Imaging in Random Media Kinetic Models

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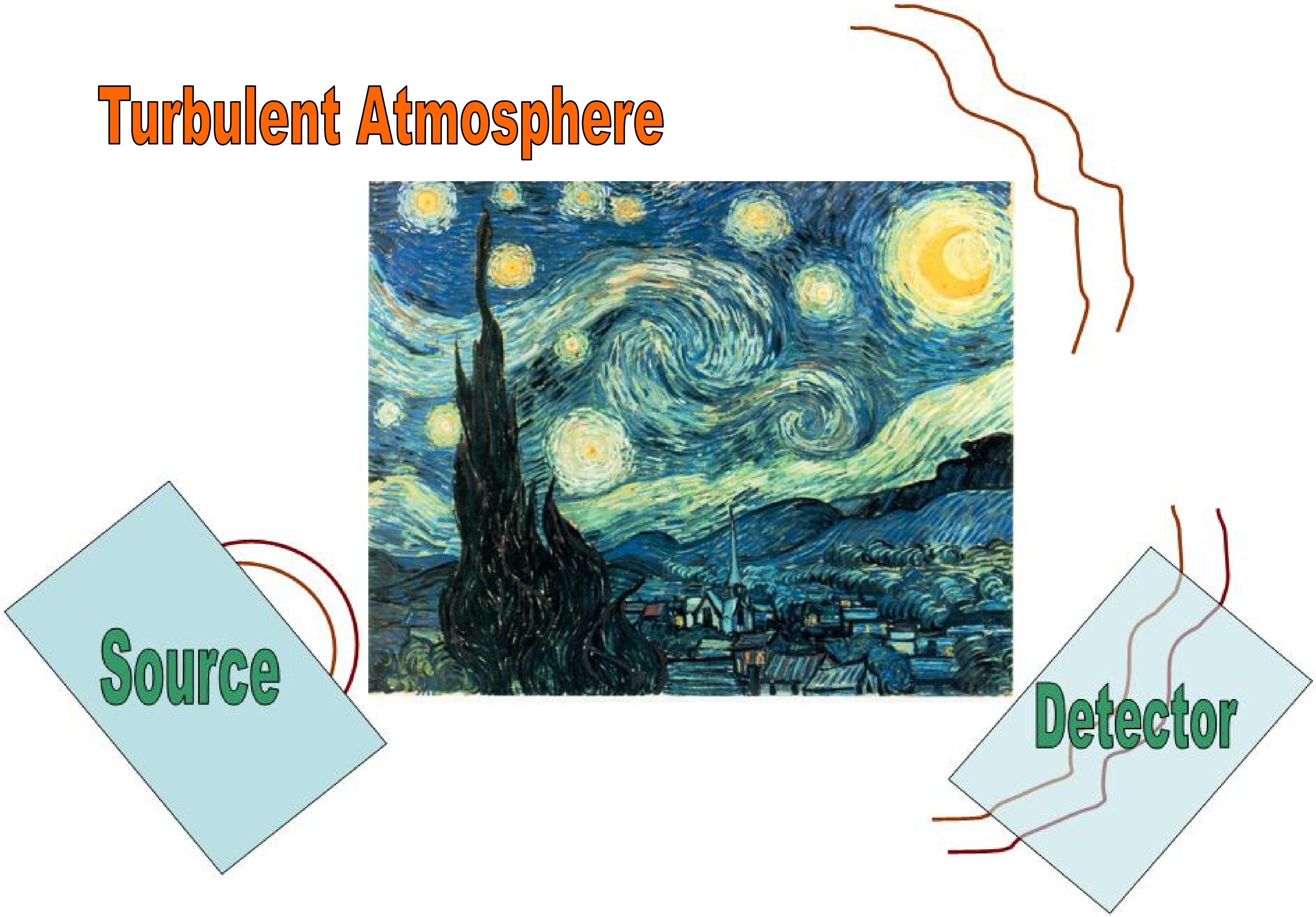
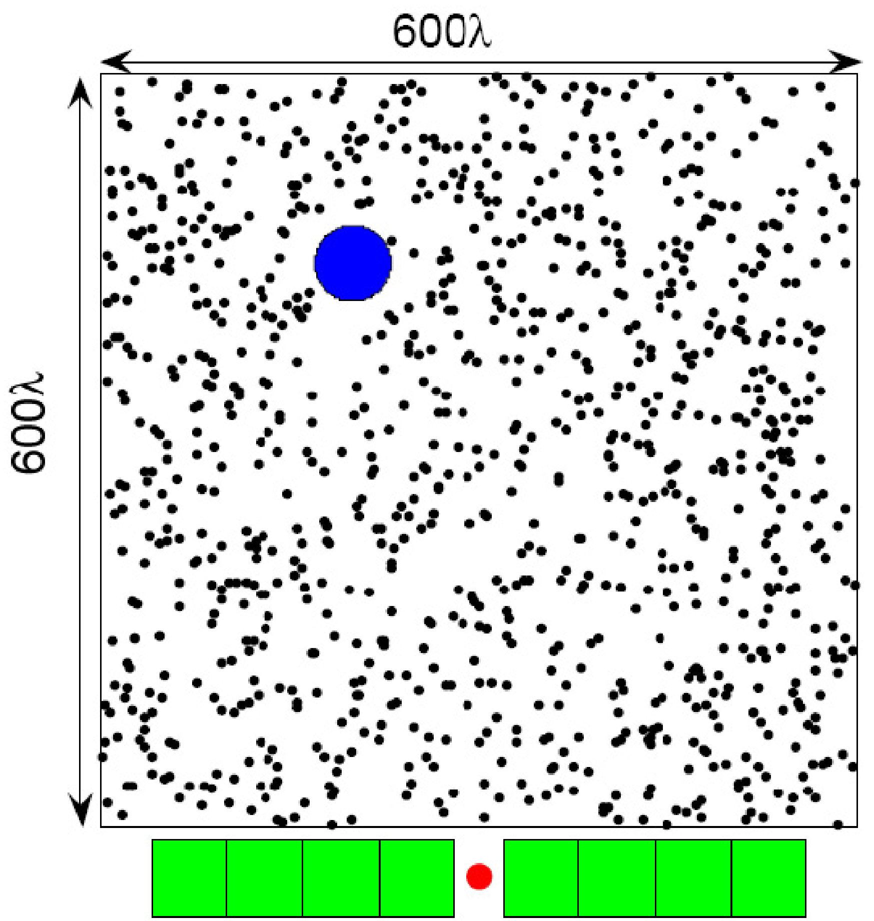
Motivations

- Imaging in Highly Heterogeneous Media
  - Statistical Properties of Random Media
  - Imaging of buried Inclusions

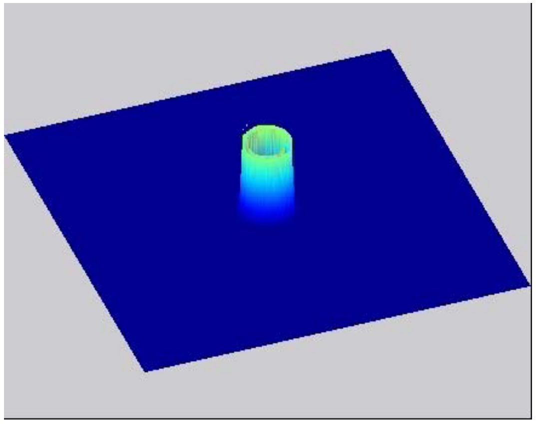
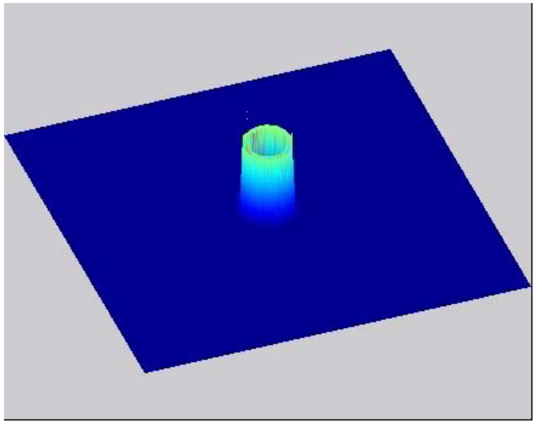
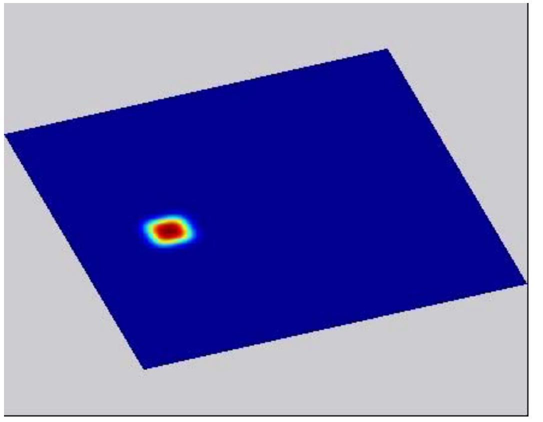
Probing heterogeneous media



Detecting Buried Inclusions



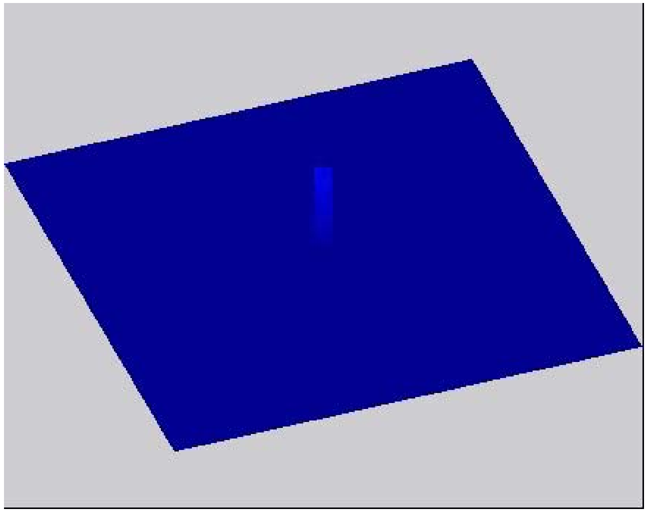
Examples of wave propagation

- Waves propagating in highly heterogeneous media

Imaging in Known Media

- When heterogeneous medium is *known*: Use **Time Reversal**:
- Time reversed waves back-propagate to their original location.
- Inclusion may be seen as secondary source.



