Stability of time reversed waves in changing media

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Abstract

We analyze the refocusing properties of time reversed waves that propagate in two different media during the forward and backward stages of a time-reversal experiment. We consider two regimes of wave propagation modeled by the paraxial wave equation with a smooth random refraction coefficient and the Itô-Schrödinger equation, respectively. In both regimes, we rigorously characterize the refocused signal in the high frequency limit and show that it is statistically stable, that is, independent of the realizations of the two media. The analysis is based on a characterization of the high frequency limit of the Wigner transform of two fields propagating in different media.

The refocusing quality of the back-propagated signal is determined by the cross correlation of the two media. When the two media decorrelate, two distinct de-focusing effects are observed. The first one is a purely absorbing effect due to the loss of coherence at a fixed frequency. The second one is a phase modulation effect of the refocused signal at each frequency. This causes de-focusing of the back-propagated signal in the time domain.

1 Introduction

The refocusing of back-propagated pulses in time-reversal experiments has attracted a lot of attention recently both in the physical and mathematical literatures; see [4, 6, 7, 10, 12, 13, 16, 19] and their references. A time reversal experiment consists of two stages. In the first stage, a signal is sent from a localized source term to an array of receiver-transducers that record the signal in time. In the second stage, the signal is time reversed and re-emitted into the medium, that is, the part that is recorded first is sent back last and vice versa. It has been observed experimentally and justified theoretically that the back-propagated signal refocuses much more tightly at the location of the original source when propagation occurs in a highly heterogeneous medium rather than in a homogeneous medium. Moreover, the shape of the back-propagated signal does not depend, under appropriate assumptions, on the realization of the underlying medium if it is modeled as a random medium.

In order to obtain a tight refocusing, it is important that the underlying media do not change during the two stages of the time reversal experiment. Several experimental studies have demonstrated that the refocusing of time reversed waves degrades as the back-propagating medium is modified [19, 25]. The modifications in the refocusing properties have been analyzed in [8] in the weak coupling regime based on the formal theories of radiative transfer and diffusion equations for time reversed waves propagating in random media [7]. They have also been rigorously analyzed in the one-dimensional setting [1] in the regime of strong fluctuations and wave localization. It has been

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shown in [1] that the re-propagated signal is both not as tightly focused and no longer statistically stable when the two media are different in the one-dimensional case.

Here, we consider time reversal in changing media for two models of wave propagation: the paraxial regime and its white noise limit. These regimes model multi-dimensional propagation of wave pulses with beam-like structure so that backscattering in the main direction of propagation of the beam can be neglected. Time reversed waves in these regimes have been analyzed in [3, 6, 22]. We characterize the modifications incurred in the radiative transfer equations modeling time reversal as the medium of back-propagation changes. They are described in terms of the cross-correlation of the two media of propagation and are similar to those derived formally in [8]. We also show that the back-propagated signal is still statistically stable, that is, independent of the realizations of the random media provided that the correlation functions remain the same. This is similar to what was obtained in [6, 15, 22] in the case when the two media are identical and is consistent with the numerical simulations in [8]. This contrasts, however, with the results obtained in the localization regime in [1], where statistical instability has been demonstrated in one dimension.

As in the pioneering paper on multi-dimensional time reversal [12], the characterization of the back-propagated signal in the high frequency limit is carried out by analyzing the correlation function and the Wigner transform of two wave fields. The main novelty is that we now consider the Wigner transform of two fields propagating in two different media [18, 23]. Time reversal is the first application where such correlations seem to be of a practical interest. Our theoretical analysis is very similar to that in [6] and is based on the construction of approximate martingales and perturbed test functions.

The rest of the paper is organized as follows. Section 2 presents the equations modeling time reversal in changing media in the paraxial regime. The main results on the characterization of the time reversed signal in the paraxial regime are given in Section 3. The theory in the Itô-Schrödinger regime is carried out in Section 4. In both cases, we observe that the focusing of the back-propagated signal at the original source location deteriorates as the cross-correlation of the two media decreases. This de-correlation is analyzed in detail in Section 5. Section 6 offers some concluding remarks.

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2 Two-media Time reversal in the paraxial regime

In this section, we generalize the time reversal setting presented in [6] to the situation where the media differ during the forward and backward propagation stages.

2.1 Paraxial wave equation and scaling

Propagation of acoustic waves is described by the scalar wave equation for the pressure field $p(z, \mathbf{x}, t)$

$$\frac{1}{c^2(z, \mathbf{x})} \frac{\partial^2 p}{\partial t^2} - \Delta p = 0. \tag{1}$$

Here, $c(z, \mathbf{x})$ is the local wave speed, which we model as a random process, and the Laplacian Δ is both in the direction of propagation z and the transverse variable $\mathbf{x} \in \mathbb{R}^d$. The physical dimension is $d = 2$ although out theory applies to any $d \geq 1$. The wave speed $c(z, \mathbf{x})$ is different during the forward and backward propagation stages of time reversal.

The paraxial (or parabolic) approximation of wave propagation consists of assuming that the wave field has a "beam-like" structure in the z direction and that back scattering in the z direction can be neglected [24]. This implies the approximation

$$p(z, \mathbf{x}, t) \approx \int_{\mathbb{R}} e^{ik(z-c_0 t)} \psi(z, \mathbf{x}, k) c_0 dk, \tag{2}$$

where the function ψ satisfies the Schrödinger equation

$$\begin{aligned} &2ik \frac{\partial \psi}{\partial z}(z, \mathbf{x}, k) + \Delta_{\mathbf{x}} \psi(z, \mathbf{x}, k) + k^2 (n^2(z, \mathbf{x}) - 1) \psi(z, \mathbf{x}, k) = 0, \\ &\psi(z = 0, \mathbf{x}, k) = \psi_0(\mathbf{x}, k) \end{aligned} \tag{3}$$

and $\Delta_{\mathbf{x}}$ is the Laplacian in the variable $\mathbf{x}$. We have defined the refraction index as $n(z, \mathbf{x}) = c_0/c(z, \mathbf{x})$ where c_0 is a reference speed. Note that (3) is an initial value problem in the z-variable. Theoretical justifications of the passage from the wave equation to the parabolic approximation can be found in [2, 9].

We analyze the high frequency regime, where waves undergo multiple interactions with the inhomogeneous medium and wave propagation may be described by macroscopic equations in appropriate limits. To quantify these limits, we introduce some scaling parameters. Let L_x and L_z be the overall propagation distances. We re-scale $\mathbf{x}$ and z as $L_x \mathbf{x}$ and $L_z z$ with the new $\mathbf{x}$ and z being non-dimensional $O(1)$ quantities. In order for the paraxial approximation (3) to be valid one has to assume that $L_x \ll L_z$.

