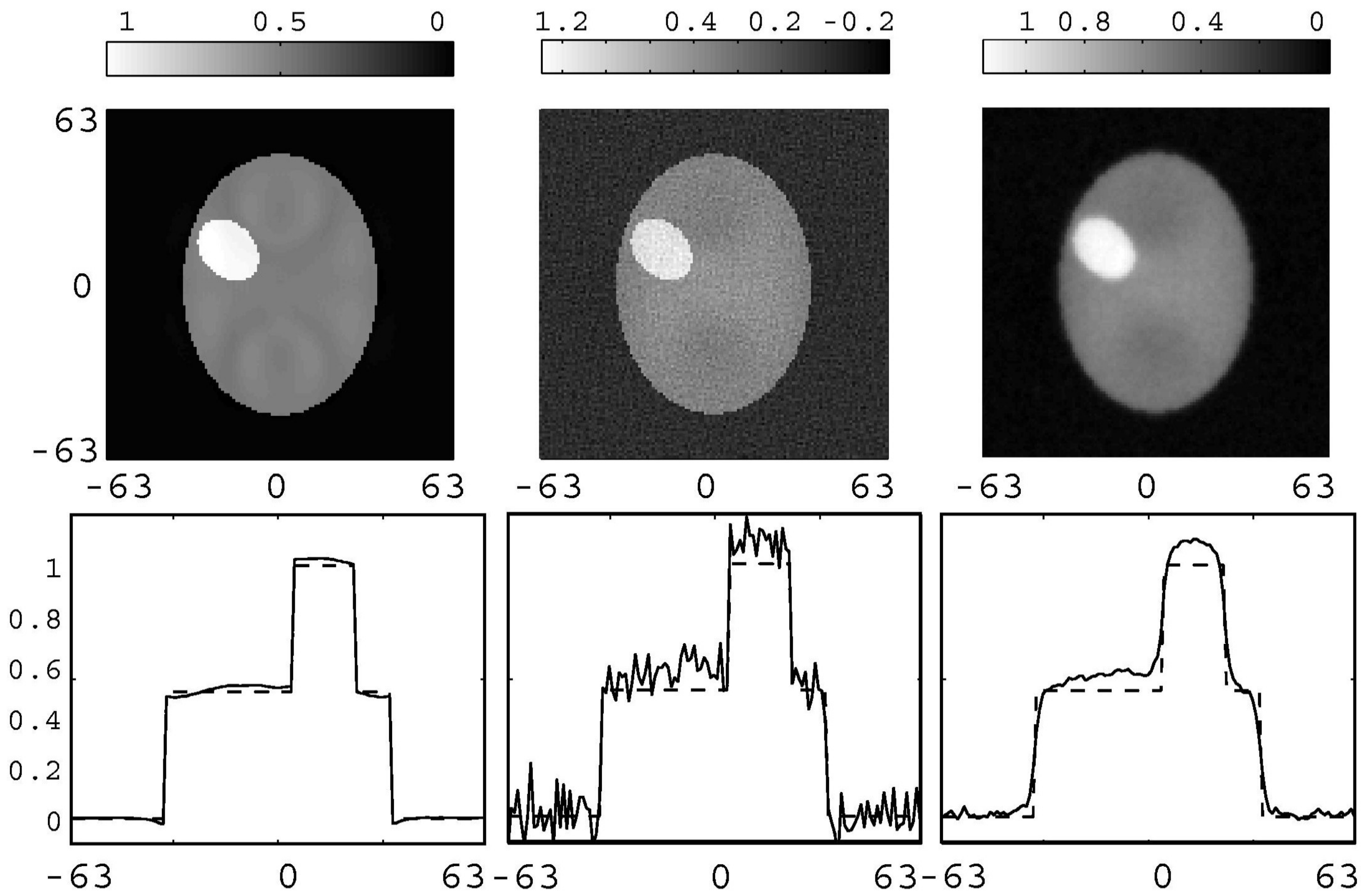
Example of reconstruction with 4 CG iterations (iii)