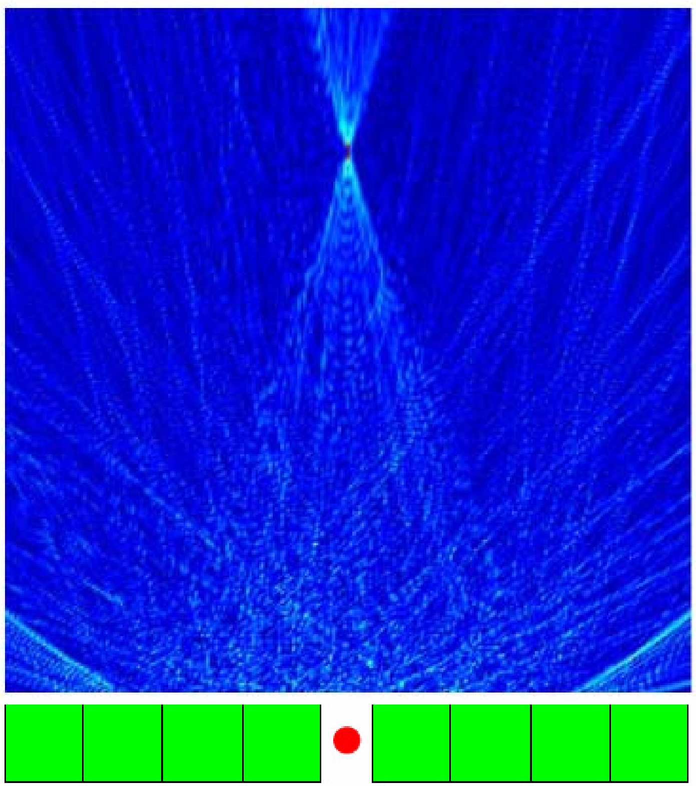
Imaging in Unknown Media

- When the (random) medium is not known:
  - Model random medium by a **homogeneous medium** with **small** random fluctuations.
  - **Model** wave propagation **macroscopically**:

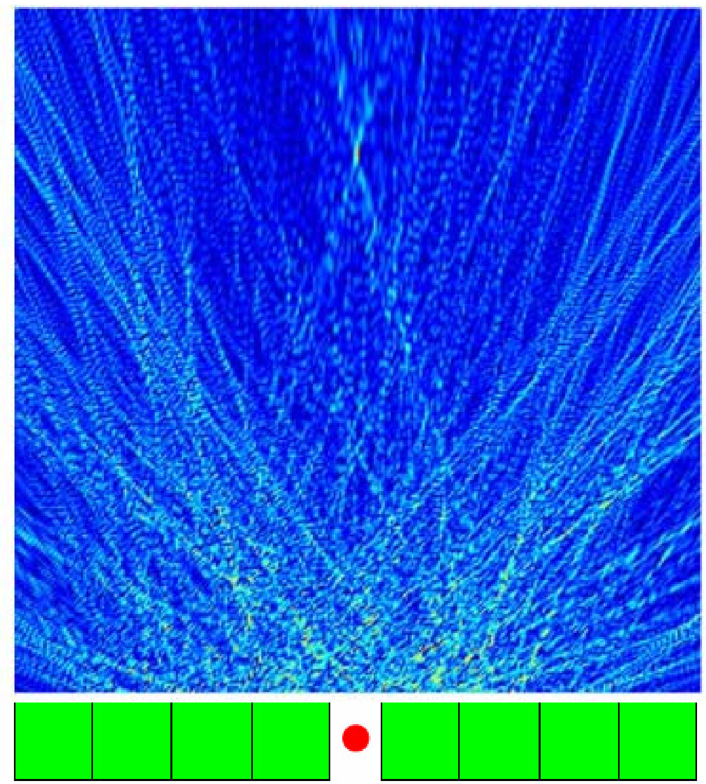
    what we are interested in today.

Homogeneous medium: Kirchhoff migration

Weakly Scattering



Strongly Scattering



High frequency waves in Random Media

- Macroscopic model: need an asymptotic regime. Here **high frequency** waves with **highly heterogeneous** media.
- High frequency waves: **Liouville** equation for the wave **energy density** $a(t, \mathbf{x}, \mathbf{k})$ **:**

$$\frac{\partial a}{\partial t} + \nabla_{\mathbf{k}}\omega \cdot \nabla_{\mathbf{x}} a - \nabla_{\mathbf{x}}\omega \cdot \nabla_{\mathbf{k}} a = 0$$

$$\omega(\mathbf{x}, \mathbf{k}) = c(\mathbf{x})|\mathbf{k}|$$

Radiative Transfer Equation

- Regime: fluctuations too large for Liouville to be valid but too small to prevent transport: perturbation that accounts for SCATTERING.

$$\frac{\partial a}{\partial t} + \nabla_{\mathbf{k}}\omega \cdot \nabla_{\mathbf{x}} a - \nabla_{\mathbf{x}}\omega \cdot \nabla_{\mathbf{k}} a = \frac{\pi\omega^2(\mathbf{x},\mathbf{k})}{2(2\pi)^d}$$
$$\times \int_{\mathbb{R}^d} \hat{R}(\mathbf{x},\mathbf{p}-\mathbf{k})\Big(a(\mathbf{p}) - a(\mathbf{k})\Big)\delta\Big(\omega(\mathbf{x},\mathbf{p}) - \omega(\mathbf{x},\mathbf{k})\Big)d\mathbf{p}$$

$\hat{R}(\mathbf{x},\mathbf{k})$ **: Power Spectrum of velocity fluctuations**

Regimes of Wave propagation

- **Weak Coupling** regime: $\delta c_\varepsilon^2(\mathbf{x}) = \sqrt{\varepsilon}\,\delta c^2\left(\dfrac{\mathbf{x}}{\varepsilon}\right)$

$$\widehat{R}(\mathbf{k})\delta(\mathbf{k}+\mathbf{p}) = c_d \mathbb{E}\{\widehat{\delta c^2}(\mathbf{k})\widehat{\delta c^2}(\mathbf{p})\}$$

- **Low Density** regime: $\widehat{R}_0 = c_d \mathbb{E}\{\tau^2\} n_0$

$$\delta c_\varepsilon^2(\mathbf{x}) = \varepsilon^{\frac{1-(\gamma+\beta)d}{2}} \sum_j \tau_j \delta c^2\left(\frac{\frac{\mathbf{x}}{\varepsilon} - \mathbf{x}_j^\varepsilon}{\varepsilon^\beta}\right)$$

$\mathbf{x}_j^\varepsilon$ Poisson P.P. with density $\varepsilon^{(\gamma-1)d} n_0$

- For larger fluctuations, waves **localize**.

Inverse Problem

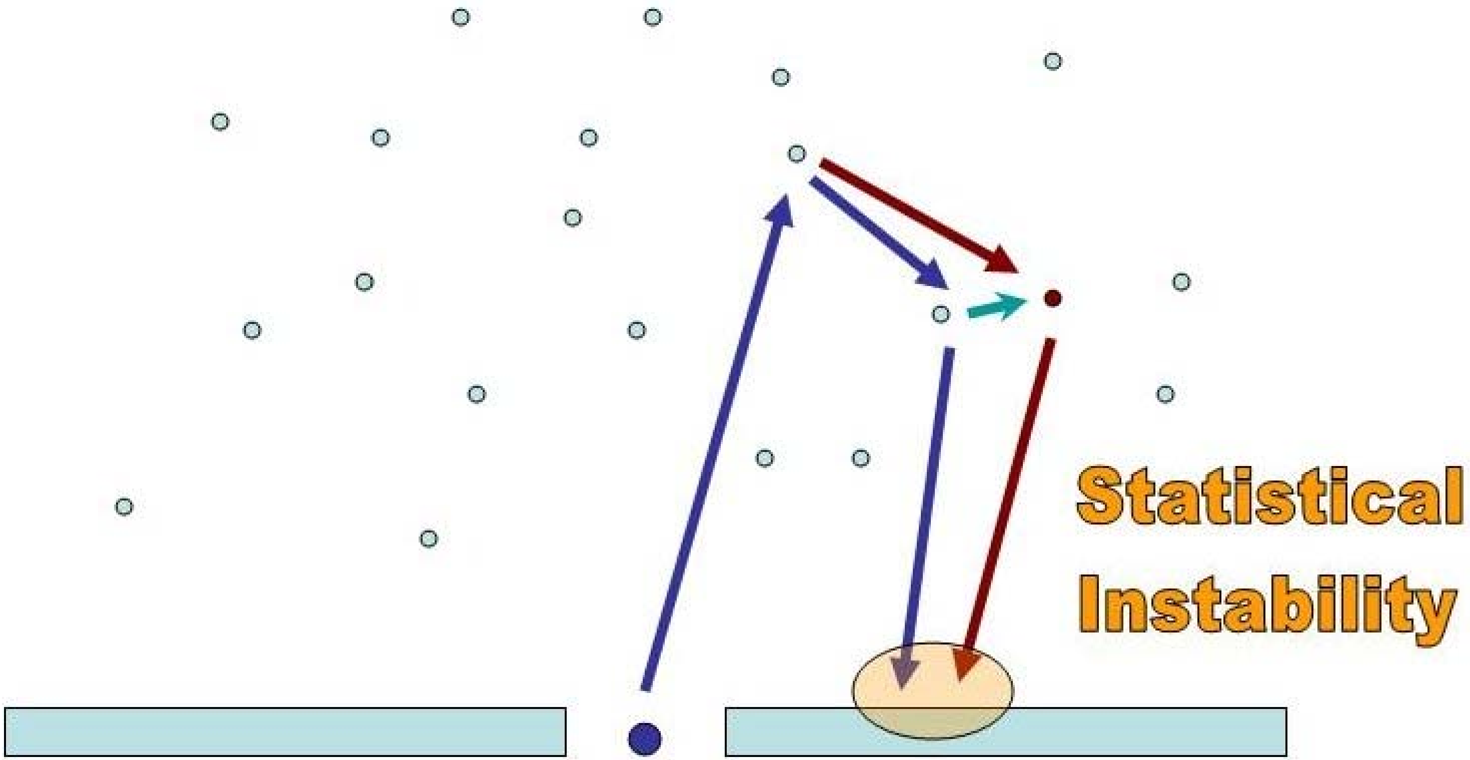
- Imaging the random media and/or buried inclusions becomes an **inverse transport** problem:

$$\frac{\partial a}{\partial t} + \nabla_{\mathbf{k}}\omega \cdot \nabla_{\mathbf{x}} a - \nabla_{\mathbf{x}}\omega \cdot \nabla_{\mathbf{k}} a = \frac{\pi\omega^2(\mathbf{x},\mathbf{k})}{2(2\pi)^d} \times \int_{\mathbb{R}^d} \hat{R}(\mathbf{x},\mathbf{p}-\mathbf{k})\Big(a(\mathbf{p}) - a(\mathbf{k})\Big)\delta\Big(\omega(\mathbf{x},\mathbf{p}) - \omega(\mathbf{x},\mathbf{k})\Big)$$

$$\omega(\mathbf{x},\mathbf{k}) = c(\mathbf{x})|\mathbf{k}|$$

- How *stable* are the measurements?

