Inverse Problems with Internal Functionals

From Calderón's problem to Hybrid Inverse Problems

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The Calderón problem

Consider the elliptic model:

$$-\nabla \cdot \gamma(x)\nabla u = 0 \quad \text{in} \quad X \qquad \text{and} \qquad u = g \quad \text{on} \quad \partial X.$$

The **Calderón problem** consists of reconstructing the unknown $\gamma(x)$ from knowledge of *all possible* Cauchy data $(u, \gamma\nu \cdot \nabla u)$ on ∂X with u solution of the above equation.

Calderón (1980) showed injectivity of the *linearized* Calderón problem using complex geometric optics (CGO) solutions. **Sylvester and Uhlmann** (1987) showed injectivity of the Calderón problem for C^2 functions γ and **Astala and Päivärinta** (2006) for L^∞ functions γ in dimension two.

Alessandrini (1988) showed that the modulus of continuity of the inverse problem was (essentially) logarithmic: severe *loss of resolution*.

CGO solutions and Qualitative Statements

Injectivity of the Calderón problem is proved by showing that $q_1 = q_2$

$$\text{when} \qquad (\Delta - q_i)u_i = 0 \qquad \text{and} \qquad \int_X (q_1 - q_2)\, u_1 u_2 \, dx = 0.$$

Statement on the **density** of products of (almost-) harmonic solutions.

CGO solutions are of the form

$$\boxed{u_\rho = e^{\rho\cdot x}\Big(1 + \psi_\rho(x)\Big)} \qquad \rho = k + ik^\perp \in \mathbb{C}^n, \quad |k| = |k^\perp|,\ k\cdot k^\perp = 0.$$

Property: $|\rho||\psi_\rho|$ is bounded (ψ_ρ is small as $|\rho| \to \infty$).
Choosing ρ_1 and ρ_2 such that $\rho_1 + \rho_2 = i\xi \in \mathbb{R}^n$ and $|\rho_1|, |\rho_2| \to \infty$:

$$\lim_{|\rho_1|,|\rho_2|\to\infty} \int_X (q_1 - q_2) u_{\rho_1} u_{\rho_2} dx = \int_X (q_1 - q_2) e^{i\xi\cdot x} dx = 0.$$

However, $|u_1|, |u_2| \sim e^{|\xi|}$ to *determine* $\widehat{q}(\xi)$: the Calderón problem is a severely ill-posed inverse problem with **low resolution** capabilities.

High Contrast and High Resolution

Optical Tomography and Electrical Impedance Tomography, modeled by the Calderón problem, are **low resolution** but **High Contrast** modalities.

High resolution modalities include Ultrasound, M.R.I., X-ray CT. These modalities are sometimes **low contrast**.

Hybrid Inverse Problems are problems resulting from the physical coupling between a **High Contrast** modality and a **High Resolution** modality.

In this lecture, the *high resolution* modality is Ultrasound. The *high contrast* comes from elastic, electrical, or optical properties of tissues.

Hybrid inverse problems and internal functionals

Hybrid inverse problems (HIP) typically involve a two-step process. In a first step, a **high resolution** inverse boundary problem is solved. This could be an *inverse wave problem* (reconstruction of an initial condition in a wave equation) or the *inversion of a Fourier transform* (similar to reconstructions in M.R.I.). We do not consider this step here.

The *outcome* of the first step is the availability of specific internal functionals of the parameters of interest. HIP theory aims to address:

- Which parameters can be uniquely determined
- With which stability (resolution)
- Under which illumination (probing) mechanism.

Quantitative Photo-Acoustic Tomography (QPAT)

In the diffusive regime, **optical radiation** is modeled by:

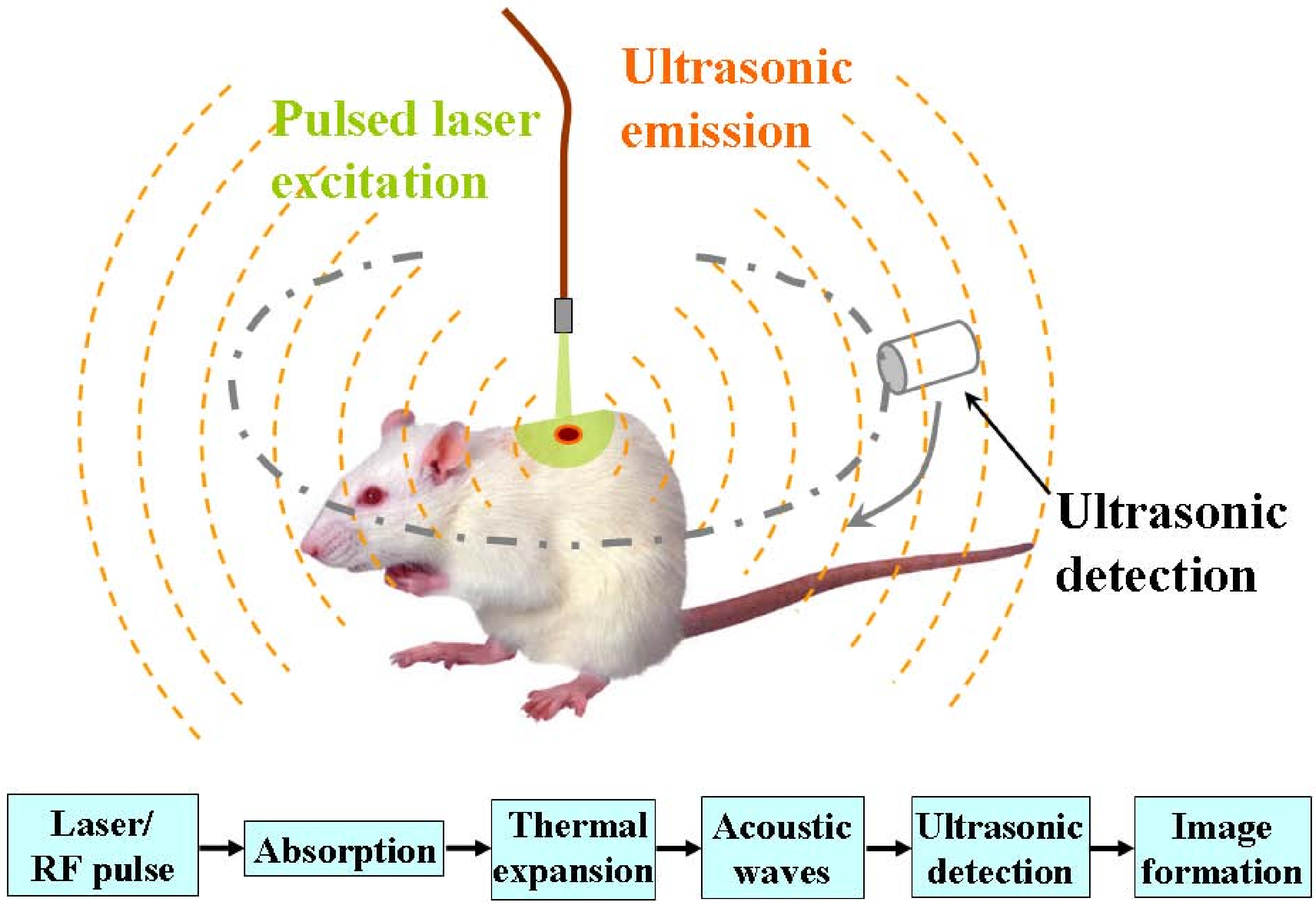
$$-\nabla\cdot\gamma(x)\nabla u+\sigma(x)u=0 \text{ in } X \qquad u=g \text{ on } \partial X \qquad \textbf{Illumination,}$$

$$H(x)=\Gamma(x)\sigma(x)u(x) \text{ in } X \qquad \textbf{Internal Functional}.$$

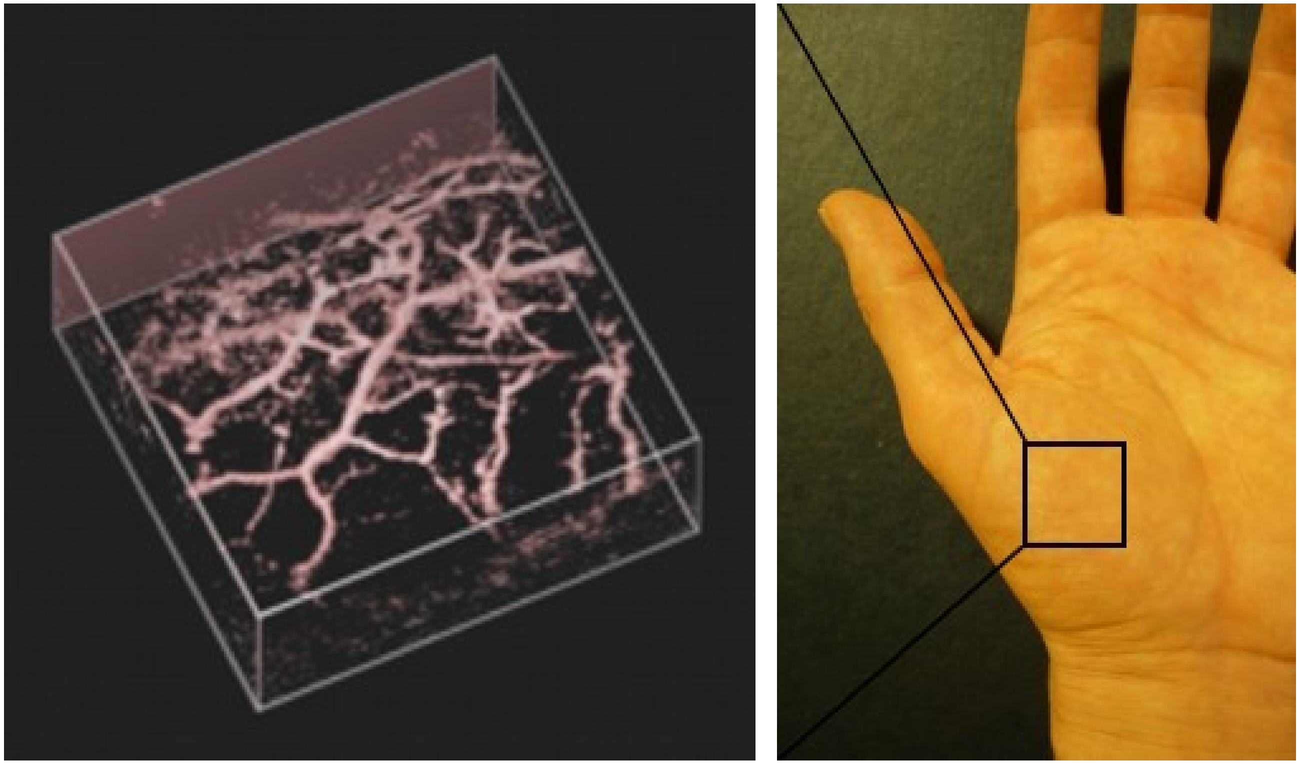
The **objectives** of *quantitative PAT* are to understand:

- What we can reconstruct of $(\gamma(x),\sigma(x),\Gamma(x))$ from knowledge of $H_j(x)$, $1\le j\le J$ obtained for illuminations $g=g_j$, $1\le j\le J$.
- How stable the reconstructions are.
- How to choose J and the illuminations g_j.