Let l_x and l_z, be the transversal and longitudinal correlation lengths of the heterogeneous medium. Upon recasting the refraction index as

$$n^2(z, \mathbf{x}) - 1 = -2\sigma V(\frac{z}{l_z}, \frac{\mathbf{x}}{l_x}), \tag{4}$$

the above equation (3) becomes in the re-scaled variables

$$\frac{2ik}{L_z} \frac{\partial \psi}{\partial z} + \frac{1}{L_x^2} \Delta_{\mathbf{x}} \psi - 2k^2 \sigma V(\frac{L_z z}{l_z}, \frac{L_x \mathbf{x}}{l_x}) \psi = 0. \tag{5}$$

Let us now assume that the medium and the typical wavelength of the propagating waves satisfy the following scaling assumptions:

$$\varepsilon = \frac{l_x}{L_x} = \frac{l_z}{L_z} \ll 1, \qquad k L_z = \frac{\kappa}{\varepsilon} \Big(\frac{L_z}{L_x} \Big)^2, \qquad \sigma = \sqrt{\varepsilon} \frac{L_x}{L_z}. \tag{6}$$

These constraints imply that we are in the high frequency regime when the non-dimensional wave number κ is of order $O(1)$. Note that there is one free parameter left in the above relations, namely

$$\frac{L_x}{L_z} = \varepsilon^\eta, \qquad \eta > 0, \tag{7}$$

where $\eta > 0$ is necessary to be compatible with the paraxial approximation and to ensure that $L_x \ll L_z$. The relations (6) quantify how the correlation length and the strength of the fluctuations are related so that the parabolic wave equation (5) in the radiative transfer scaling is given by

$$i\kappa\varepsilon \frac{\partial \psi}{\partial z} + \frac{\varepsilon^2}{2} \Delta_{\mathbf{x}} \psi - \kappa^2 \sqrt{\varepsilon} V\Big(\frac{z}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}\Big) \psi = 0. \tag{8}$$

The above equation is our model for wave propagation in this section. We will see a different scaling in Section 4. This equation is a Schrödinger equation with "time"-dependent potential, as the potential depends here also on the variable z.

The above choice of scaling implies that

$$\frac{l_x}{l_z} = \frac{L_x}{L_z} = \varepsilon^\eta \ll 1, \tag{9}$$

so that the medium is physically anisotropic: fluctuations in the longitudinal and transversal directions are not defined at the same scale. Only in the limit $L_x/L_z \to 1$, i.e., $\eta \to 0$ do we recover a statistically isotropic medium. This limit, which is more relevant in many practical problems, is much more difficult to handle mathematically [7, 23]. The paraxial approximation in the radiative transfer regime presented in this section shares most of the physical aspects of the isotropic model and is much more amenable to a rigorous mathematical treatment.

2.2 Time reversal modeling

We are interested in the refocusing properties of tightly localized pulses. We assume that the center of our pulse is a point $\mathbf{x}_0$ and that its spatial width is ε, so that the typical wavelength in the system is ε. We thus scale our initial condition for the Schrödinger equation as

$$\psi(z=0,\mathbf{x},\kappa) = \psi_0\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon},\kappa\Big). \tag{10}$$

During the forward propagation phase, we assume that the medium is described by fluctuations $V_1(z,\mathbf{x})$. The Green function associated to (8) is then the unique solution to

$$\begin{aligned} &i\kappa\varepsilon\frac{\partial G_f(z,\mathbf{x},\kappa;\mathbf{y})}{\partial z} + \frac{\varepsilon^2}{2}\Delta_{\mathbf{x}} G_f(z,\mathbf{x},\kappa;\mathbf{y}) - \kappa^2\sqrt{\varepsilon}V_1\Big(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\varepsilon}\Big)G_f(z,\mathbf{x},\kappa;\mathbf{y}) = 0 \\ &G_f(0,\mathbf{x},\kappa;\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{11}$$

Let us assume that waves propagate for a distance $z = L = c_0 T$ along the z axis, or equivalently for a time T. The solution at $z = L$ is given by

$$\psi_-(L,\mathbf{x},\kappa) = \int_{\mathbb{R}^d} G_f(L,\mathbf{x},\kappa;\mathbf{y})\psi_0\Big(\frac{\mathbf{y}-\mathbf{x}_0}{\varepsilon},\kappa\Big)d\mathbf{y}. \tag{12}$$

The signal is then recorded on a domain of small (but of order $O(1)$) aperture – this is modeled by multiplication of the signal by a compactly supported function $\chi(\mathbf{x})$. We also allow for some blurring at the detectors so that the re-emitted signal after time reversal is given by

$$\psi_+(L,\mathbf{x},\kappa) = \chi(\mathbf{x})\int_{\mathbb{R}^d} \varepsilon^{-d} f(\frac{\mathbf{x}-\mathbf{y}}{\varepsilon})\chi(\mathbf{y})\psi_-^*(L,\mathbf{y},\kappa)d\mathbf{y}. \tag{13}$$

Here * denotes complex conjugation and corresponds to time reversal. Indeed, the time reversal $t \to -t$ in the time domain amounts to complex conjugation $e^{i\omega t} \to e^{-i\omega t}$ in the frequency domain. The blurring must be controlled at the scale of the wavelength ε for otherwise all the coherent signal would be irretrievably lost. The case with no blurring is modeled by $f(\mathbf{x}) = \delta(\mathbf{x})$. Note that $f(\mathbf{x})$ will be required to be smoother than the $\delta(\mathbf{x})$-function in what follows.

It now remains to model back-propagation to the hyperplane $z = 0$, that is, again for a duration T. The back-propagation takes place in a different medium described by the random potential $V_2(z,\mathbf{x})$ whose Green's function satisfies

$$\begin{aligned} &i\kappa\varepsilon\frac{\partial G_b(z,\mathbf{x},\kappa;\mathbf{y})}{\partial z} + \frac{\varepsilon^2}{2}\Delta_{\mathbf{x}} G_b(z,\mathbf{x},\kappa;\mathbf{y}) - \kappa^2\sqrt{\varepsilon}V_2\Big(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\varepsilon}\Big)G_b(z,\mathbf{x},\kappa;\mathbf{y}) = 0 \\ &G_b(0,\mathbf{x},\kappa;\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{14}$$

After back-propagation for a distance L along the z-axis and a second time reversion (complex conjugation in the frequency domain) we obtain that the re-propagated signal takes the form

$$\tilde{\psi}^B(\mathbf{x},\kappa) = \int_{\mathbb{R}^d} G_b^*(L,\mathbf{x},\kappa;\mathbf{y})\psi_+^*(L,\mathbf{y},\kappa)d\mathbf{y}. \tag{15}$$

The second conjugation is performed so that when full measurements are available, that is, $\chi \equiv 1$, and the detectors are perfect, so that $f(\mathbf{x}) = \delta(\mathbf{x})$, we recover the original signal exactly: $\tilde{\psi}^B(\mathbf{x},\kappa) = \psi(z=0,\mathbf{x},\kappa)$.