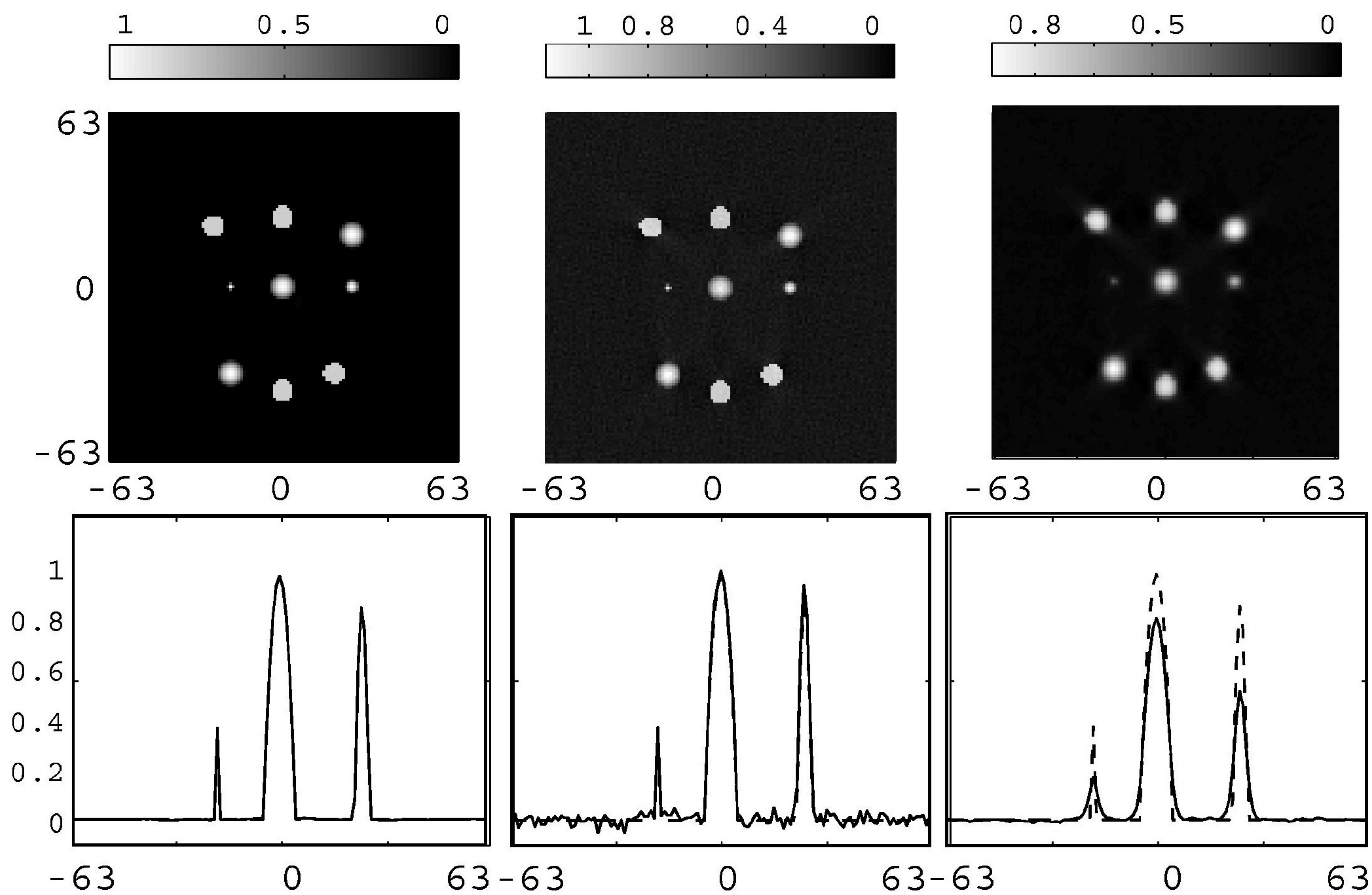
Example of reconstruction with 0 CG iterations



More reconstructions (CG=4)



More reconstructions (still CG=4)