Experiments in Photo-acoustics

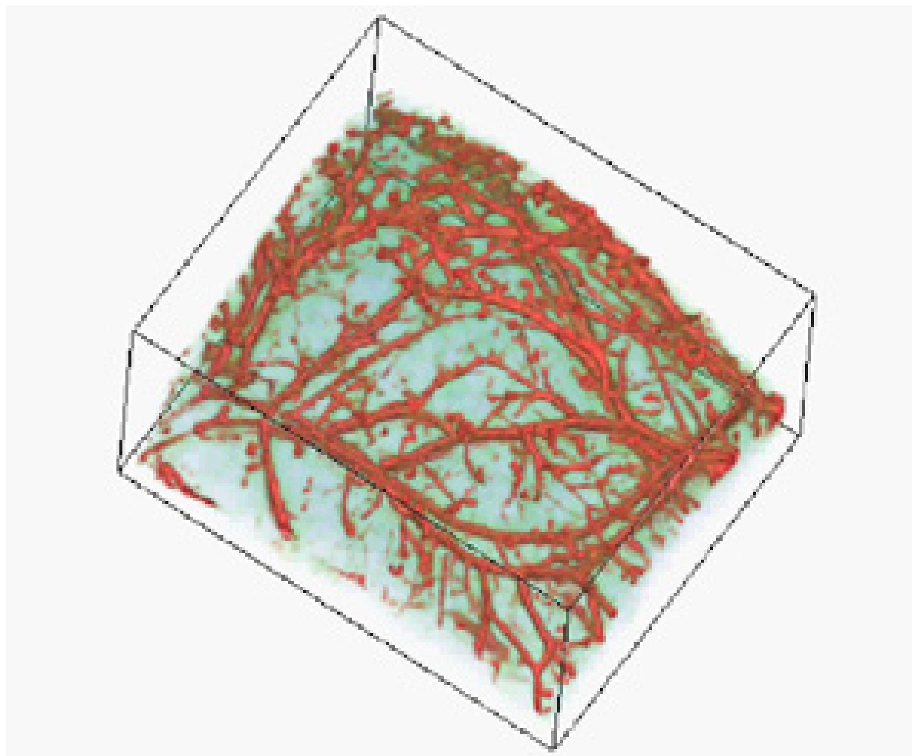


Experimental results in Photoacoustics



Courtesy UCL (Paul Beard's Lab).

Experimental results in Photoacoustics



From Lihong Wang's lab (Wash. Univ.)

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Quantitative Thermo-Acoustic Tomography (QTAT)

In **Thermo-Acoustic Tomography**, low-frequency radiation is used.

Using a (scalar) **Helmholtz model** for radiation, **quantitative TAT** is

$$\Delta u + n(x)k^2 u + ik\sigma(x)u = 0 \text{ in } X, \qquad u = g \text{ on } \partial X \qquad \textbf{Illumination,}$$

$$H(x) = \sigma(x)|u|^2(x) \text{ in } X \qquad \textbf{Internal Functional}.$$

QTAT consists of uniquely and stably reconstructing $\sigma(x)$ from knowledge of $H(x)$ for appropriate illuminations g.

Ultrasound Modulation

In **Ultrasound modulated** Optical Tomography (UMOT) or Electrical Impedance Tomography (UMEIT), ultrasonic waves are used to **modify** electrical or optical properties tissues.

After modeling (à la MRI; see e.g., [B.-Schotland PRL'10]), the UMEIT and UMOT HIP take the form:

$-\nabla \cdot \gamma(x) \nabla u + \sigma(x) u = 0$ in X $\qquad u = g$ on ∂X $\qquad$ **Illumination,**

$H(x) = \alpha_1 \gamma(x) |\nabla u|^2(x) + \alpha_2 \sigma(x) |u|^2(x)$ in X $\qquad$ **Internal Functional.**

The objective is to *reconstruct* $\gamma(x)$ and $\sigma(x)$ from knowledge of **internal functionals** $H(x)$ for **one or several illuminations** $g(x)$ on ∂X.