We are interested in the back-propagated signal in the vicinity of $\mathbf{x}_0$ and define

$$\psi_\varepsilon^B(\boldsymbol{\xi},\kappa;\mathbf{x}_0) = \tilde{\psi}^B(\mathbf{x}_0+\varepsilon\boldsymbol{\xi},\kappa). \tag{16}$$

Summarizing the successive steps described above, we can relate the back-propagated signal to the initial signal as

$$\psi_\varepsilon^B(\boldsymbol{\xi},\kappa;\mathbf{x}_0) = \int_{\mathbb{R}^{3d}} G_b^*(L,\mathbf{x}_0+\varepsilon\boldsymbol{\xi},\kappa;\boldsymbol{\eta})G_f(L,\mathbf{x}_0+\varepsilon\boldsymbol{\zeta},\kappa,\mathbf{y})\chi(\boldsymbol{\eta},\mathbf{y})\psi_0(\boldsymbol{\zeta},\kappa)d\boldsymbol{\zeta}d\boldsymbol{\eta}d\mathbf{y}, \tag{17}$$

where we have used that $G(L,\mathbf{x},\kappa;\mathbf{y}) = G(L,\mathbf{y},\kappa;\mathbf{x})$ as can be seen from the equation satisfied by the Green function and where we have defined

$$\chi(\boldsymbol{\eta},\mathbf{y}) = \chi(\boldsymbol{\eta})\chi(\mathbf{y})f\Big(\frac{\boldsymbol{\eta}-\boldsymbol{\zeta}}{\varepsilon}\Big) = \chi(\boldsymbol{\eta})\chi(\boldsymbol{\zeta})\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{f}(\mathbf{q})e^{i\boldsymbol{\eta}\cdot\mathbf{q}/\varepsilon}e^{-i\boldsymbol{\zeta}\cdot\mathbf{q}/\varepsilon}d\mathbf{q}. \tag{18}$$

The above notation implicitly defines our convention for the Fourier transform $\hat{f}(\mathbf{q})$ of $f(\mathbf{x})$. We observe that the back-propagated signal in (17) involves the product of two Green's functions at nearby points. The Wigner transform is thus a very natural tool to understand the statistical properties of this two point correlation [18, 23]. Following [6, 7] we introduce the functions $Q_{f,b}$ as

$$Q_{f,b}(L,\mathbf{x},\kappa;\mathbf{q}) = \int_{\mathbb{R}^d} G_{f,b}(L,\mathbf{x},\kappa;\mathbf{y})\chi(\mathbf{y})e^{-i\mathbf{q}\cdot\mathbf{y}/\varepsilon}d\mathbf{y}, \tag{19}$$

which solve the initial value problems

$$\begin{aligned} &i\varepsilon\kappa\frac{\partial Q_{f,b}}{\partial z}(z,\mathbf{x},\kappa;\mathbf{q}) + \frac{\varepsilon^2}{2}\Delta_{\mathbf{x}}Q_{f,b}(z,\mathbf{x},\kappa;\mathbf{q}) - \kappa^2\sqrt{\varepsilon}V_{f,b}\left(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\varepsilon}\right)Q_{f,b}(z,\mathbf{x},\kappa;\mathbf{q}) = 0,\\ &Q_{f,b}(z=0,\mathbf{x},\kappa;\mathbf{q}) = \chi(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}/\varepsilon}. \end{aligned} \tag{20}$$

We then define the Wigner measure W_ε as

$$W_\varepsilon(z,\mathbf{x},\mathbf{k},\kappa) = \int_{\mathbb{R}^d}\hat{f}(\mathbf{q})U_\varepsilon(z,\mathbf{x},\mathbf{k},\kappa;\mathbf{q})d\mathbf{q}, \tag{21}$$

where U_ε is the Wigner transform of the auxiliary functions $Q_{f,b}$ defined by

$$U_\varepsilon(z,\mathbf{x},\mathbf{k},\kappa;\mathbf{q}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}Q_f(z,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2},\kappa;\mathbf{q})Q_b^*(z,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},\kappa;\mathbf{q})\frac{d\mathbf{y}}{(2\pi)^d}. \tag{22}$$

The main reason for introducing the above notation is that the back-propagated signal can be recast in terms of the Wigner measure as

$$\psi_\varepsilon^B(\boldsymbol{\xi},\kappa;\mathbf{x}_0) = \int_{\mathbb{R}^{2d}} e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})}W_\varepsilon(L,\mathbf{x}_0+\varepsilon\frac{\mathbf{y}+\boldsymbol{\xi}}{2},\mathbf{k},\kappa)\psi_0(\mathbf{y},\kappa)\frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^d}. \tag{23}$$

Thus in order to understand the macroscopic properties of the time reversed signal ψ_ε^B in the high frequency limit, i.e., as $\varepsilon \to 0$, it suffices to analyze the Wigner measure W_ε in the same limit. This task is taken up in the following section.

3 Stability of waves in changing environment

3.1 The main result

We consider in this section the general problem of the correlation of solutions of the linear paraxial Schrödinger equations in two different albeit correlated random media. We let $\psi_\varepsilon(z,\mathbf{x})$ and $\phi_\varepsilon(z,\mathbf{x})$ be the solutions of the family of Cauchy problems

$$\begin{aligned}
&i\varepsilon\kappa\frac{\partial\psi_\varepsilon}{\partial z}+\frac{\varepsilon^2}{2}\Delta\psi_\varepsilon-\kappa^2\sqrt{\varepsilon}V_1\left(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\varepsilon}\right)\psi_\varepsilon=0\\
&\psi_\varepsilon(0,\mathbf{x})=\psi_\varepsilon^0(\mathbf{x};\zeta)
\end{aligned}\tag{24}$$

and

$$\begin{aligned}
&i\varepsilon\kappa\frac{\partial\phi_\varepsilon}{\partial z}+\frac{\varepsilon^2}{2}\Delta\phi_\varepsilon-\kappa^2\sqrt{\varepsilon}V_2\left(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\varepsilon}\right)\phi_\varepsilon=0,\\
&\phi_\varepsilon(0,\mathbf{x})=\phi_\varepsilon^0(\mathbf{x};\zeta)
\end{aligned}\tag{25}$$

with two different random potentials V_1 and V_2. The initial data depend on an additional random variable ζ defined over a state space S with a probability measure $d\varpi(\zeta)$. It accounts for the consideration of a mixture of states rather than the single solution of the Schrödinger equation. The mixture of states arises naturally in the time-reversal set-up, because of the integration over the wave vector $\mathbf{q}$ in (21). This introduces additional regularity into the problem, which is crucial to obtain statistical stability.

The cross Wigner transform is defined by

$$W_\varepsilon(z,\mathbf{x},\mathbf{k})=\int_{\mathbb{R}^d\times S}e^{i\mathbf{k}\cdot\mathbf{y}}\psi_\varepsilon\left(z,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2};\zeta\right)\bar{\phi}_\varepsilon\left(z,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2};\zeta\right)\frac{d\mathbf{y}}{(2\pi)^d}d\varpi(\zeta).$$

The evolution equation for the Wigner transform is

$$\frac{\partial W_\varepsilon}{\partial z}+\frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}}W_\varepsilon=\frac{\kappa}{i\sqrt{\varepsilon}}\int_{\mathbb{R}^d}e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon}\left[\tilde{V}_1\left(\frac{z}{\varepsilon},\mathbf{p}\right)W_\varepsilon\left(\mathbf{k}-\frac{\mathbf{p}}{2}\right)-\tilde{V}_2\left(\frac{z}{\varepsilon},\mathbf{p}\right)W_\varepsilon\left(\mathbf{k}+\frac{\mathbf{p}}{2}\right)\right]\frac{d\mathbf{p}}{(2\pi)^d}.\tag{26}$$