Reconstruction from partial measurements

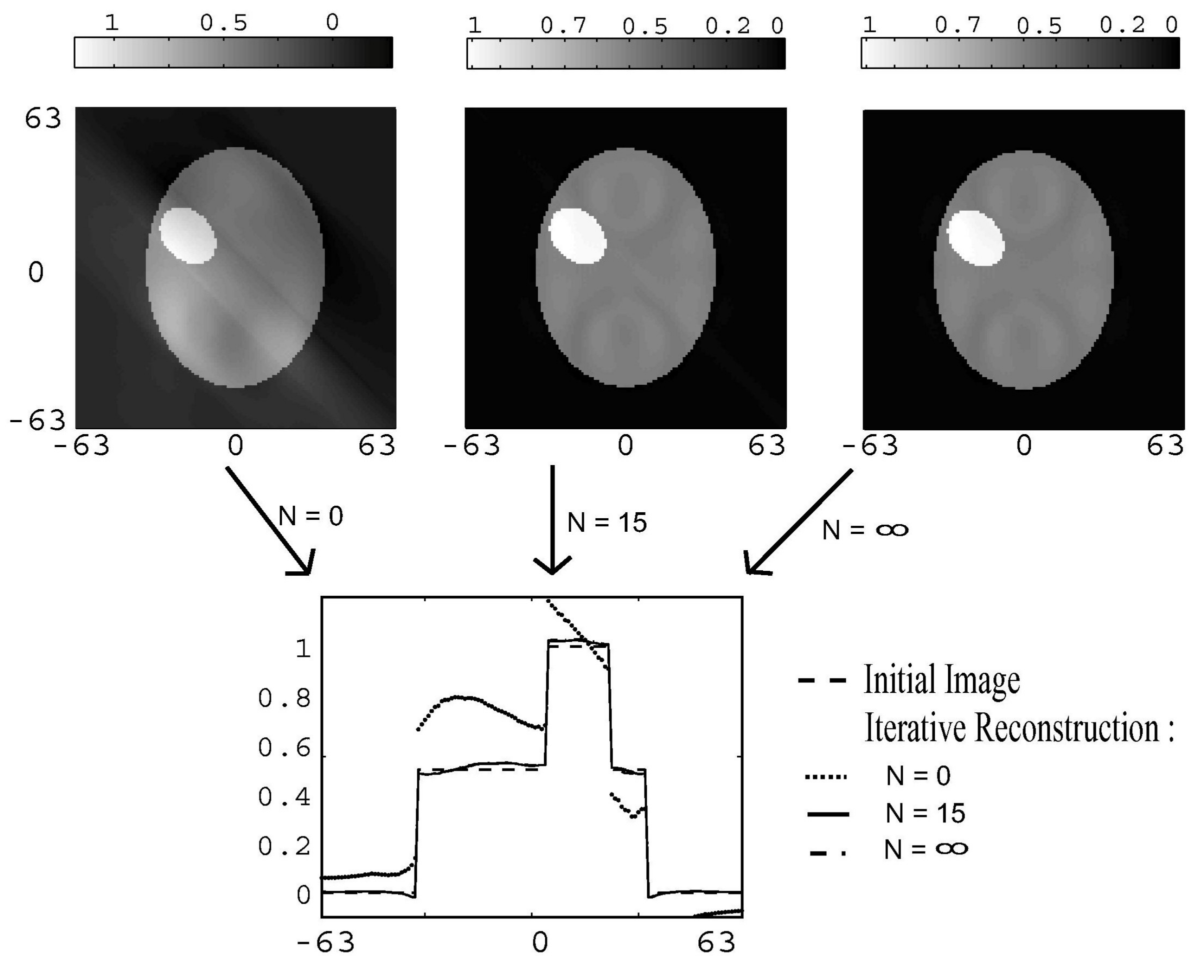
Let us assume the measurements are made on $\Theta_1 \cup \Theta_2$ only. The discretization of

$$f(\mathbf{x}) = d(\mathbf{x}) + F^a f(\mathbf{x}) + F^s f(\mathbf{x}), \qquad d(\mathbf{x}) = F^d f(\mathbf{x}),$$

is performed as before and can be calculated in $O(Nn^2 \log n)$ operations provided that the Fourier coefficients of $e^{D_\theta a(\mathbf{x})}$ are *precalculated*.

In the cases taken from the literature, we have always observed that $\|F^a + F^s\|_2 < 1$. However, once discretized, the latter operator may not remain of norm less than 1. In the reconstruction, it is important that the spectral radius of the discrete iterative procedure be close to the spectral radius of the continuous iterative procedure. We can then rely on accurate reconstruction techniques based on CG and zero-padding.

Example of reconstruction (ZP=1 for $F^a + F^s$)



Conclusions

Two (and only two) *spatially independent* source terms can be reconstructed from the AtRT (extension of the Novikov fornula).

Under some smallness condition on the gradient of the absorption map $a(\mathbf{x})$, the spatially dependent source term $f(\mathbf{x})$ can uniquely be reconstructed from half of the AtRT measurements. There is an explicit iterative procedure to do so.

A generalization of the fast slant stack algorithm allows us to obtain fast (in specific cases), *robust and accurate* reconstructions of the source term from full and partial measurements. A good accuracy may rely on using conjugate gradient iterations or on zero-padding the initial image into a bigger one.