Ultrasound Modulation

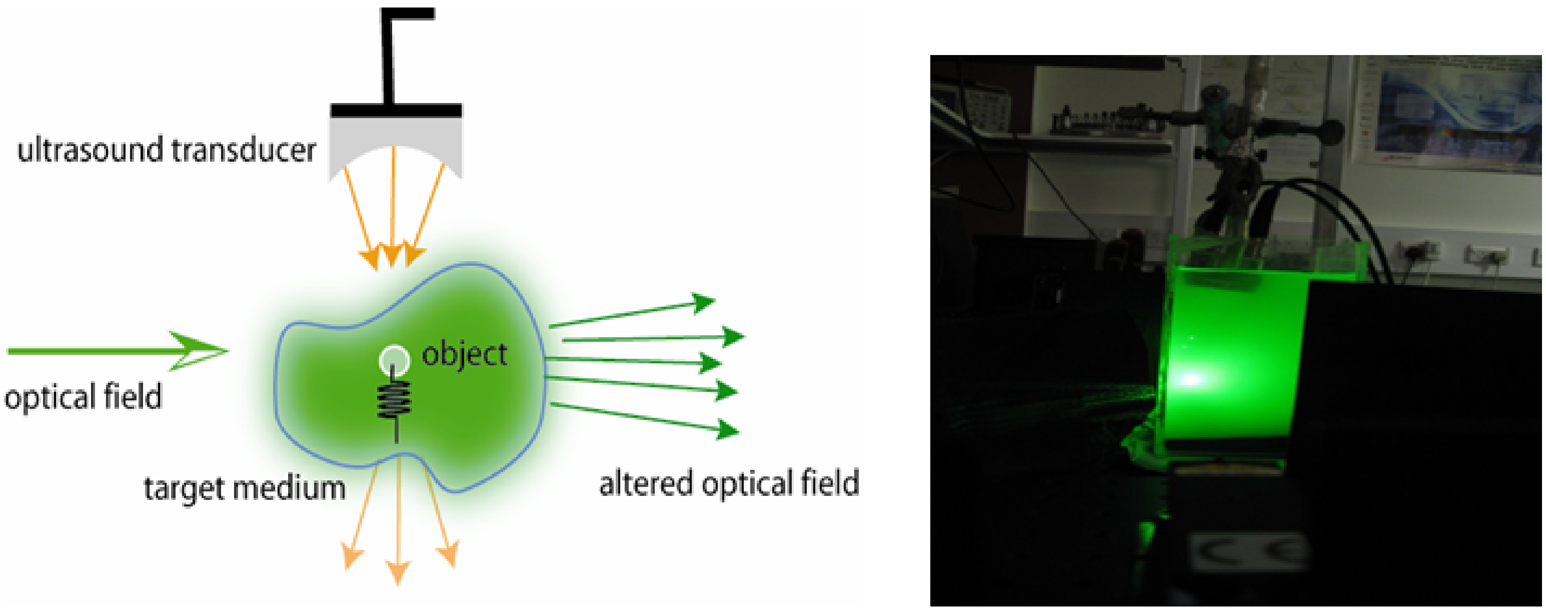

Figure 6: An illustrative diagram (left) and a photo(right) of our UOT system with a 532nm laser, a 5MHz ultrasound transducer, and a CCD camera.

Courtesy Dr. Tang, Imperial College, London.

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Solutions to HIP: a Roadmap

HIP starts with *unknown* coefficients, *unknown* elliptic solutions for *known* (elliptic) models, and *known* internal functionals.

1. We **eliminate** unknowns to focus on one.

2. **IF** some **qualitative** properties of elliptic solutions are satisfied, then we obtain **unique** and **stable** reconstructions for *some* coefficients.

3. We verify the **IF** for **well-chosen** illuminations g. Typically done by means of CGO solutions in dimension $n \geq 3$.

QPAT (and MRE/TE) with two/more measurements

$$-\nabla\cdot\gamma(x)\nabla u+\sigma(x)u=0 \text{ in } X,\quad u=g \text{ on } \partial X,\quad H(x)=\Gamma(x)\sigma(x)u(x).$$

Let (g_1,g_2) providing (H_1,H_2). Define $\beta=H_1^2\nabla\frac{H_2}{H_1}$. **IF:** $|\beta|\geq c_0>0$, then

Theorem[B.-Uhlmann'10, B.-Ren'11] (i) (H_1,H_2) uniquely determine the whole measurement operator $g\in H^{\frac{1}{2}}(\partial X)\mapsto\mathcal{H}(g)=H\in H^1(X)$.
(ii) The measurement operator $\mathcal{H}$ uniquely determines

$$\chi(x):=\frac{\sqrt{\gamma}}{\Gamma\sigma}(x),\qquad q(x):=-\Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}+\frac{\sigma}{\gamma}\Big)(x).$$

(iii) (χ,q) uniquely determine (H_1,H_2).

Two well-chosen measurements suffice to reconstruct (χ,q) and thus (γ,σ,Γ) up to transformations leaving (χ,q) invariant.

Quantitative PAT, transport, and diffusion

The proof of (i) & (ii) is based on the *elimination* of σ to get

$$-\nabla\cdot\chi^2\Big[H_1^2\nabla\frac{H}{H_1}\Big]=0 \text{ in } X \quad (\chi,H) \text{ known on } \partial X.$$

$$\text{Then we verify that } q:=-\Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}+\frac{\sigma}{\gamma}\Big)(x)=-\frac{\Delta(\chi H_1)}{\chi H_1}.$$

$$\text{(iii) Finally, define } (\Delta+q)v_j=0 \text{ to get } H_j=\frac{v_j}{\chi}.$$

The **IF** implies that **vector field** $H_1^2\nabla\frac{u_2}{u_1}\neq 0$. This is a qualitative statement on the absence of critical points of elliptic solutions.

Theorem[B.-Ren'11] When *one* coefficient in (γ,σ,Γ) is known, then the other two are **uniquely** determined by the two measurements (H_1,H_2).

Stability of the reconstruction

Assuming **IF** satisfied, then the reconstruction of (e.g.) χ is **stable**.