Here $\tilde{V}(z,\mathbf{p})$ is the partial Fourier transform of $V(z,\mathbf{x})$ in $\mathbf{x}$ only. We will assume that the initial data $W_\varepsilon(0,\mathbf{x},\mathbf{k})$ converges strongly in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ to a limit $W_0(\mathbf{x},\mathbf{k})$. This is possible due to the introduction of the mixture of states – the integration against the measure $\varpi(d\xi)$– although the Wigner transform of a pure state is not uniformly bounded in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ [20]. The evolution equation (26) preserves the L^2-norm so that in order to identify the limit of W_ε as $\varepsilon\to 0$, it suffices to consider initial data

$$W_\varepsilon(0,\mathbf{x},\mathbf{k})=W_0(\mathbf{x},\mathbf{k})\tag{27}$$

that are independent of the parameter ε. In the time reversal application, the initial condition for the Wigner transform is as follows:

$$\begin{aligned}
W_\varepsilon(0,\mathbf{x},\mathbf{k})&=\int_{\mathbb{R}^{2d}}e^{i\mathbf{k}\cdot\mathbf{y}+i\mathbf{q}\cdot\mathbf{y}}\hat{f}(\mathbf{q})\chi(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\chi(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\frac{d\mathbf{y}d\mathbf{q}}{(2\pi)^d}\\
&=\int_{\mathbb{R}^d}e^{-i\mathbf{k}\cdot\mathbf{y}}f(\mathbf{y})\chi(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\chi(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})d\mathbf{y}.
\end{aligned}\tag{28}$$

The limit as $\varepsilon\to 0$ is given by $W_0(\mathbf{x},\mathbf{k})=\hat{f}(\mathbf{k})\chi^2(\mathbf{x})$ for sufficiently smooth functions $f(\mathbf{x})$ and $\chi(\mathbf{x})$. We have the following result:

Lemma 3.1 *Let us assume that $f(\mathbf{x}) \in L^2(\mathbb{R}^d)$ and that $\chi(\mathbf{x}) \in L^4(\mathbb{R}^d)$. Then*

$$\lim_{\varepsilon\to 0} \|W_\varepsilon(0,\mathbf{x},\mathbf{k}) - W_0(\mathbf{x},\mathbf{k})\|_{L^2(\mathbb{R}^{2d})} = 0. \tag{29}$$

Proof. Let $\chi_n(\mathbf{x}) \in C_c(\mathbb{R}^d)$ be a sequence of compactly supported continuous functions converging to $\chi(\mathbf{x})$ so that $\|\chi - \chi_n\|_{L^4} \to 0$ as $n \to \infty$. Let us define $\phi_n = \chi - \chi_n$ and

$$I_n(\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{y}} f(\mathbf{y})\phi_n(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2})\chi(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2})d\mathbf{y}.$$

We verify that

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} d\mathbf{x}d\mathbf{k}|I_n(\mathbf{x},\mathbf{k})|^2 \\
&= \int_{\mathbb{R}^{4d}} d\mathbf{y}d\mathbf{y}_1 d\mathbf{x}d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{y}_1 - i\mathbf{k}\cdot\mathbf{y}} f(\mathbf{y})\overline{f(\mathbf{y}_1)}\phi_n(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2})\chi(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2})\phi_n(\mathbf{x} - \frac{\varepsilon\mathbf{y}_1}{2})\chi(\mathbf{x} + \frac{\varepsilon\mathbf{y}_1}{2}) \\
&= (2\pi)^d \int_{\mathbb{R}^{2d}} d\mathbf{x}d\mathbf{y}|f(\mathbf{y})|^2 \left|\phi_n(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2})\right|^2 \left|\chi(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2})\right|^2 \le 2(2\pi)^d \|f\|_{L^2}^2 \|\phi_n\|_{L^4}^2 \|\chi\|_{L^4}^2,
\end{aligned}$$

by the Cauchy-Schwarz inequality. Notice that the bound in independent of ε. This implies that for all $\eta > 0$ we can find n such that

$$\|W_\varepsilon(0,\mathbf{x},\mathbf{k}) - W_{\varepsilon n}(0,\mathbf{x},\mathbf{k})\|_{L^2(\mathbb{R}^{2d})}^2 + \|W_0(\mathbf{x},\mathbf{k}) - W_{0n}(\mathbf{x},\mathbf{k})\|_{L^2(\mathbb{R}^{2d})}^2 \le \eta,$$

uniformly in ε, where $W_{\varepsilon n}(0,\mathbf{x},\mathbf{k})$ and $W_{0n}(\mathbf{x},\mathbf{k})$ are defined as $W_\varepsilon(0,\mathbf{x},\mathbf{k})$ and $W_0(\mathbf{x},\mathbf{k})$, respectively, with $\chi(\mathbf{x})$ replaced by $\chi_n(\mathbf{x})$. The same calculation as above shows that

$$E_{n\varepsilon} = \|W_{\varepsilon n}(0,\mathbf{x},\mathbf{k}) - W_{0n}(\mathbf{x},\mathbf{k})\|_{L^2(\mathbb{R}^{2d})}^2 = \int_{\mathbb{R}^{2d}} d\mathbf{x}d\mathbf{y}|f(\mathbf{y})|^2 \Big|\chi_n(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2})\chi_n(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}) - \chi_n^2(\mathbf{x})\Big|^2.$$

Up to an error on $E_{n\varepsilon}$ bounded by η, we can replace $f(\mathbf{x})$ above by an approximation $f_n(\mathbf{x}) \in C_c(\mathbb{R}^d)$ by density. Now, the function $h_{n_\varepsilon}(\mathbf{x},\mathbf{y}) = |f_n(\mathbf{y})|^2 \big|\chi_n(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2})\chi_n(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}) - \chi_n^2(\mathbf{x})\big|^2$ converges to 0 pointwise in $\mathbb{R}^{2d}$. By the Lebesgue dominated convergence theorem, this implies that $E_{n\varepsilon} \to 0$ as $\varepsilon \to 0$. We thus deduce that

$$\overline{\lim_{\varepsilon\to 0}} \|W_\varepsilon(0,\mathbf{x},\mathbf{k}) - W_0(\mathbf{x},\mathbf{k})\|_{L^2(\mathbb{R}^{2d})}^2 \le 2\eta,$$

for all $\eta > 0$. This concludes the proof of the lemma. □

It remains to model the random potentials. We assume that the random processes $V_{1,2}(z)$ are statistically homogeneous in space $\mathbf{x}$ and "time" z, have mean zero and rapidly decaying correlation functions $R_{ij}(s,\mathbf{y})$:

$$\mathbb{E}\left\{V_i(z,\mathbf{x})\right\} = 0, \quad \mathbb{E}\left\{V_i(z+s,\mathbf{x}+\mathbf{y})V_j(z,\mathbf{x})\right\} = R_{ij}(s,\mathbf{y}), \quad i,j = 1,2.$$

We denote by $\hat{R}_{ij}(\omega,\mathbf{p})$ the corresponding power spectra:

$$\mathbb{E}\left\{\hat{V}_i(\omega,\mathbf{p})\hat{V}_j(\omega',\mathbf{q})\right\} = (2\pi)^{d+1}\hat{R}_{ij}(\omega,\mathbf{p})\delta(\omega+\omega')\delta(\mathbf{p}+\mathbf{q}), \quad \hat{R}_{ij}(\omega,\mathbf{p}) = \int e^{-i\omega t - i\mathbf{p}\cdot\mathbf{x}} R_{ij}(t,\mathbf{x})dtd\mathbf{x}.$$

We will also assume that the partial Fourier transforms $\tilde{V}_j(z,\mathbf{p})$ in $\mathbf{x}$ only are almost surely supported in a deterministic compact set $\{\|\mathbf{p}\| \le C\}$ and the total mass is also almost surely uniformly bounded:

$$\int |d\tilde{V}_j(z,\mathbf{p})| \le C,$$

with a deterministic constant C. We denote the state space of such spectral measure by $\mathcal{V}$.