Statistical Stability



The Energy Density IS Statistically Stable

- **Result**: *Under appropriate assumptions*, the energy density **converges**, as the wavelength goes to 0, **weakly and in probability**, to its **deterministic limit**.
- Weakly means we have to average energy over a **sufficiently large region** compared to the wavelength.
- Result shows that the RTE indeed provides a model suitable for inversion: **Measurements** are ***independent*** of the ***unknown realization*** of the random medium.

Summary so far

- Random medium and buried inclusions are modeled as constitutive parameters in a transport equation, which models the (macroscopic) **wave energy density**.

- In the high frequency limit, measurements over sufficiently large detectors are ***statistically stable***.

Inverse Transport

- With **spatial & angular** measurements, **InvRTE** is **mildly ill-posed** (Hölder stability). With only **spatial** measurements, **InvRTE** is **severely ill-posed** (as in Calderón's problem).

- Practical measurements often in latter category.

- Important to find imaging scenarios that is as much immune to **statistical noise** as possible (High SNR).

Energies and Correlations

- RTE can be used to model more general ***field-field correlations*** (these are energies when the fields are the same).
- Applications: monitor **turbulent region** as a function of time, image time-varying buried inclusions.

Generalized RTE for Correlations

Correlation Function $\mathcal{C}(t,\mathbf{x}) = \int_{\mathbb{R}^d} a(t,\mathbf{x},\mathbf{k})\,d\mathbf{k}$

$$\frac{\partial a}{\partial t} + c_0\hat{\mathbf{k}}\cdot\nabla a + (\Sigma(\mathbf{k}) + i\Pi(\mathbf{k}))a$$
$$= \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\int_{\mathbb{R}^d}\hat{R}^{12}(\mathbf{k}-\mathbf{q})a(\mathbf{q})\delta\Big(\omega_+(\mathbf{q})-\omega_+(\mathbf{k})\Big)d\mathbf{q}$$

$$\Sigma(\mathbf{k}) = \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{R}^{11}+\hat{R}^{22}}{2}(\mathbf{k}-\mathbf{q})\delta\Big(\omega_+(\mathbf{q})-\omega_+(\mathbf{k})\Big)d\mathbf{q}$$

$$i\Pi(\mathbf{k}) = \frac{i\pi\sum_{j=\pm}}{4(2\pi)^d}\ \text{p.v.}\int_{\mathbb{R}^d}\Big(\hat{R}^{11}-\hat{R}^{22}\Big)(\mathbf{k}-\mathbf{q})\frac{\omega_j(\mathbf{k})\omega_+(\mathbf{q})}{\omega_j(\mathbf{q})-\omega_+(\mathbf{k})}d\mathbf{q}$$

Imaging Scenarios

- Scenario 1: Image from **Direct Energy** Measurements (with inclusion)
- Scenario 2: Image from **Energy** Measurements **With and Without** Inclusion
- Scenario 3: Image from **Wave Field** Measurements **With and Without** Inclusion

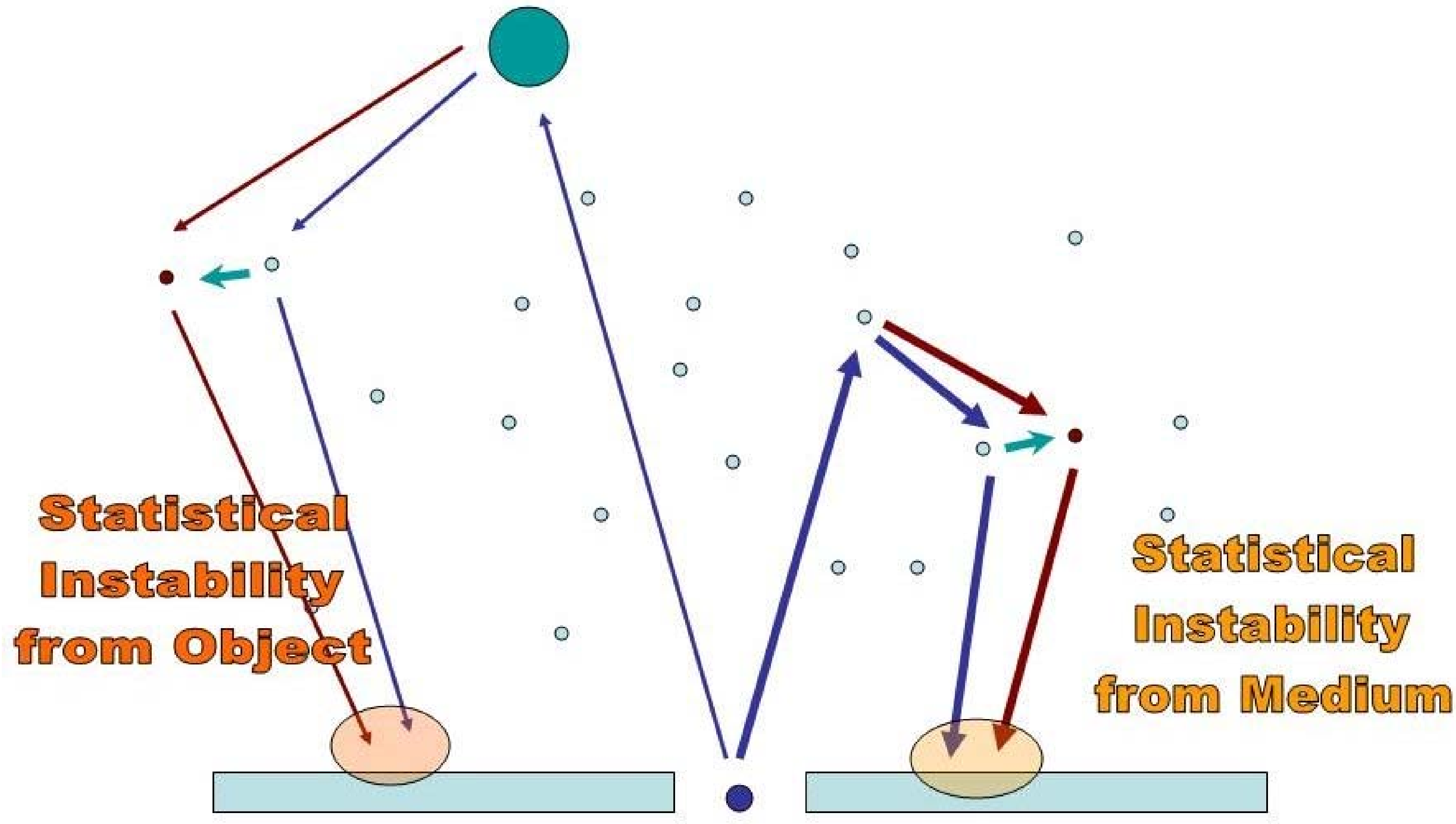
Technologically More Demanding

Direct versus Differential Measurements

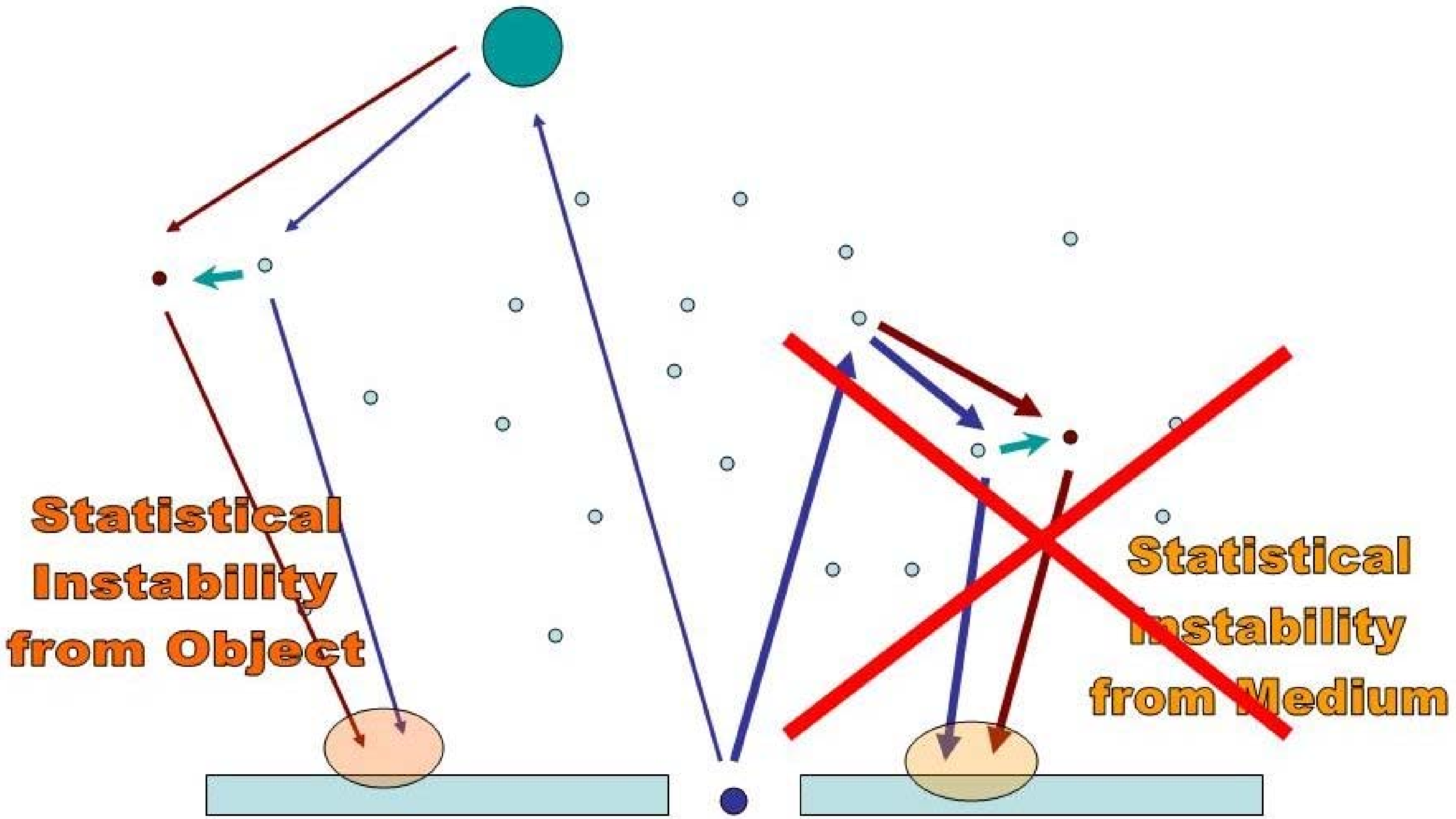
- Scenario 1 suffers from **large statistical instability** caused by our **lack of knowledge** of the random medium.