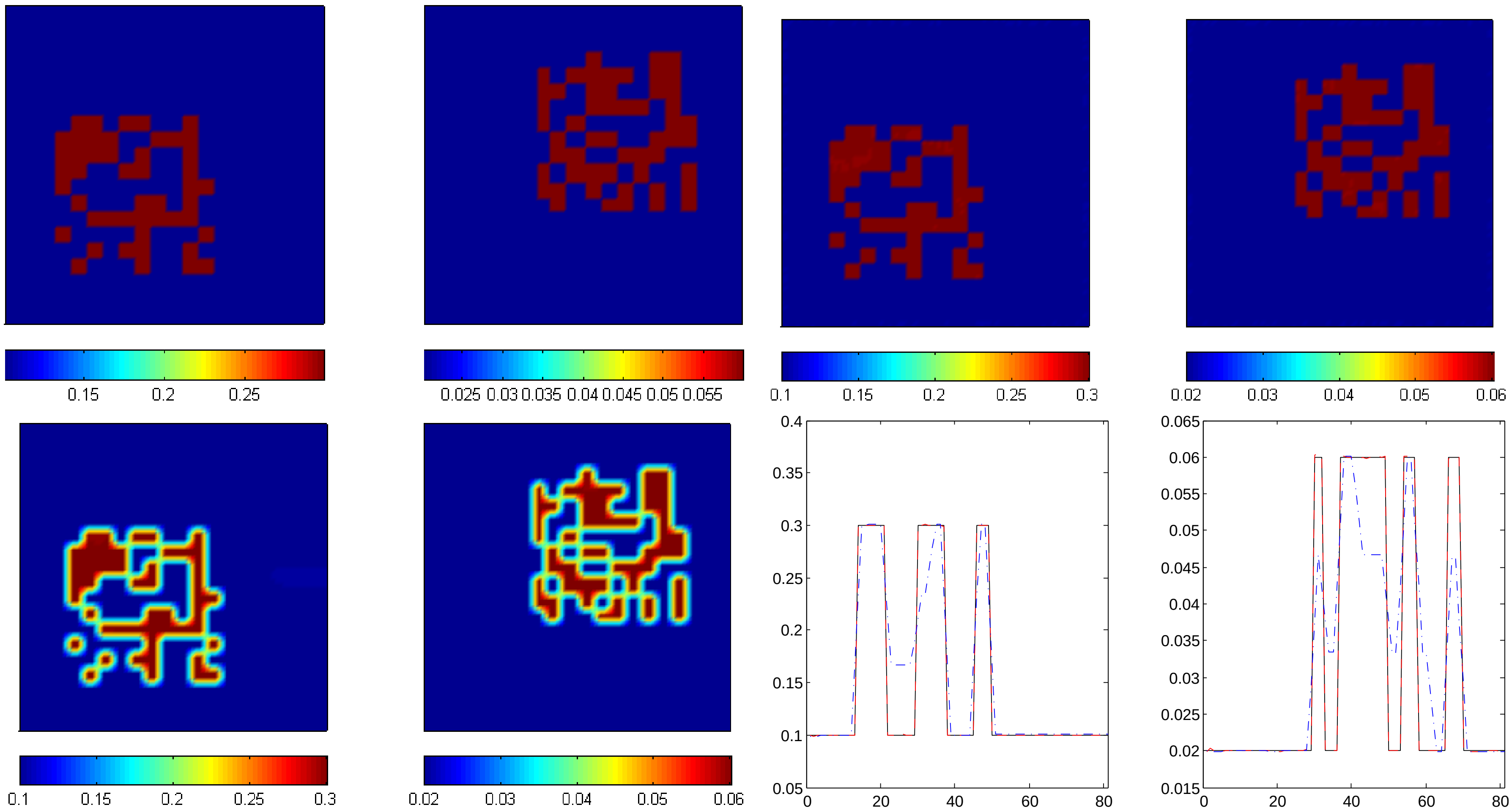
CGO method. Analyzing the transport equation by the method of characteristics and using CGO solutions, we show that for appropriate illuminations (and for $k \geq 3$):

$$\|\chi - \tilde{\chi}\|_{C^{k-1}(X)} \leq C\|H - \tilde{H}\|_{(C^k(X))^2}.$$

Transport method. Analyzing the transport equation directly and the renormalization property ($\varphi(\rho)$ satisfies a transport equation when ρ does) we obtain under appropriate regularity assumptions that

$$\|\chi - \tilde{\chi}\|_{L^\infty(X)} \leq C\|H - \tilde{H}\|_{(L^{\frac{p}{2}}(X))^2}^{\frac{p}{3(n+p)}}, \qquad \text{for all } 2 \leq p < \infty.$$

Reconstruction of two discontinuous parameters



Stability result for QTAT

$$\Delta u + k^2 u + i\sigma(x) u = 0 \text{ in } X, \quad u = g \text{ on } \partial X, \quad H(x) = \sigma(x)|u|^2.$$

Theorem [B.,Ren,Uhlmann,Zhou'11] Let σ and $\tilde{\sigma}$ be uniformly bounded functions in $Y = H^p(X)$ for $p > n$ with X the bounded support of the unknown conductivity.