We further assume that the joint random process $V(z) = (V_1(z), V_2(z))$ is Markovian in the variable z with a generator Q (written in the Fourier domain) that is bounded on $L^\infty(\mathcal{V})$, has a unique invariant measure $\pi(\hat{V})$ and a spectral gap $\alpha > 0$. This means that

$$Q^*\pi = 0,$$

and if $\langle g, \pi\rangle = 0$, then

$$\|e^{rQ}g\|_{L^\infty_{\mathcal{V}}} \le C\|g\|_{L^\infty_{\mathcal{V}}} e^{-\alpha r}. \tag{30}$$

Given (30), the Fredholm alternative holds for the Poisson equation

$$Qf = g,$$

provided that g satisfies $\langle \pi, g\rangle = 0$. It has a unique solution f with $\langle \pi, f\rangle = 0$ and $\|f\|_{L^\infty_{\mathcal{V}}} \le C\|g\|_{L^\infty_{\mathcal{V}}}$. The solution f is given explicitly by

$$f(\hat{V}) = -\int_0^\infty dr e^{rQ} g(\hat{V}),$$

and the integral converges absolutely because of (30).

The main result of this section is that under the above assumptions, the following theorem holds. Let us define the operator

$$\begin{aligned}\mathcal{L}f(\mathbf{x},\mathbf{k}) = &\int_{\mathbb{R}^d}\left[\hat{R}_{12}(\frac{\mathbf{p}^2-\mathbf{k}^2}{2},\mathbf{p}-\mathbf{k})W_0(\mathbf{p}) - \frac{\hat{R}_{11}(\frac{\mathbf{p}^2-\mathbf{k}^2}{2},\mathbf{p}-\mathbf{k}) + \hat{R}_{22}(\frac{\mathbf{p}^2-\mathbf{k}^2}{2},\mathbf{p}-\mathbf{k})}{2}W_0(\mathbf{k})\right]\frac{d\mathbf{p}}{(2\pi)^d}\\ &- i\Pi(\mathbf{k})W_0(\mathbf{k})\end{aligned} \tag{31}$$

with

$$\begin{aligned}\Pi(\mathbf{k}) &= \frac{1}{i}\int_{\mathbb{R}} dr \int_{\mathbb{R}^d}\frac{d\mathbf{p}}{(2\pi)^d}\frac{\tilde{R}_{22}(r,\mathbf{p}) - \tilde{R}_{11}(r,\mathbf{p})}{2}\exp\{ir(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}\}\mathrm{sgn}(r)\\ &= \int_{\mathbb{R}^d}\mathrm{p.v.}\int_{\mathbb{R}}\frac{\hat{R}_{22}(\omega,\mathbf{k}-\mathbf{p}) - \hat{R}_{11}(\omega,\mathbf{k}-\mathbf{p})}{\omega - \frac{|\mathbf{p}|^2-|\mathbf{k}|^2}{2}}\frac{d\omega d\mathbf{p}}{(2\pi)^{d+1}}.\end{aligned} \tag{32}$$

Here, $\tilde{R}(r,\mathbf{p})$ is the partial Fourier transform of R in $\mathbf{x}$ only. We denote the standard inner product on $L^2(\mathbb{R}^{2d})$ by $\langle f, g\rangle = \int_{\mathbb{R}^{2d}} f(\mathbf{x},\mathbf{k})\bar{g}(\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k}$. Then we have the following result.

Theorem 3.2 *Under the above assumptions, the Wigner distribution W_ε converges in probability and weakly in $L^2(\mathbb{R}^{2d})$ to the solution $\overline{W}$ of the transport equation*

$$\kappa\frac{\partial\overline{W}}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\overline{W} = \kappa^2\mathcal{L}\overline{W}. \tag{33}$$

More precisely, for any test function $\lambda \in L^2(\mathbb{R}^{2d})$ the process $\langle W_\varepsilon(z), \lambda\rangle$ converges to $\langle\overline{W}(z), \lambda\rangle$ in probability as $\varepsilon \to 0$, uniformly on finite intervals $0 \le z \le Z$.

3.2 Proof of Theorem 3.2

The strategy of the proof is very similar to that in [6]. Observe first that since the Wigner equation preserves the L^2-norm, the joint process $(W_\varepsilon(z), V(z))$ is a Markov process on $\mathcal{X} \times \mathcal{V}$, where $\mathcal{X} = \{\|W\|_2 \leq C\}$ is an appropriate ball in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. The corresponding family of measures P^ε on the right-continuous paths on $\mathcal{X}$ is tight, as can be shown in a way identical to [5] and [6] (see also [11] for a detailed calculation in a similar setting).

Given a test function $\lambda(z, \mathbf{x}, \mathbf{k})$ we will show that the functional

$$G_\lambda(z) = \langle W, \lambda\rangle - \int_0^z \left\langle W, \left(\frac{\partial}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}} + \kappa\mathcal{L}^*\right)\lambda\right\rangle(s)ds \tag{34}$$

is an approximate P_ε-martingale. More precisely, we show that

$$\left|\mathbb{E}^{P_\varepsilon}\left\{G_\lambda[W](z)|\mathcal{F}_s\right\} - G_\lambda[W](s)\right| \leq C_{\lambda,Z}\sqrt{\varepsilon} \tag{35}$$

uniformly for all $W \in C([0, Z]; X)$ and $0 \leq s < z \leq Z$, with a deterministic constant $C_{\lambda,Z}$. The weak convergence of the probability measures P_ε together with (35) imply that $\mathbb{E}\{W^\varepsilon\}$ converges to $\overline{W}$. In order to establish (35) we will construct another functional G^ε_λ that is an exact martingale and that is uniformly close to G_λ. This is done by the perturbed test function method. A similar argument applied to $\langle W, \lambda\rangle^2$ implies that $\mathbb{E}\{W^\varepsilon \otimes W^\varepsilon\}$ converges weakly to $\overline{W}\otimes\overline{W}$. This implies convergence in probability. In order to simplify the notation we set $\kappa = 1$ throughout the proof.