- Scenarios 2&3 suffer from statistical instability **proportional to changes** in the differential measurements.

Direct Measurements



Differential Measurements



Energies versus Correlations

- Comparison of Scenarios 2&3 in Highly Scattering regime:

In highly scattering media (in the diffusive regime), the perturbation in the energy caused by a void inclusion is given by

$$\delta\mathcal{E}(t,\mathbf{x}) = d\pi D_0 \mathsf{R}^d \int_0^t \nabla_{\mathbf{x}} u_0(t-s,\mathbf{x}_b)\cdot\nabla_{\mathbf{x}_b} G(s,\mathbf{x},\mathbf{x}_b)\,ds.$$

Here d is dimension and $G(s,\mathbf{x},\mathbf{x}_b)$ the background Green's function.

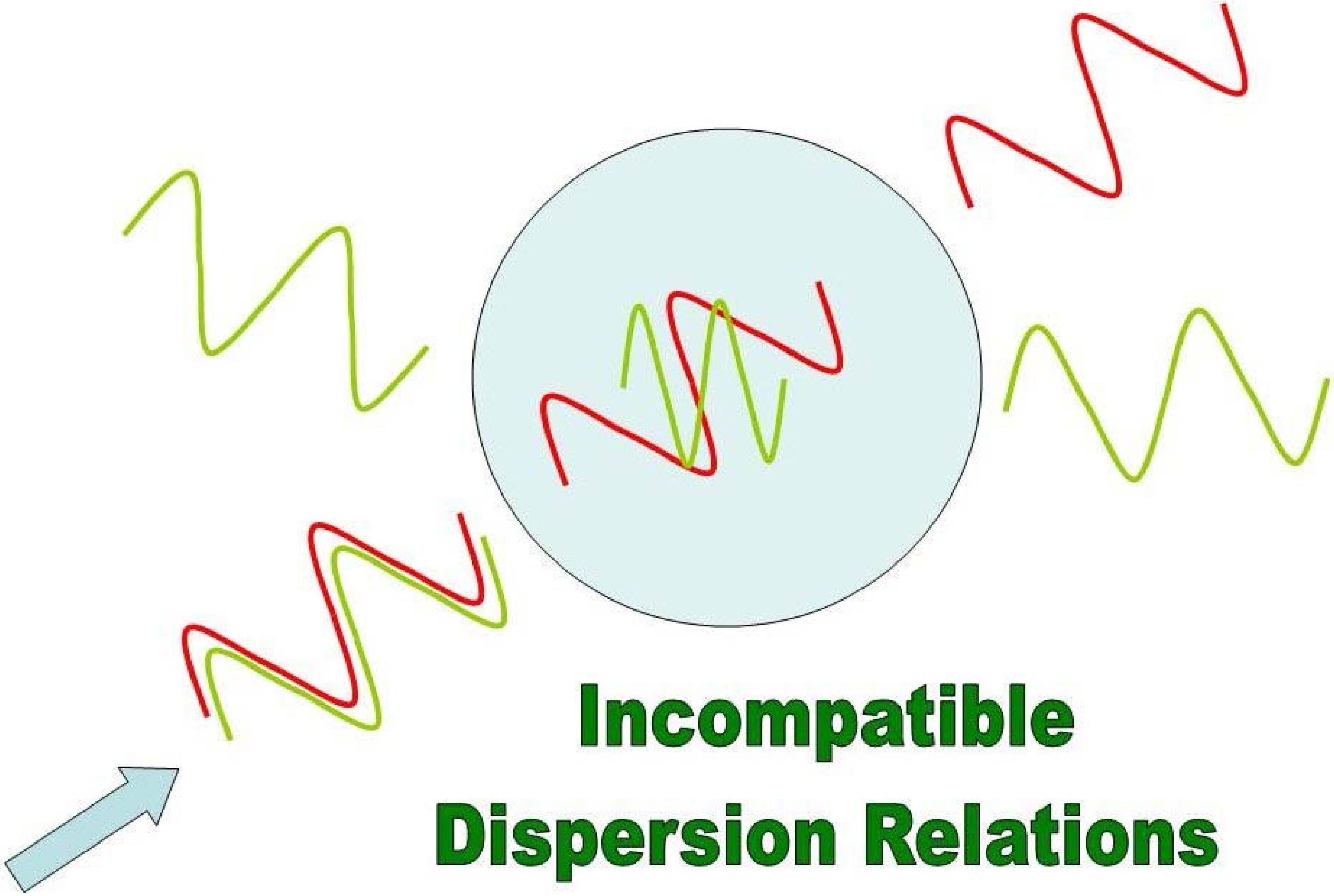
The perturbation of the two-field correlation is given by

$$\delta\mathcal{C}(t,\mathbf{x}) = -4\pi\mathsf{R}\int_0^t u_0(t-s,\mathbf{x}_b)G(s,\mathbf{x},\mathbf{x}_b)\,ds + o(\mathsf{R}), \qquad d=3$$

$$\delta\mathcal{C}(t,\mathbf{x}) = \frac{2\pi}{\ln\mathsf{R}}\int_0^t u_0(t-s,\mathbf{x}_b)G(s,\mathbf{x},\mathbf{x}_b)\,ds + o\left(\frac{1}{|\ln\mathsf{R}|}\right), \qquad d=2.$$

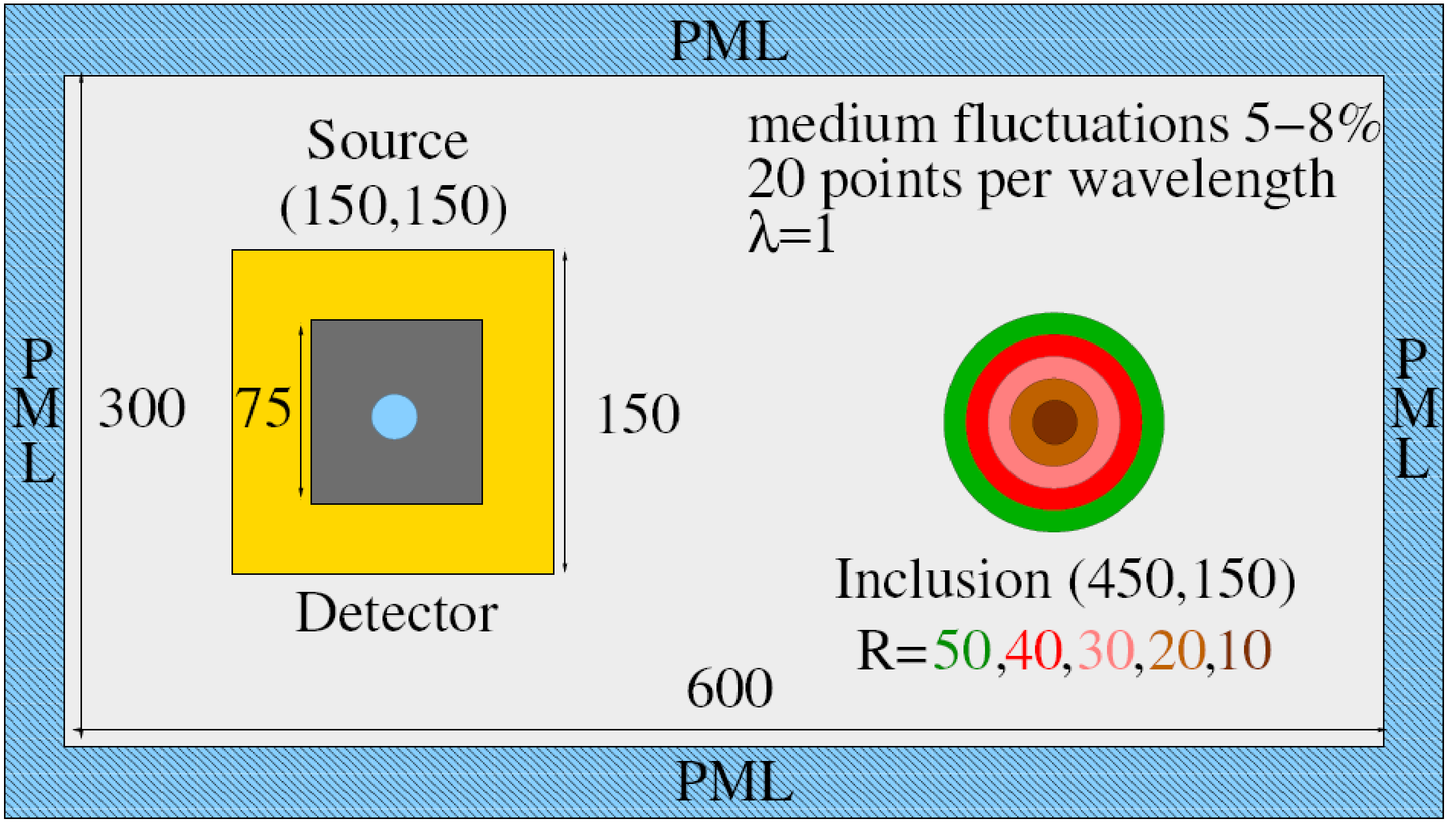
- In moderately scattering regime, both are of order R^{d-1}.