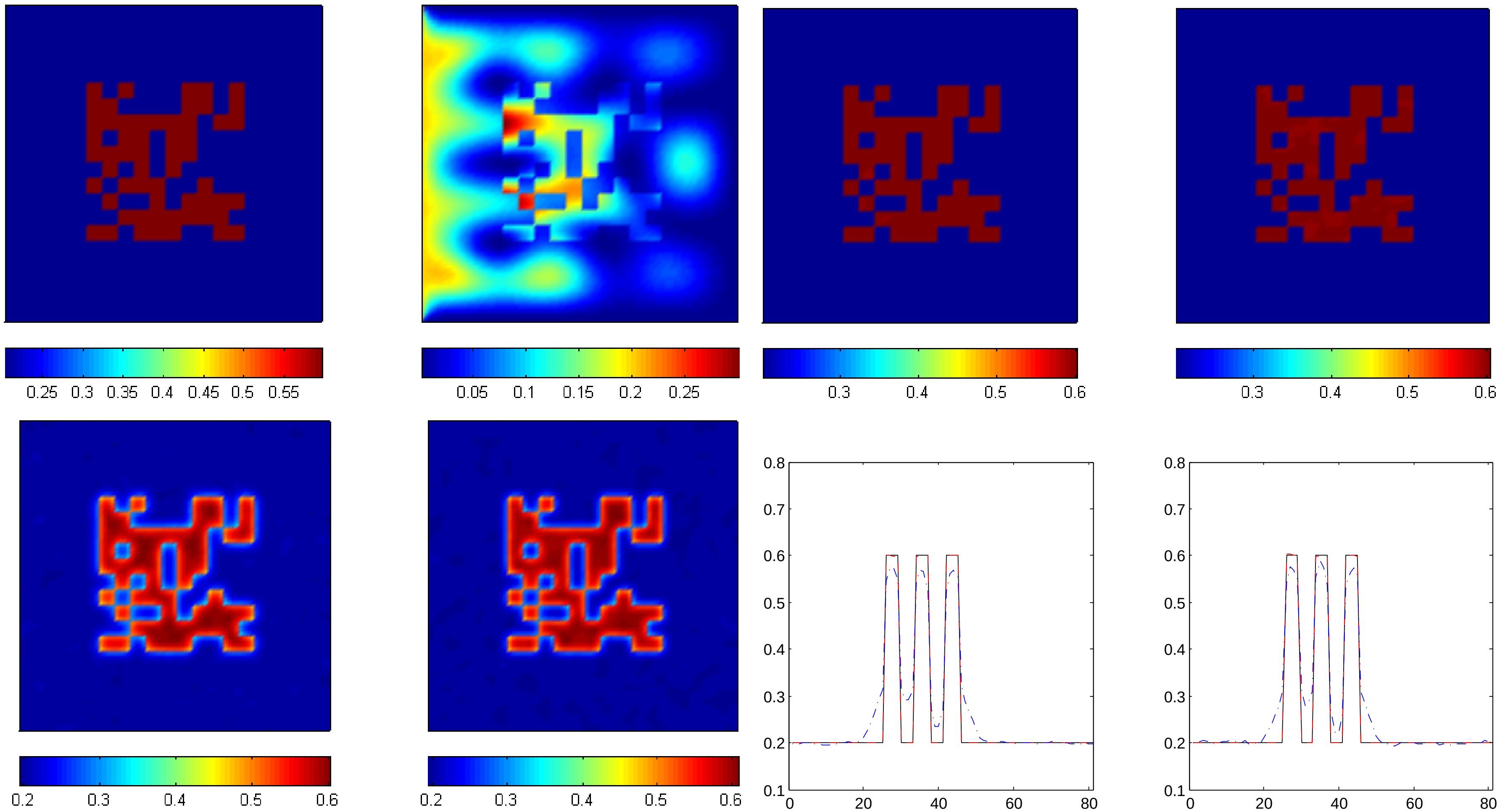
Then there is an **open set of illuminations** g such that

$$H(x) = \tilde{H}(x) \text{ in } Y \qquad \text{implies that} \qquad \sigma(x) = \tilde{\sigma}(x) \text{ in } Y.$$

Moreover, there exists C such that $\boxed{\|\sigma - \tilde{\sigma}\|_Y \le C\|H - \tilde{H}\|_Y.}$

The **inverse scattering problem with internal data** is **well posed**. We apply a Banach fixed point **IF** appropriate functional is a contraction.

Discontinuous conductivity in TAT



Nonlinear PDEs and non-uniqueness in UMOT

$$\Delta u = \sigma u \text{ in } X, \quad u = g \text{ on } \partial X, \qquad H(x) = \sigma(x) u^2(x).$$

As for many hybrid inverse problems, we can *eliminate* σ and recast HIP as the nonlinear PDE $\boxed{u\Delta u = H(x) \text{ in } X, \quad u = g \text{ on } \partial X.}$

Theorem [B-Ren'11]. The solution to this semilinear equation is **not unique** in general. (Similar to Ambrosetti-Prodi theory.)

Take $\sigma(x)$ such that $\Delta\phi + \sigma\phi = 0$ in X, $\phi = 0$ on ∂X, $\phi \not\equiv 0$. Define

$$\sigma_\delta(x) = \sigma(x)\frac{u - \delta\phi}{u + \delta\phi}, \ 0 < \delta < \delta_0. \qquad \text{Then } \ H_\delta = \sigma_\delta u_\delta^2 = \sigma_{-\delta} u_{-\delta}^2 = H_{-\delta}.$$

UMEIT and the 0−Laplacian

$$-\nabla\cdot\gamma(x)\nabla u = 0 \text{ in } X, \quad u = g \text{ on } \partial X, \qquad H(x) = \gamma(x)|\nabla u|^2(x).$$

The elimination of γ is straightforward and yields the 0-Laplace equation

$$-\nabla\cdot\frac{H(x)}{|\nabla u|^{2-p}}\nabla u = 0 \text{ in } X, \qquad u = g \text{ on } \partial X, \qquad p = 0.$$

For $1 < p < \infty$, the above problem is **elliptic** and associated to the strictly convex functional $J(x) = \int_X H(x)|\nabla u|^p dx$. When $p = 1$ (with applications in the HIP: CDII and MREIT), the problem is **degenerate elliptic**.

For $p < 1$, or $p = 0$ as in UMEIT, the problem is **hyperbolic**. We thus modify the HIP. and assume that the current $j = \partial_\nu u$ is also known. This becomes a **0-Laplacian with Cauchy data**.