Step 1. Convergence of the expectation. Given a function $F(W, \hat{V})$ let us define the conditional expectation

$$\mathbb{E}^{\tilde{P}_\varepsilon}_{W,\hat{V}_0,z}\left\{F(W,\hat{V})\right\}(\tau) = \mathbb{E}^{\tilde{P}_\varepsilon}\left\{F(W(\tau),\hat{V}(\tau))|\; W(z) = W, \hat{V}(z) = \hat{V}\right\}, \quad \tau \geq z,$$

where $\tilde{P}_\varepsilon$ is the joint probability measure of V and W_ε. The weak form of the infinitesimal generator of the Markov process generated by $V_{1,2}$ and W_ε is given by

$$\frac{d}{dh}\mathbb{E}^{\tilde{P}_\varepsilon}_{W,\hat{V},z}\left\{\langle W, \lambda(\hat{V})\rangle\right\}(z+h)\bigg|_{h=0} = \frac{1}{\varepsilon}\langle W, Q\lambda\rangle + \left\langle W, \left(\frac{\partial}{\partial t} + \mathbf{k}\cdot\nabla_{\mathbf{x}} - \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V}, \frac{\mathbf{x}}{\varepsilon}]\right)\lambda\right\rangle, \tag{36}$$

hence

$$G^\varepsilon_\lambda = \langle W, \lambda(\hat{V})\rangle(z) - \int_0^z \left\langle W, \left(\frac{1}{\varepsilon}Q + \frac{\partial}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}} - \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V}, \frac{\mathbf{x}}{\varepsilon}]\right)\lambda\right\rangle(s)ds \tag{37}$$

is a martingale. The skew-symmetric operator $\mathcal{K}$ is defined by

$$\mathcal{K}[\hat{V}, \boldsymbol{\xi}]\psi(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}, \hat{V}) = \frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\psi(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k} - \frac{\mathbf{p}}{2}) - \frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\psi(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k} + \frac{\mathbf{p}}{2}). \tag{38}$$

The generator (36) results from the Wigner equation written in the form

$$\frac{\partial W_\varepsilon}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}W_\varepsilon = \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\tilde{V}(\frac{z}{\varepsilon}), \frac{\mathbf{x}}{\varepsilon}]W_\varepsilon. \tag{39}$$

The following lemma is the key element to show that $\mathbb{E}\{W_\varepsilon\} \to \overline{W}$, solution of (33).

Lemma 3.3 *Let $\lambda(z, \mathbf{x}, \mathbf{k}) \in C^1([0, Z]; \mathcal{S})$ be a deterministic test function, and let the functionals G^ε_λ and G_λ be defined by (34) and (37), respectively. There exists a deterministic constant $C_\lambda > 0$ and a family of perturbed random test functions λ_ε so that $\|\lambda_\varepsilon - \lambda\|_2 \leq C_\lambda\sqrt{\varepsilon}$ almost surely and*

$$\|G^\varepsilon_{\lambda_\varepsilon}(z) - G_\lambda(z)\|_{L^\infty(\mathcal{V})} \leq C_\lambda\sqrt{\varepsilon} \tag{40}$$

uniformly for all distances $z \in [0, Z]$.

The proof of this lemma is presented in Appendix A. The weak convergence of the probability measures P_ε and Lemma 3.3 imply that $\mathbb{E}\{W_\varepsilon\} \to \overline{W}$, weak solution of

$$\langle \overline{W}(z), \lambda(z)\rangle - \langle W_0, \lambda(0)\rangle - \int_0^z ds \left\langle \overline{W}, \left(\frac{\partial}{\partial s} + \mathbf{k}\cdot\nabla_{\mathbf{x}} + \mathcal{L}^*\right)\lambda \right\rangle(s) = 0, \tag{41}$$

which is nothing but the weak form of (33).

Step 2. Convergence in probability. We now look at the second moment $\mathbb{E}\left\{\langle W_\varepsilon, \lambda\rangle^2\right\}$ and show that it converges to $\langle \overline{W}, \lambda\rangle^2$. This implies convergence in probability. The calculation is similar to that for $\mathbb{E}\left\{\langle W_\varepsilon, \lambda\rangle\right\}$ and is based on constructing an approximate martingale for the functional $\langle W\otimes W, \mu\rangle$, where $\mu(z, \mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2)$ is a test function, and $W\otimes W(z, \mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2) = W(z, \mathbf{x}_1, \mathbf{k}_1)W(z, \mathbf{x}_2, \mathbf{k}_2)$. As before we consider functionals of W and $\hat{V}$ of the form $F(W, \hat{V}) = \langle W\otimes W, \mu(\hat{V})\rangle$, where μ is a given function. The infinitesimal generator acts on such functions as

$$\frac{d}{dh}\mathbb{E}^{P_\varepsilon}_{W,\hat{V},z}\left\{\langle W\otimes W, \mu(\hat{V})\rangle\right\}(z+h)\Big|_{h=0} = \frac{1}{\varepsilon}\langle W\otimes W, Q\mu\rangle + \langle W\otimes W, \mathcal{H}_2^\varepsilon\mu\rangle, \tag{42}$$

where

$$\mathcal{H}_2^\varepsilon\mu = \mathbf{k}^j\cdot\nabla_{\mathbf{x}^j}\mu - \sum_{j=1}^2 \frac{1}{\sqrt{\varepsilon}}\mathcal{K}_j\left[\hat{V}, \frac{\mathbf{x}^j}{\varepsilon}\right]\mu, \tag{43}$$

with

$$\mathcal{K}_1[\hat{V}, \boldsymbol{\xi}_1]\mu = \frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i(\mathbf{p}\cdot\boldsymbol{\xi}_1)}\mu(\mathbf{k}_1 - \frac{\mathbf{p}}{2}, \mathbf{k}_2) - \frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d}e^{i(\mathbf{p}\cdot\boldsymbol{\xi}_1)}\mu(\mathbf{k}_1 + \frac{\mathbf{p}}{2}, \mathbf{k}_2)$$

and

$$\mathcal{K}_2[\hat{V}, \boldsymbol{\xi}_2]\mu = \frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i(\mathbf{p}\cdot\boldsymbol{\xi}_2)}\mu(\mathbf{k}_1, \mathbf{k}_2 - \frac{\mathbf{p}}{2}) - \frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d}e^{i(\mathbf{p}\cdot\boldsymbol{\xi}_2)}\mu(\mathbf{k}_1, \mathbf{k}_2 + \frac{\mathbf{p}}{2}).$$

Therefore the functional

$$\begin{aligned} G_\mu^{2,\varepsilon} &= \langle W\otimes W, \mu(\hat{V})\rangle(z) \\ &- \int_0^z \left\langle W\otimes W, \Big(\frac{1}{\varepsilon}Q + \frac{\partial}{\partial z} + \mathbf{k}_1\cdot\nabla_{\mathbf{x}_1} + \mathbf{k}_2\cdot\nabla_{\mathbf{x}_2} - \frac{1}{\sqrt{\varepsilon}}(\mathcal{K}_1[\hat{V}, \frac{\mathbf{x}_1}{\varepsilon}] - \mathcal{K}_2[\hat{V}, \frac{\mathbf{x}_2}{\varepsilon}])\Big)\mu\right\rangle(s)ds \end{aligned} \tag{44}$$

is a P^ε martingale. The following lemma is proved in Appendix B.