Correlations vanish at the inclusion's boundary



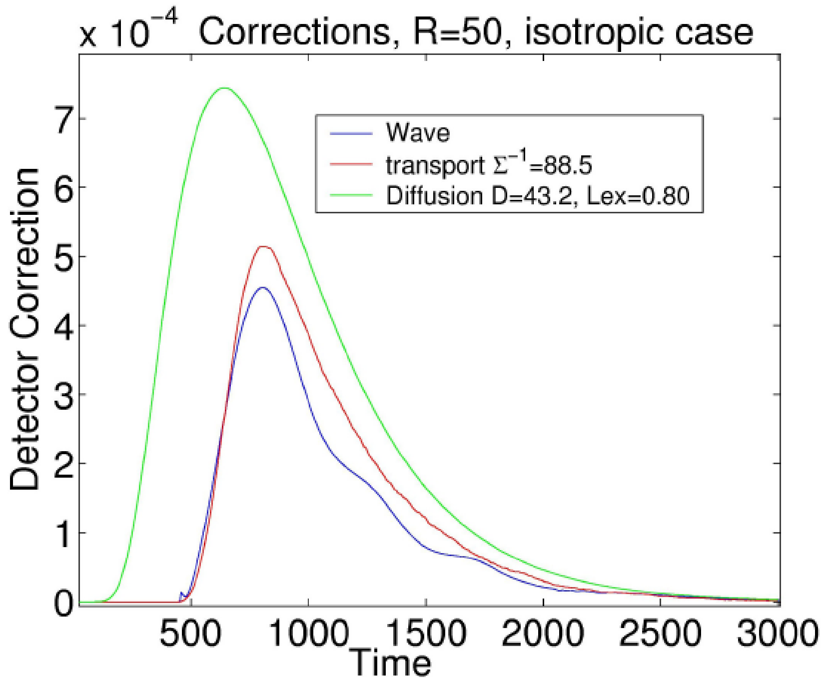
Incompatible Dispersion Relations

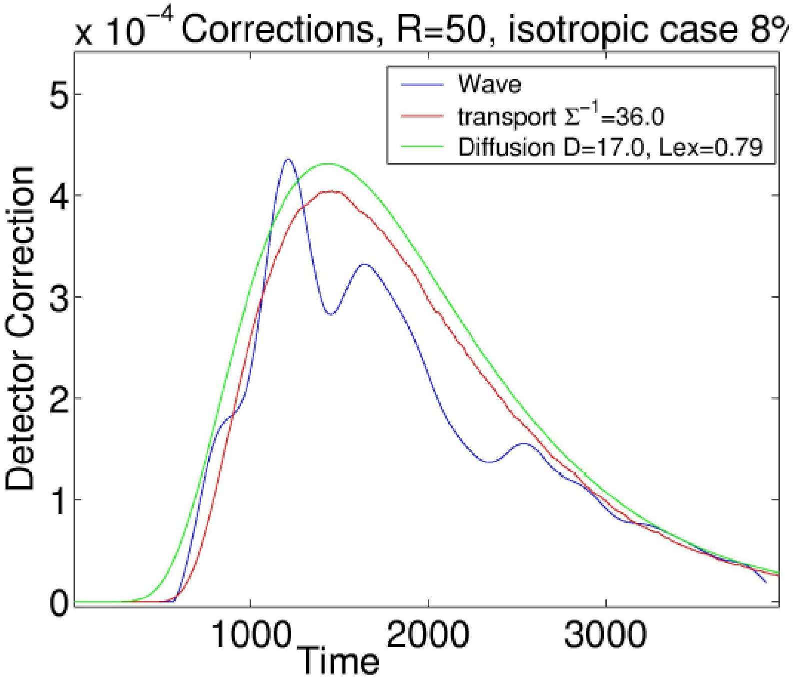
Numerical Simulations



- Waves solved by Finite Differences
- Transport solved by Monte Carlo

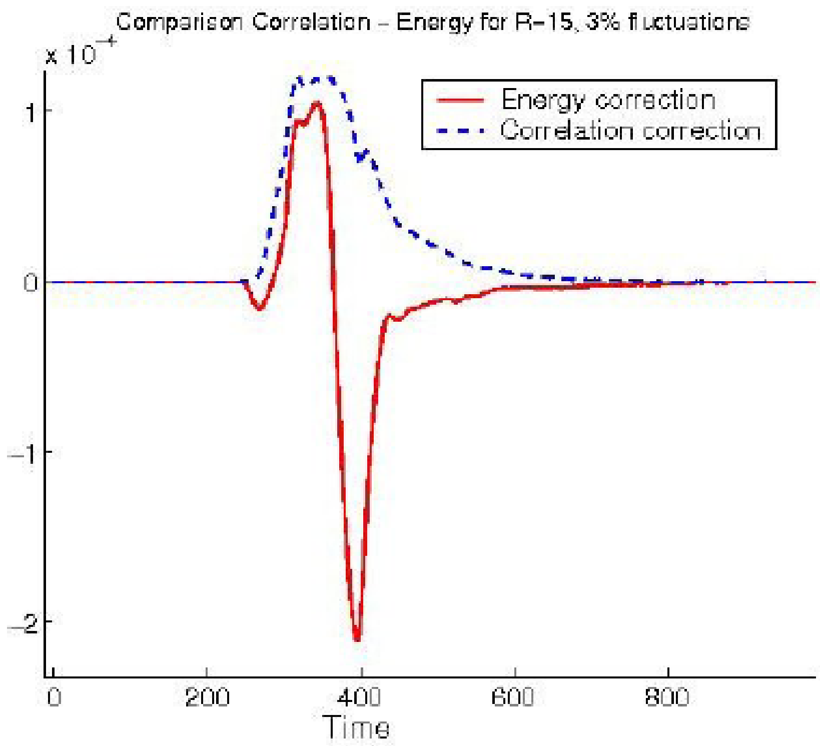
Effect of Void inclusions

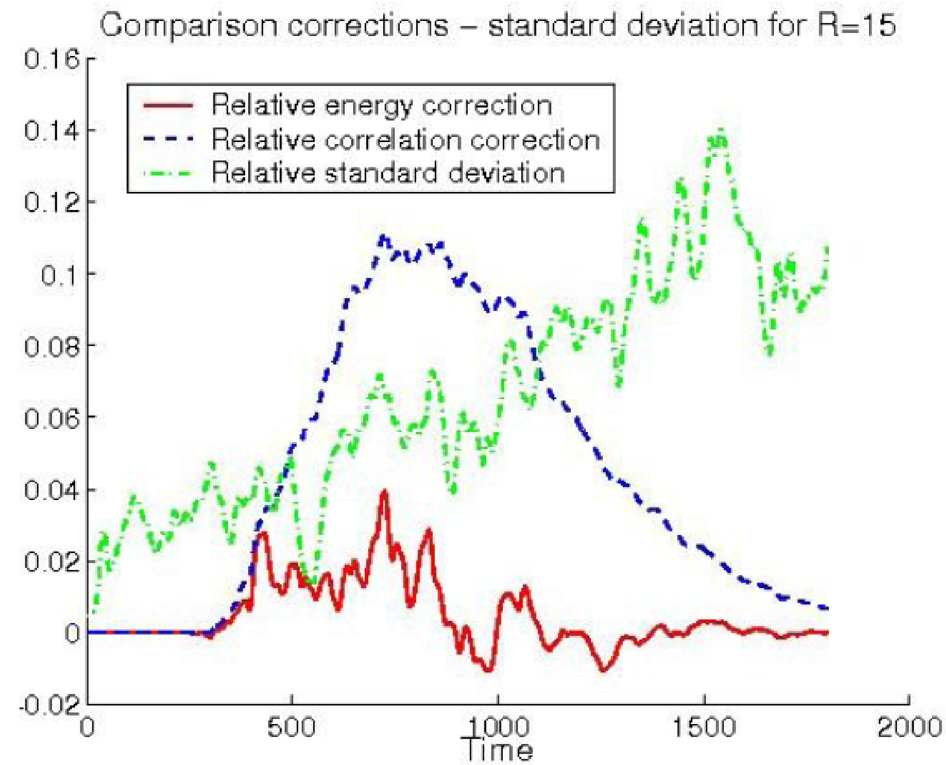




- **Transport theory** accurately predicts the **influence** of an inclusion on the energy measurement.

Energies versus correlations

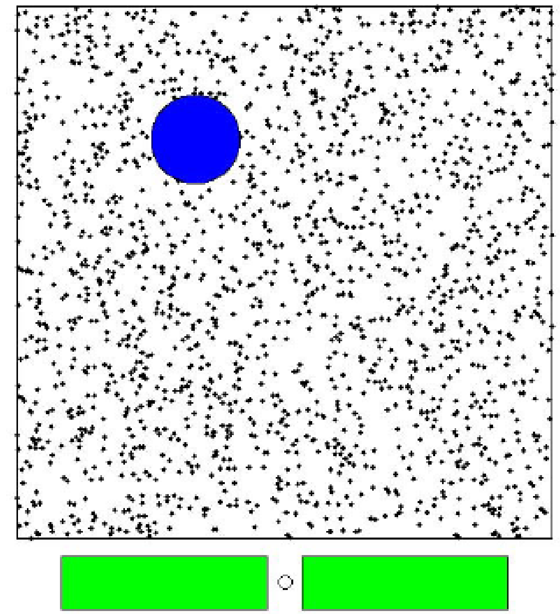




- Correlation fluctuations (blue) versus energy fluctuations (red) in *weakly* (left) and *strongly* (right) scattering media.