Nonlinear Hyperbolic Problem

The above equation may be transformed as

$$(I-2\widehat{\nabla u}\otimes\widehat{\nabla u}):\nabla^2 u+\nabla \ln H\cdot\nabla u=0 \text{ in } X, \qquad u=f \text{ and } \frac{\partial u}{\partial\nu}=j \text{ on } \partial X.$$

Here $\widehat{\nabla u}=\frac{\nabla u}{|\nabla u|}$. With

$$g^{ij}=g^{ij}(\nabla u)=-\delta^{ij}+2(\widehat{\nabla u})_i(\widehat{\nabla u})_j \quad \text{and} \quad k^i=-(\nabla \ln H)_i,$$

the above equation is in coordinates

$$g^{ij}(\nabla u)\partial^2_{ij}u+k^i\partial_i u=0 \text{ in } X, \qquad u=f \text{ and } \frac{\partial u}{\partial\nu}=j \text{ on } \partial X.$$

Here g^{ij} is a definite matrix of signature $(1,n-1)$ so that we have quasilinear strictly hyperbolic equation with $\widehat{\nabla}u(x)$ the "time" direction. Stable Cauchy data must be on "space-like" part of ∂X for the *metric* g.

Stability on domain of influence

Let u and $\tilde{u}$ be two solutions of the hyperbolic equation and $v = u - \tilde{u}$.

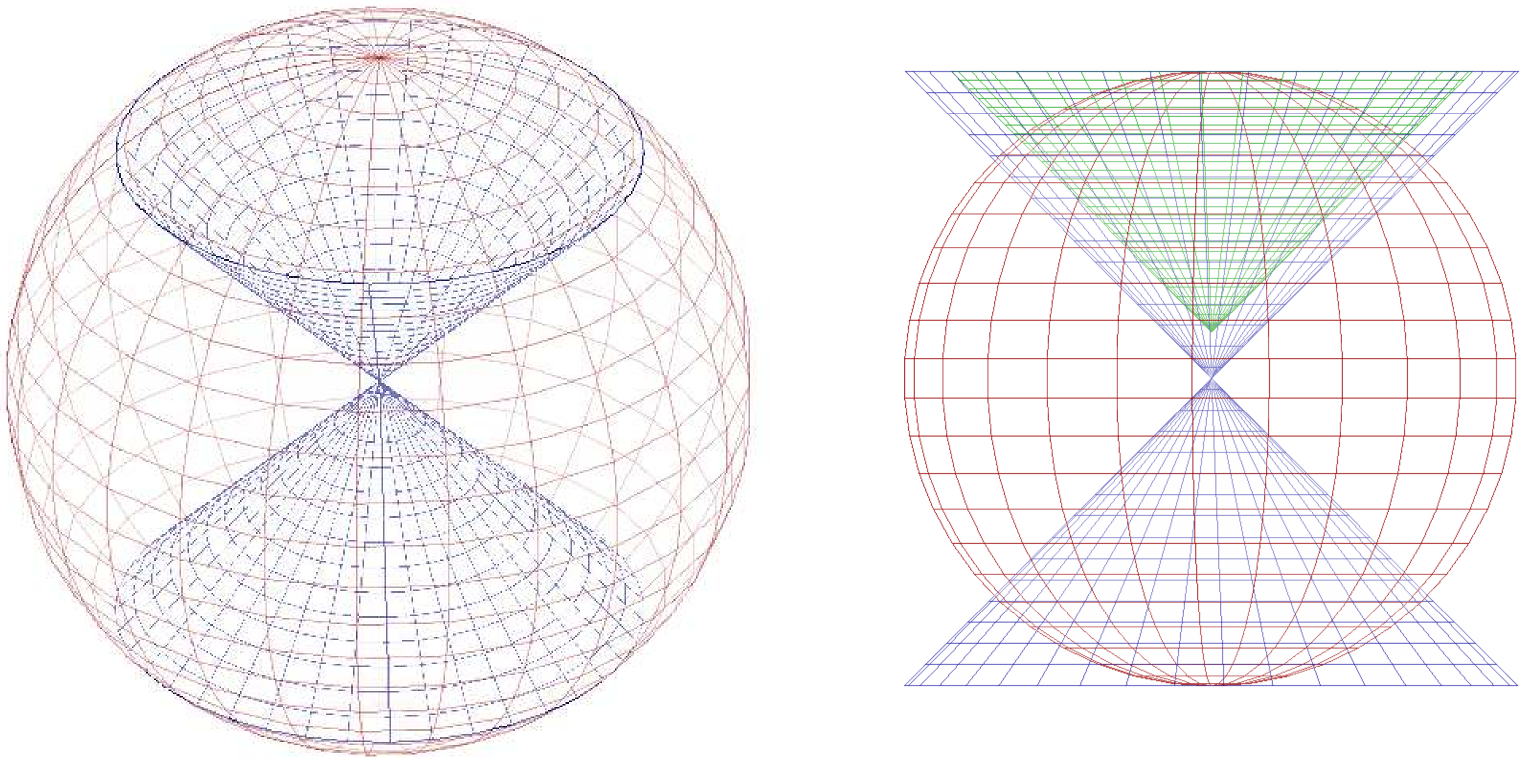
IF (appropriate) Lorentzian metric is strictly hyperbolic, then:

Theorem [B.'11]. Let $\Sigma_1 \subset \Sigma_g$ the space-like component of ∂X and $\mathcal{O}$ the **domain of influence** of Σ_1. For θ the distance of $\mathcal{O}$ to the boundary of the domain of influence of Σ_g, we have the local stability result:

$$\int_{\mathcal{O}} |v^2| + |\nabla v|^2 + (\gamma - \tilde{\gamma})^2 \, dx \leq \frac{C}{\theta^2} \Big(\int_{\Sigma_1} |f - \tilde{f}|^2 + |j - \tilde{j}|^2 \, d\sigma + \int_{\mathcal{O}} |\nabla \delta H|^2 \, dx \Big),$$

where $\gamma = \frac{H}{|\nabla u|^2}$ and $\tilde{\gamma} = \frac{\tilde{H}}{|\nabla \tilde{u}|^2}$ are the reconstructed conductivities.