Lemma 3.4 *Let $\mu(z, \mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2)$ be a deterministic test function and let the functional $G_\mu^{2,\varepsilon}$ be defined by (44). Then there exists a deterministic constant $C > 0$ so that*

$$|G_\mu^{2,\varepsilon} - \bar{G}_\mu^{2,\varepsilon}| \le C\sqrt{\varepsilon} \tag{45}$$

with

$$\bar{G}_\mu^{2,\varepsilon} = \langle W\otimes W, \mu\rangle(z) - \int_0^z \left\langle W\otimes W, \frac{\partial}{\partial z} + \mathbf{k}_1\cdot\nabla_{\mathbf{x}_1} + \mathbf{k}_2\cdot\nabla_{\mathbf{x}_2} + \mathcal{L}^*_{2,\varepsilon})\Big)\mu\right\rangle(s)ds \tag{46}$$

and with a deterministic operator $\mathcal{L}_{2,\varepsilon}$ such that $\|\mathcal{L}^*_{2,\varepsilon} - \mathcal{L}^*\otimes\mathcal{L}^*\|_{L^2\to L^2} \to 0$ as $\varepsilon\to 0$.

Lemma 3.4 implies immediately that for any test function μ we have $\mathbb{E}\left\{\langle W_\varepsilon\otimes W_\varepsilon, \mu\rangle\right\} \to \langle \overline{W}\otimes\overline{W}, \mu\rangle$. If we take $\mu = \lambda\otimes\lambda$ we get $\mathbb{E}\left\{\langle W_\varepsilon, \lambda\rangle^2\right\} \to \langle \overline{W}, \lambda\rangle^2$ and hence $\langle W_\varepsilon, \lambda\rangle \to \langle \overline{W}, \lambda\rangle$ in probability. This finishes the proof of Theorem 3.2.

4 The Itô-Schrödinger regime

We consider in this section the regime where the ratio l_z/L_z of the correlation length l_z of the fluctuations in the z direction to the propagation distance L_z is the smallest parameter in the system.

4.1 Itô-Schrödinger equation

Let us recall the Schrödinger equation (5)

$$\frac{2ik}{L_z}\frac{\partial\psi}{\partial z} + \frac{1}{L_x^2}\Delta_{\mathbf{x}}\psi - 2k^2\sigma V(\frac{L_z z}{l_z}, \frac{L_x \mathbf{x}}{l_x})\psi = 0. \tag{47}$$

The scaling assumptions (6) are now replaced by

$$\varepsilon = \frac{l_x}{L_x} \ll 1, \qquad \frac{l_z}{L_z} = \varepsilon^{1+\alpha},\ \alpha > 0, \qquad kL_z = \frac{\kappa}{\varepsilon}\Big(\frac{L_z}{L_x}\Big)^2, \qquad \sigma = \varepsilon^{\frac{1-\alpha}{2}}\frac{L_x}{L_z}. \tag{48}$$

The constraint $\alpha > 0$ indeed implies that l_z/L_z is smaller than any other dimensionless term in the system. With these assumptions, (47) may be recast as

$$\frac{\partial\psi}{\partial z} = \frac{i\varepsilon}{2\kappa}\Delta_{\mathbf{x}}\psi - i\kappa\frac{1}{\varepsilon^{\frac{1+\alpha}{2}}}V\Big(\frac{z}{\varepsilon^{1+\alpha}}, \frac{\mathbf{x}}{\varepsilon}\Big)\psi. \tag{49}$$

Because the variations in z of the potential are faster than any other quantity in the above equation, we can formally replace

$$\frac{-i\kappa}{\varepsilon^{\frac{1+\alpha}{2}}}V\Big(\frac{z}{\varepsilon^{1+\alpha}}, \frac{\mathbf{x}}{\varepsilon}\Big)dz \qquad \text{by} \qquad i\kappa B(dz, \frac{\mathbf{x}}{\varepsilon}), \tag{50}$$

its white noise limit, where $B(dz, \mathbf{x})$ is the Wiener measure described by the statistics

$$\mathbb{E}\{B(\mathbf{x}, z)B(\mathbf{y}, z')\} = K(\mathbf{x}-\mathbf{y})z \wedge z'. \tag{51}$$

Here, $\mathbb{E}\{\cdot\}$ means mathematical expectation with respect to the Wiener measure, $K(\mathbf{x})$ is the correlation function of the random fluctuations and $z \wedge z' = \min(z, z')$. The paraxial Schrödinger equation then becomes the following stochastic equation

$$d\psi(z, \mathbf{x}) = \frac{i\varepsilon}{2\kappa}\Delta_{\mathbf{x}}\psi(z, \mathbf{x})dz + i\kappa\psi(z, \mathbf{x}) \circ B(dz, \frac{\mathbf{x}}{\varepsilon}). \tag{52}$$

Here, the notation $\circ$ means that the stochastic equation is understood in the Stratonovich sense [17, 21]. In the Itô formalism, it becomes the following Itô-Schrödinger equation

$$d\psi(z, \mathbf{x}) = \frac{1}{2}\Big(\frac{i\varepsilon}{\kappa}\Delta_{\mathbf{x}} - \kappa^2 K(\mathbf{0})\Big)\psi(z, \mathbf{x})dz + i\kappa\psi(z, \mathbf{x})B(dz, \frac{\mathbf{x}}{\varepsilon}). \tag{53}$$

We do not justify the derivation of (53) here. It was shown in [2] that the paraxial approximation and the white noise limit can be taken consistently in the one-dimensional case.

As in the paraxial regime, we still have one parameter left, namely L_x/L_z, which we choose as in (7). We then verify that

$$\frac{l_x}{l_z} = \frac{l_x}{L_x}\frac{L_x}{L_z}\frac{L_z}{l_z} = \varepsilon^{\eta-\alpha}. \tag{54}$$

Thus with the choice $\eta = \alpha$, the Itô-Schrödinger equation (53) can be used to model isotropic fluctuations.

4.2 Time reversed waves in changing media

The formalism presented in Section 2.2 applies in the white noise limit as well. We can still define the functions $Q_{f,b}$, which now solve

$$\begin{aligned} dQ_{f,b}(z,\mathbf{x},\kappa;\mathbf{q}) &= \frac{1}{2}\Big(\frac{i\varepsilon}{\kappa}\Delta_{\mathbf{x}} - \kappa^2 K_{1,2}(\mathbf{0})\Big)Q_{f,b}(z,\mathbf{x},\kappa;\mathbf{q})dz + i\kappa Q_{f,b}(z,\mathbf{x},\kappa;\mathbf{q})B_{1,2}(dz,\frac{\mathbf{x}}{\varepsilon}), \\ Q_{f,b}(0,\mathbf{x},\kappa;\mathbf{q}) &= \chi(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{q}/\varepsilon}, \end{aligned} \tag{55}$$

where the Wiener measures $B_{1,2}$ are described by different statistics $K_{1,2}$ for the forward propagation (index 1) and the backward propagation (index 2). The *cross-correlation* of the two media, is defined by

$$\mathbb{E}\{B_m(\mathbf{x},z)B_n(\mathbf{y},z')\} = K_{mn}(\mathbf{x}-\mathbf{y})z\wedge z', \qquad 1\le m,n\le 2. \tag{56}$$

We will see in what follows that the relative strength of the cross-correlation K_{12} compared to the auto-correlation functions K_{mm} determines the quality of time-reversal.