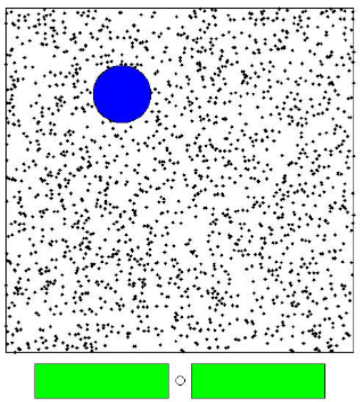
Inverse monochromatic transport

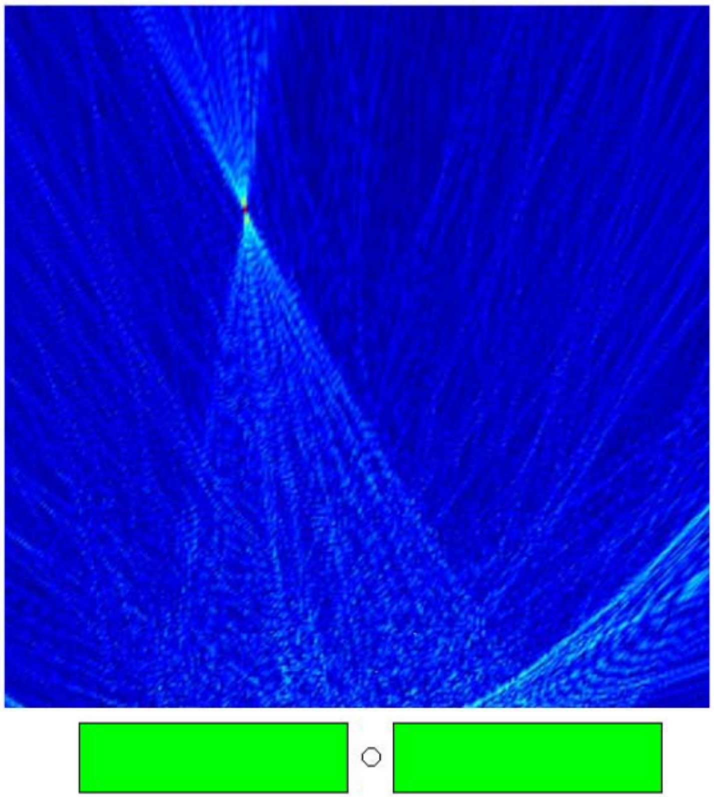
- Monochromatic waves
- **Foldy Lax** to model **point scatterers** and solve for wave fields
- Forward and inverse **transport problems** solved by **Monte Carlo** method
- Random medium parameterized by **mean free path**:

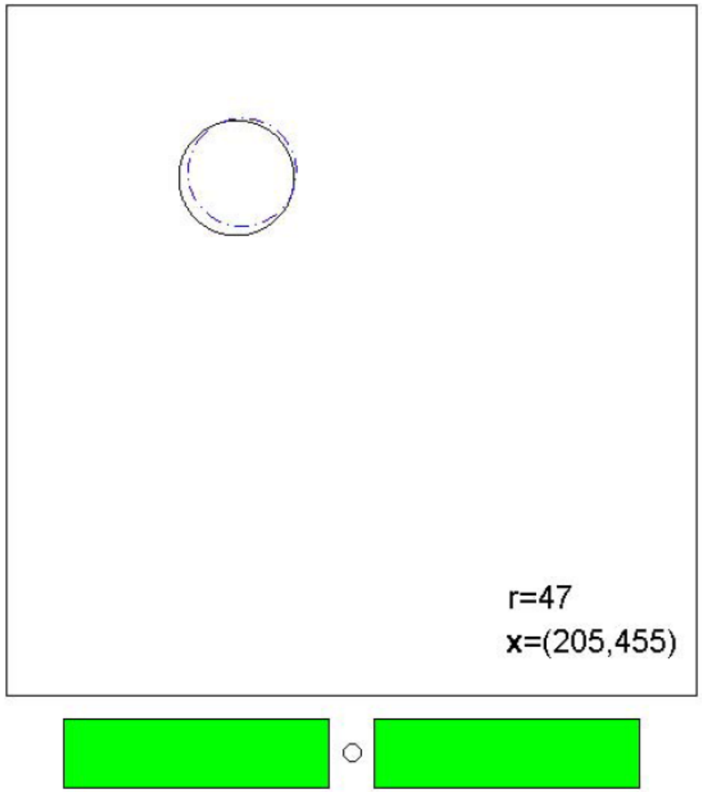


$$l^{*}_{2\mathrm{D}}(k) \approx \frac{1}{\tau^{2} k^{3}}$$

Weak Scattering reconstructions

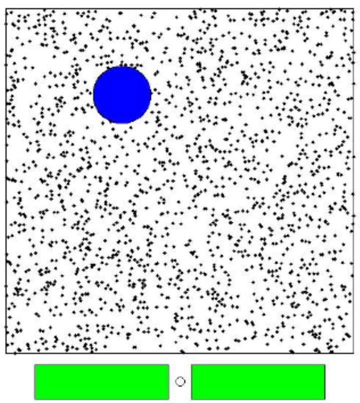


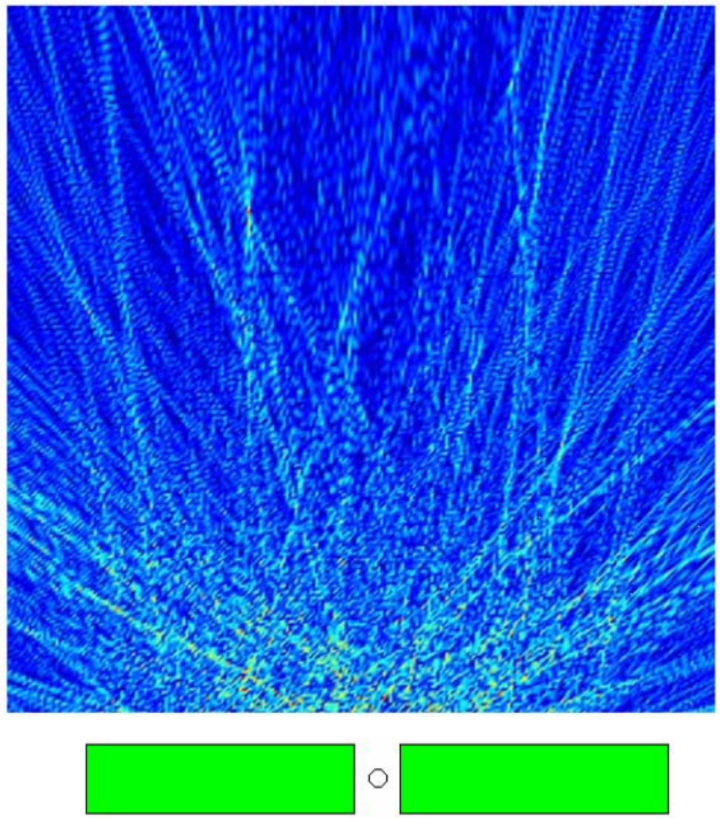


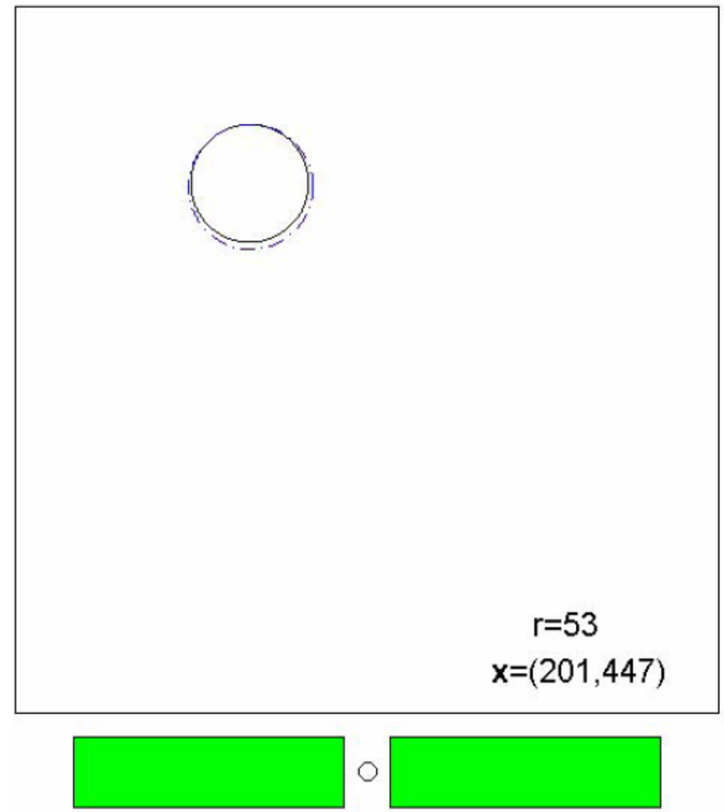


- **Kirchhoff** (middle) versus **Transport** (right) reconstructions

Strong Scattering reconstructions

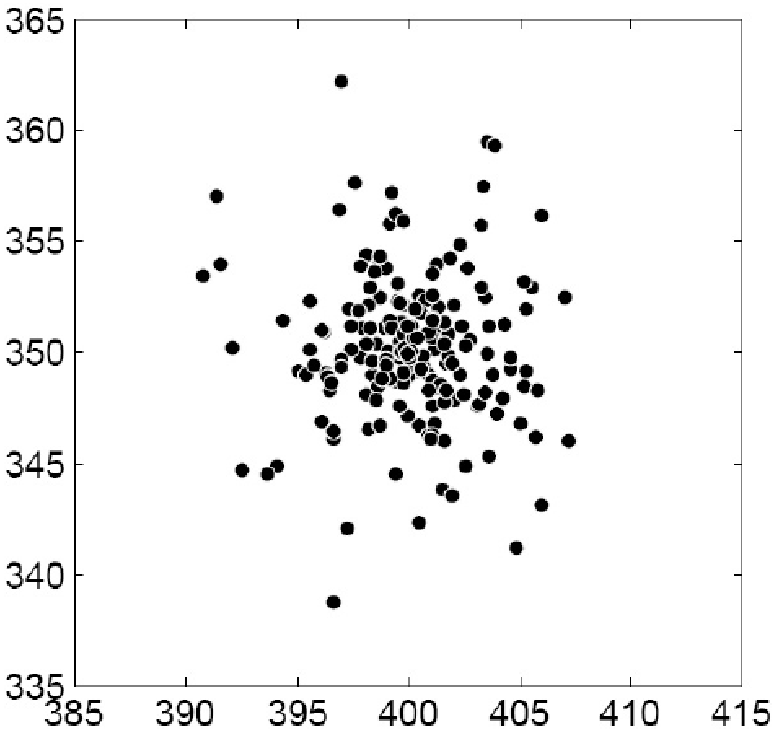


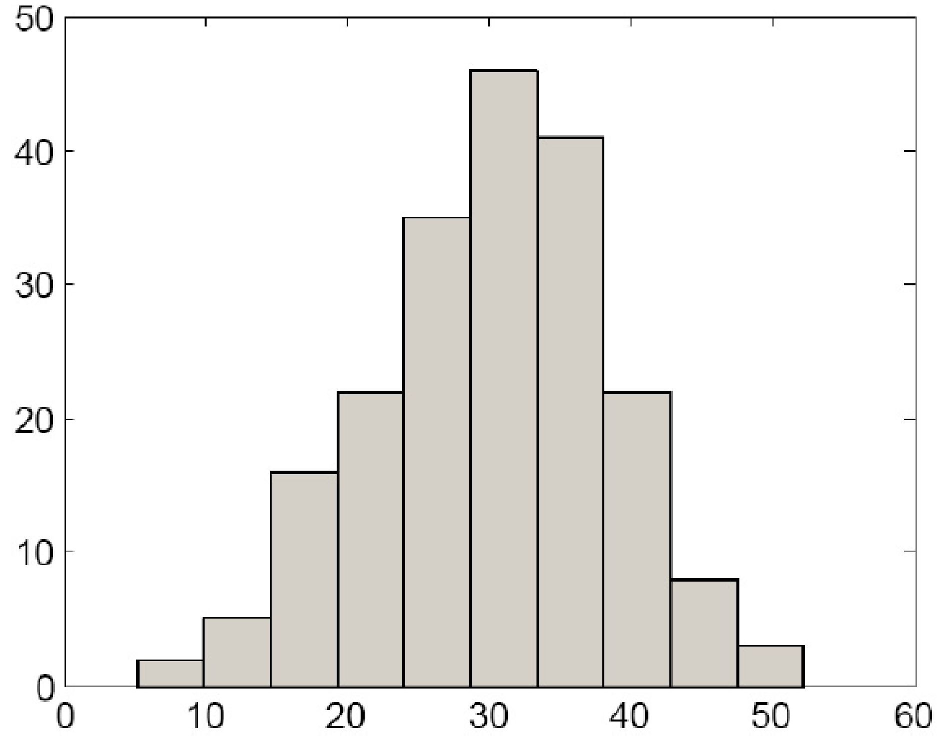


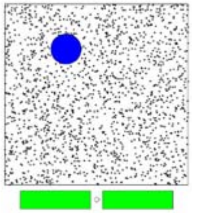


- **Kirchhoff** (middle) versus **Transport** (right) reconstructions

Reconstruction from Direct Measurements

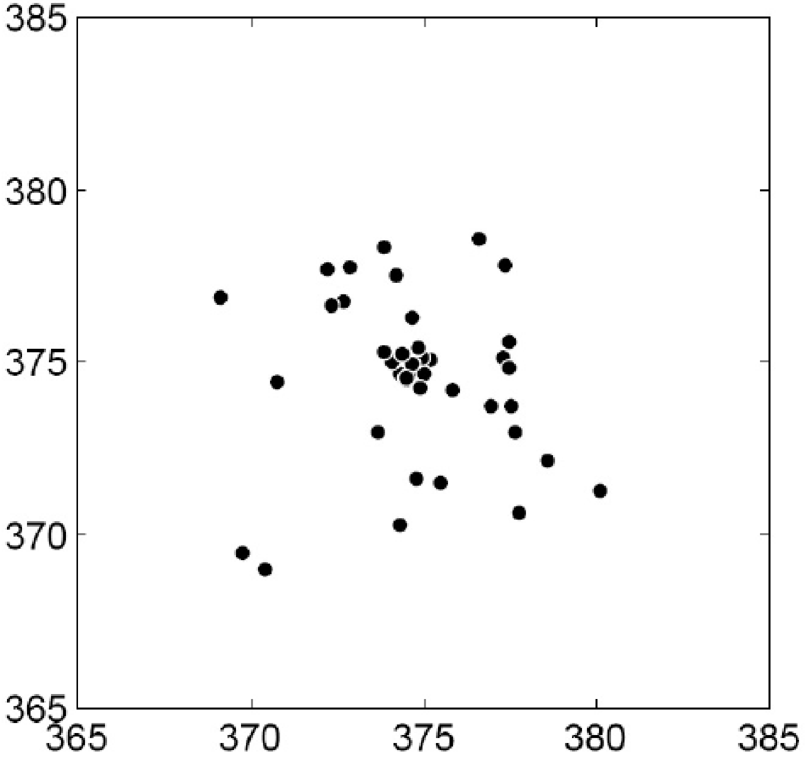


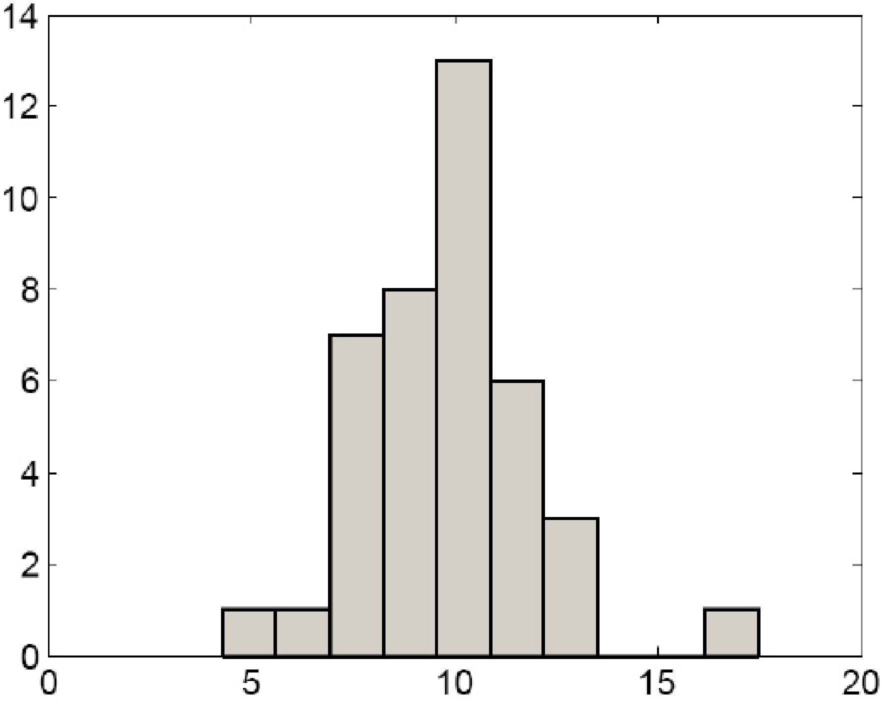


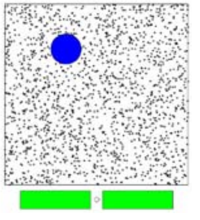


- Inclusion of radius R=30

Reconstruction from Differential Measurements

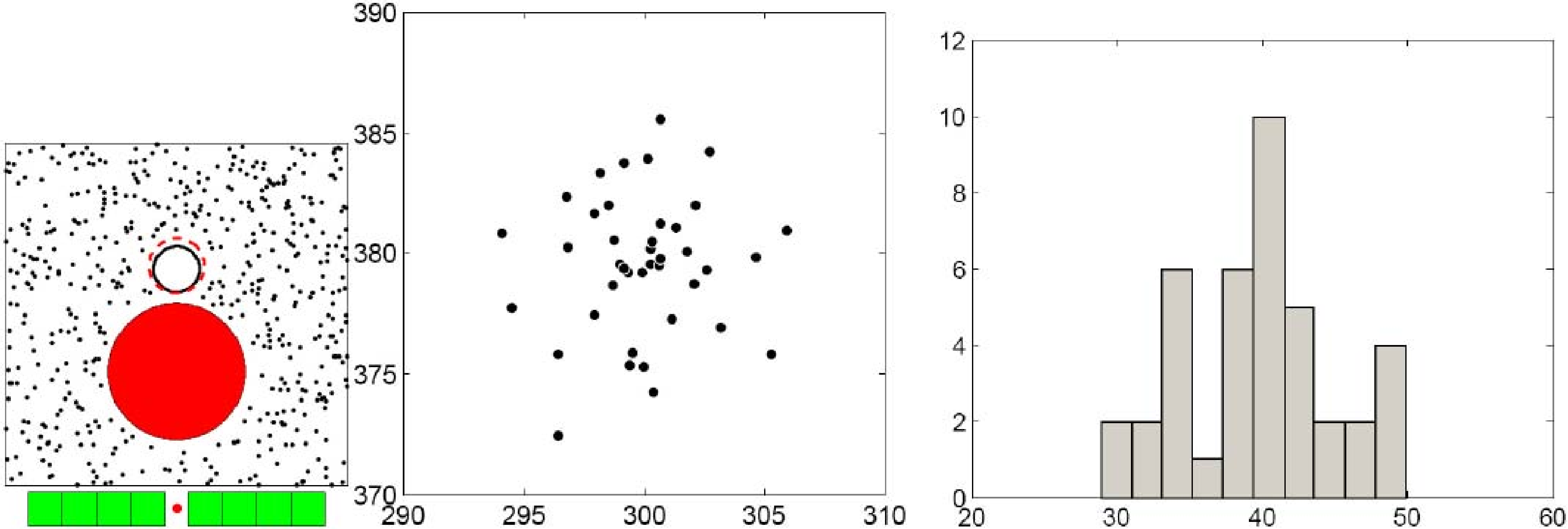






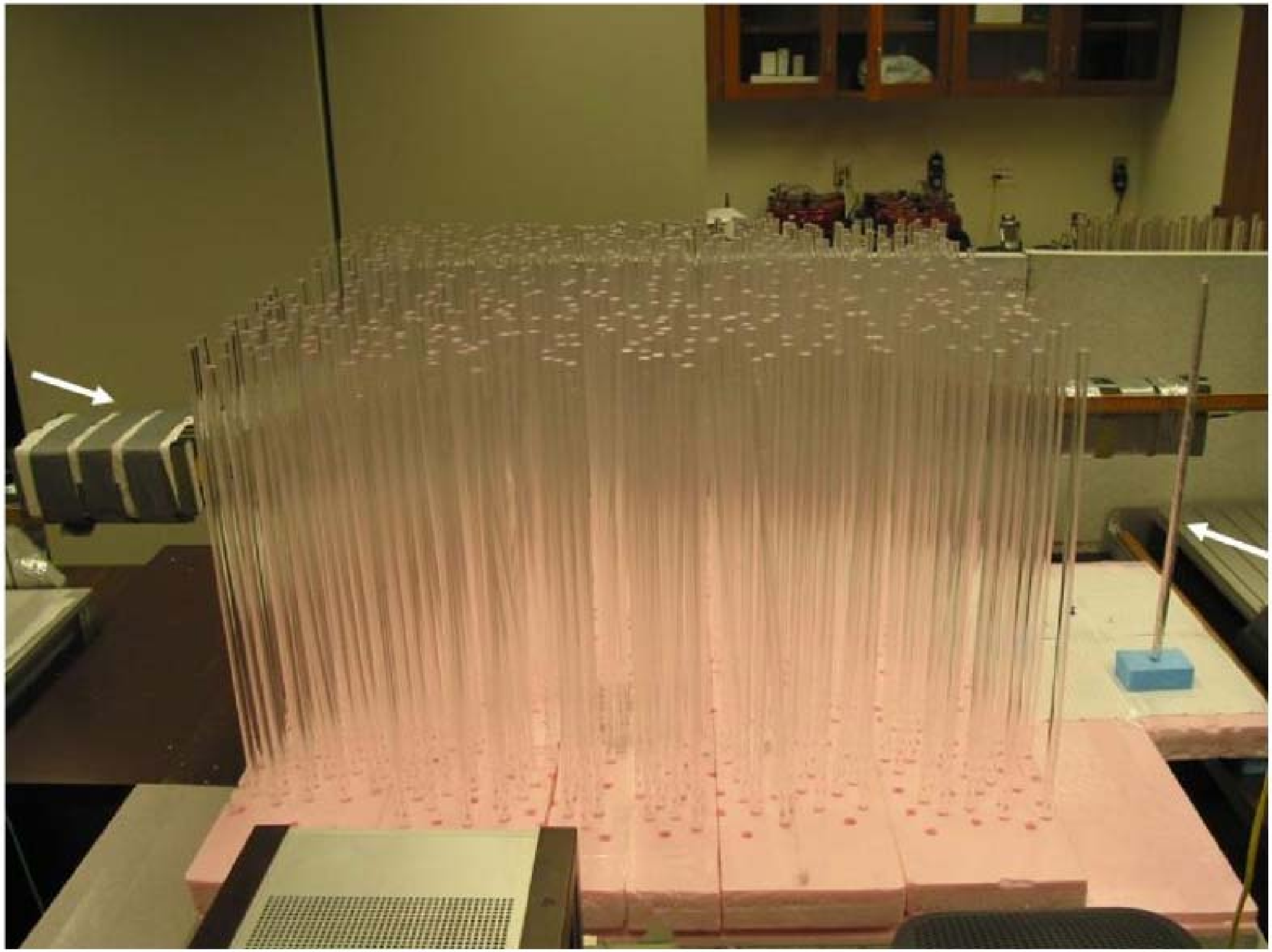
- Inclusion of radius R=10

Hidden Inclusions (by known blocker)

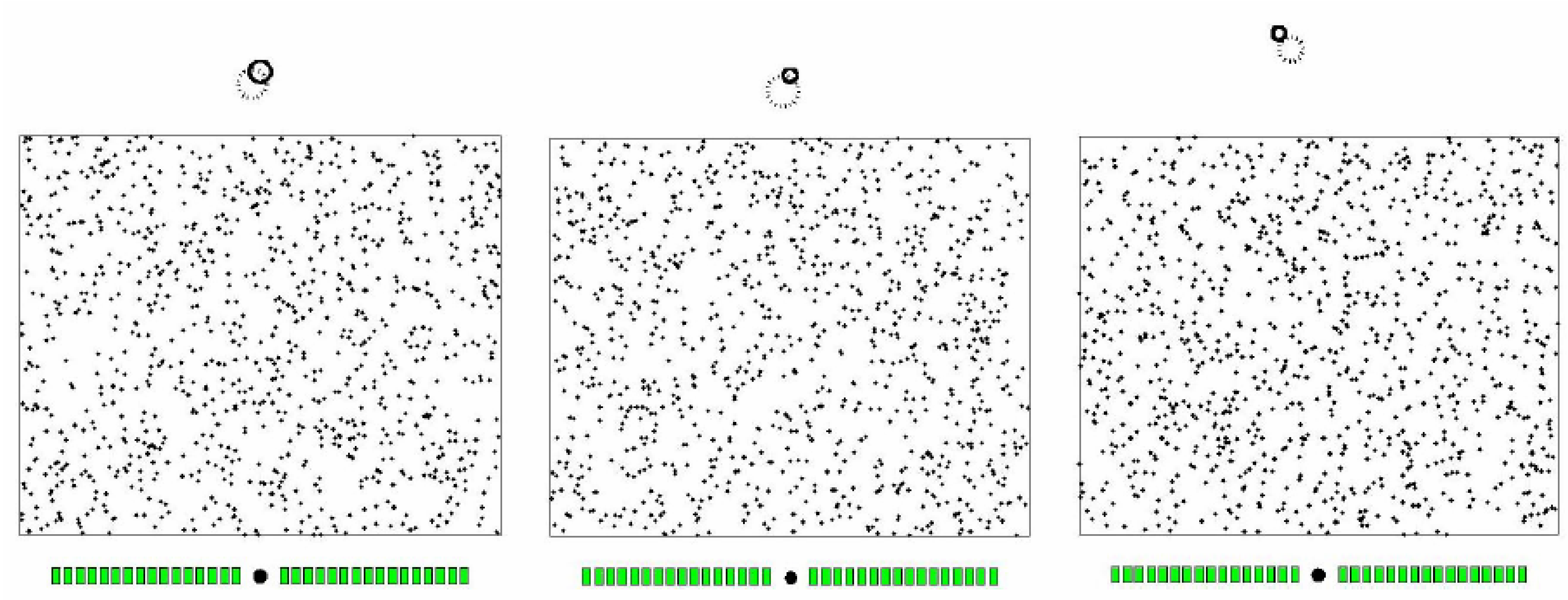


- Reconstruction of inclusions in the absence of line of sight (coherent) measurements.