Domain of Influence



Domain of influence (blue) for metric $g = 2e_z \otimes e_z - I$ on sphere (red). Null-like vectors (surface of cone) generate instabilities. Right: Sphere (red), domains of uniqueness (blue) and with controlled stability (green).

Multiple Measurement UMEIT

$$-\nabla\cdot\gamma(x)\nabla u_j = 0 \ X, \quad u_j = g_j \ \partial X, \qquad H_{ij}(x) = \gamma(x)\nabla u_i\cdot\nabla u_j(x), \ 1 \le i,j \le J.$$

Global reconstructions in UMEIT are obtained by acquiring **redundant** *internal functionals* $H_{ij} = S_i \cdot S_j(x)$ with $S_i(x) = \sqrt{\gamma}\,\nabla u_i(x)$. Then

$$\nabla \cdot S_j = -F \cdot S_j, \qquad dS_j^\flat = F^\flat \wedge S_j^\flat, \qquad 1 \le j \le J, \qquad F = \nabla(\log\gamma).$$

Strategy: (i) *Eliminate* F and find closed-form equation for $S = (S_1|\dots|S_n)$ or equivalently for the $SO(n;\mathbb{R})$-valued matrix $R = H^{-\frac{1}{2}}(S_1|\dots|S_n)$.
(ii) Solve for the redundant system of ODEs for S or R.

Works **IF** H is invertible in $\mathcal{M}(n;\mathbb{R})$, i.e., $\det(\nabla u_1,\dots,\nabla u_n) \neq 0$. This qualitative property on elliptic solutions holds for well-chosen $\{g_j\}$.

Elimination and system of ODEs in UMEIT

Lemma [B.-Bonnetier-Monard-Triki'11; Monard-B.'11]. Let $\Omega \subset X$.
IF $\inf_{x\in\Omega} \det(S_1(x),\dots,S_n(x)) \geq c_0 > 0$, then with $D(x) = \sqrt{\det H(x)}$,

$$F(x) = \frac{1}{nD}\sum_{i,j=1}^{n} \Big(\nabla(DH^{ij}) \cdot S_i(x)\Big) S_j(x), \qquad H^{-1} = (H^{ij})_{i,j}.$$

Theorem [idem]. There exists an open set of illuminations g_j for $J = n$ in even dimension and $J = n+1$ in odd dimension such that for γ and γ' the conductivities corresponding to H and H', we have the following **global stability** result:

$$\|\log\gamma - \log\gamma'\|_{W^{1,\infty}(X)} \leq C\Big(\varepsilon_0 + \|H - H'\|_{W^{1,\infty}(X)}\Big)$$
$$\varepsilon_0 = |\log\gamma(x_0) - \log\gamma'(x_0)| + \textstyle\sum_{i=1}^{J} \|S_i - S_i'\|.$$

The IFs and the CGOs

Several HIPs require to verify qualitative properties of elliptic solutions:

- the absence of critical points in QPAT (and ET and UMOT)
- the contraction of appropriate functionals in QTAT
- the hyperbolicity of a given Lorentzian metric in UMOT
- the linear independence of gradients of elliptic solutions in UMOT.

In dimension $n = 2$, critical points of elliptic solutions are *isolated*. This greatly simplifies the analysis of the above statements.

In dimension $n \geq 3$, the existence of open sets of **illuminations** g_j such that these properties hold is obtained by means of **CGO** solutions.

Vector fields and complex geometrical optics

• Take $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$. Then $\Delta e^{\rho \cdot x} = 0$. For $u_j = e^{\rho_j \cdot x}$, $j = 1, 2$:

$$\Im\Big(e^{-(\rho_1+\rho_2)\cdot x} u_1^2 \nabla \frac{u_2}{u_1}\Big) = \Im(\rho_2 - \rho_1),$$

is a constant vector field $2k$ for $\rho_1 = k + ik^\perp$ and $\rho_2 = \bar\rho_1$.

• Let $u_\rho(x) = e^{\rho\cdot x}\big(1 + \psi_\rho(x)\big)$ solution of $\Delta u_\rho + q u_\rho = 0$.

Theorem[B.-Uhlmann'10]. For q sufficiently smooth and $k \geq 0$, we have

$$|\rho| \|\psi_\rho\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} + \|\psi_\rho\|_{H^{\frac{n}{2}+k+1+\varepsilon}(X)} \leq C \|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)}.$$

• For **illuminations** g on ∂X close to *traces of* **CGO** *solutions* constructed in $\mathbb{R}^d$, we obtain "nice" vector fields $|\beta| \geq c_0 > 0$ and thus an open set of illuminations g for which stable reconstructions are guaranteed.

Conclusions

• Mathematically, many **hybrid imaging modalities** are stable inverse problems combining **high resolution** with **high contrast**.

• **Explicit** reconstructions for **one** or **several** coefficients are obtained by solving linear or nonlinear **transport**, **elliptic**, or **hyperbolic** equations or by using *Banach fixed point*. **Non-uniqueness** results exist.

• Reconstructions require qualitative properties of elliptic solutions. These properties hold true for appropriate **illuminations** constructed by means of **Complex Geometric Optics** solutions.