Duke U. experimental Setup

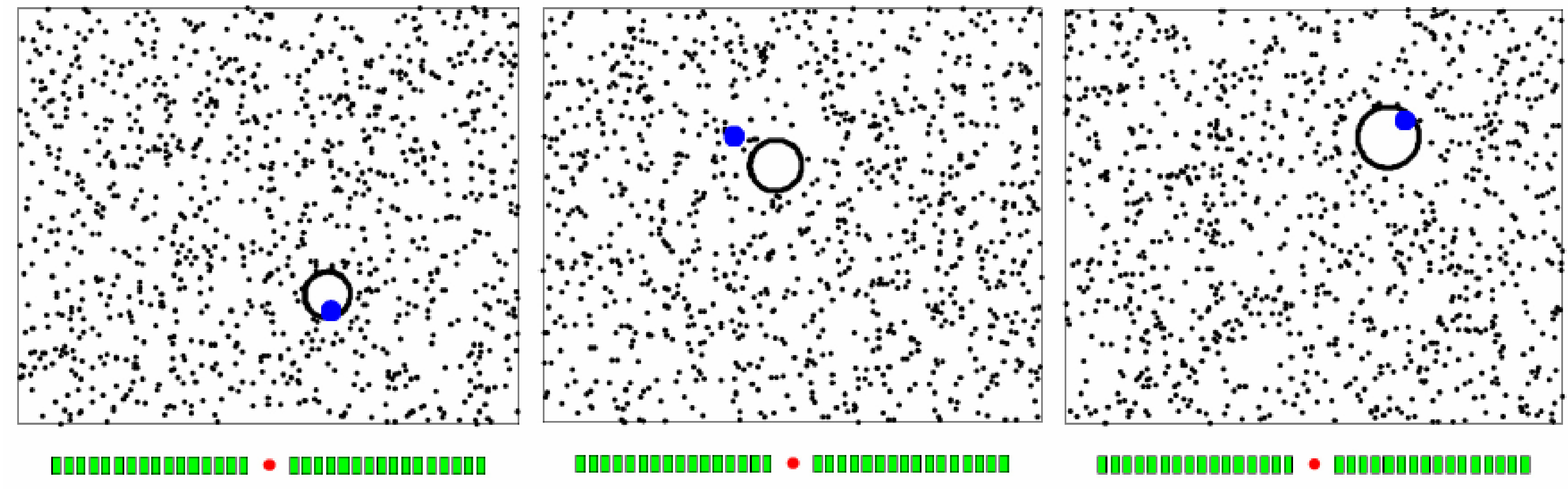


Reconstructions from Experimental Data



- Reconstructions based on differential data (Scenario 2).
- 10 GHz data. Medium is **2.5 mean free paths** thick.

Reconstruction of voids



- Reconstructions based on differential data (Scenario 2).
- 10 GHz data. Medium is **2.5 mean free paths** thick.

Conclusions

- **Transport equations** offer an accurate generalization of the **Liouville** equation in the regime of sufficiently small fluctuations.
- In that regime, the **energy density** and the **field-field correlations** are ***statistically stable.***
- Thus **inverse transport** a good model to obtain the statistical properties of random media and image buried inclusions.

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References available at www.columbia.edu/~gb2030/pubs.html :

⋆ Transport-based imaging in random media (with K. Ren),

⋆ Experimental validation of a transport-based imaging method in highly scattering environments (with L. Carin, D. Liu, and K. Ren),

⋆ Kinetic Models for Imaging in Random Media (with O. Pinaud).