Upon defining

$$U_\varepsilon(z,\mathbf{x},\mathbf{k},\kappa;\mathbf{q}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}Q_f(z,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2},\kappa;\mathbf{q})Q_b^*(z,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},\kappa;\mathbf{q})\frac{d\mathbf{y}}{(2\pi)^d}, \tag{57}$$

as in (22) and

$$W_\varepsilon(z,\mathbf{x},\mathbf{k},\kappa) = \int_{\mathbb{R}^d}\hat f(\mathbf{q})U_\varepsilon(z,\mathbf{x},\mathbf{k},\kappa;\mathbf{q})d\mathbf{q}, \tag{58}$$

as in (21), we obtain that the back-propagated signal is given as in (23) by

$$\psi_\varepsilon^B(\boldsymbol{\xi},\kappa;\mathbf{x}_0) = \int_{\mathbb{R}^{2d}} e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})}W_\varepsilon(L,\mathbf{x}_0+\varepsilon\frac{\mathbf{y}+\boldsymbol{\xi}}{2},\mathbf{k},\kappa)\psi_0(\mathbf{y},\kappa)\frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^d}. \tag{59}$$

The high frequency limit of the time reversed signal is thus again modeled by the limit $\varepsilon\to 0$ in the above equation.

4.3 High frequency limit of time reversed waves in changing media

In the high frequency limit, we have the following result

Theorem 4.1 *Let $\kappa\in\mathbb{R}$ fixed. Let us assume that the initial condition $\psi_0(\mathbf{y},\kappa)\in L^2(\mathbb{R}^d)$, the filter $f(\mathbf{x})\in L^2(\mathbb{R}^d)$, and the recorder function $\chi(\mathbf{x})\in L^4(\mathbb{R}^d)$. Then $\psi_\varepsilon^B(\boldsymbol{\xi},\kappa;\mathbf{x}_0)$ converges weakly and in probability to the deterministic signal*

$$\psi^B(\boldsymbol{\xi},\kappa;\mathbf{x}_0) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\bar W(L,\mathbf{x}_0,\mathbf{k},\kappa)\hat\psi_0(\mathbf{k},\kappa)d\mathbf{k}, \tag{60}$$

where $\bar W(L,\mathbf{x}_0,\mathbf{k},\kappa)$ solves the following radiative transfer equation

$$\begin{aligned} &\frac{\partial W}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}}W + \kappa^2\frac{K_{11}(\mathbf{0})+K_{22}(\mathbf{0})}{2}W = \kappa^2\int_{\mathbb{R}^d}\hat K_{12}(\mathbf{p}-\mathbf{k})W(\mathbf{p})d\mathbf{p} \\ &W(0,\mathbf{x},\mathbf{k},\kappa) = \hat f(\mathbf{k})\chi^2(\mathbf{x}). \end{aligned} \tag{61}$$

Moreover for a smooth test function of the form $\lambda(\boldsymbol{\xi},\mathbf{x}_0) = \tilde\lambda(\mathbf{x}_0)\mu(\boldsymbol{\xi})$, we have an error estimate of the form

$$\mathbb{E}\{(\psi_\varepsilon^B - \mathbb{E}\{\psi_\varepsilon^B\})^2\} \le C\varepsilon^d\|\lambda\|^2_{L^2(\mathbb{R}^d)}\|\psi_0\|^2_{L^2(\mathbb{R}^d)}, \tag{62}$$

uniformly in L on compact intervals.

The main steps of the proof of the theorem are very similar to that in the paraxial regime. However the mathematical analysis is substantially simplified by the fact that statistical moments of the field ψ_ε^B and the associated Wigner transform W_ε satisfy closed-form equations. We refer the reader to [14, 17, 26] for basic results about the stochastic partial differential equation (59). The proof of the above theorem can be carried out as in [3]. We highlight the differences that appear because of the change of media during the forward and backward propagation.

Let ψ_1 and ψ_2 satisfy

$$d\psi_m(z,\mathbf{x}) = \frac{1}{2}\Big(\frac{i\varepsilon}{\kappa}\Delta_{\mathbf{x}} - \kappa^2 K_m(\mathbf{0})\Big)\psi_m(z,\mathbf{x})dz + i\kappa\psi_m(z,\mathbf{x})B_m(dz,\frac{\mathbf{x}}{\varepsilon}), \qquad m=1,2. \tag{63}$$

We define the second moment $m_2(\mathbf{x},\mathbf{y})$ as

$$m_2(z,\mathbf{x},\mathbf{y},\kappa) = \mathbb{E}\{\psi_1(z,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},\kappa)\psi_2^*(z,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2},\kappa)\}. \tag{64}$$

By an application of the Itô calculus [21] we obtain that

$$d(\psi_1(z,\mathbf{x})\psi_2^*(z,\mathbf{y})) = \psi_1(z,\mathbf{x})d\psi_2^*(z,\mathbf{y}) + d\psi_1(z,\mathbf{x})\psi_2^*(z,\mathbf{y}) + d\psi_1(z,\mathbf{x})d\psi_2^*(z,\mathbf{y}).$$

We insert (63) into the above formula and taking mathematical expectation, obtain after some algebra [3] an equation for m_2:

$$\frac{\partial m_2}{\partial z} = \frac{1}{\kappa}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}} m_2(z) - \kappa^2\Big(\frac{K_{11}(\mathbf{0})+K_{22}(\mathbf{0})}{2} - K_{12}(\mathbf{y})\Big)m_2(z). \tag{65}$$

Now, defining the Wigner transform of the two fields as

$$W_{12}(z,\mathbf{x},\mathbf{k},\kappa) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{x}}\psi_1(z,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2},\kappa)\psi_2^*(z,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},\kappa)dy, \tag{66}$$

we find that

$$m_2(z,\mathbf{x},\mathbf{y},\kappa) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}\mathbb{E}\{W_{12}\}(z,\mathbf{x},\mathbf{k},\kappa)d\mathbf{k}. \tag{67}$$

Therefore, $\mathbb{E}\{W_{12}\}$ solves the following equation:

$$\frac{\partial W}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}} W + \kappa^2\frac{K_{11}(\mathbf{0})+K_{22}(\mathbf{0})}{2}W = \kappa^2\int_{\mathbb{R}^d}\hat{K}_{12}(\mathbf{p}-\mathbf{k})W(\mathbf{p})d\mathbf{p}. \tag{68}$$

This is the integro-differential equation in (61). By construction, $\mathbb{E}\{U_\varepsilon\}$ defined in (57), whence $\mathbb{E}\{W_\varepsilon\}$ defined in (58), satisfy the same equation.

Let us now consider the fourth-order moment

$$m_4(z,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\kappa) = \mathbb{E}\{\psi_1(z,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},\kappa)\psi_2^*(z,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2},\kappa)\psi_1(z,\mathbf{z}+\frac{\varepsilon\mathbf{t}}{2},\kappa)\psi_2^*(z,\mathbf{z}-\frac{\varepsilon\mathbf{t}}{2},\kappa)\}. \tag{69}$$

We deduce from the application of Itô calculus to four arbitrary functions

$$d(\psi_1\psi_2^*\psi_3\psi_4^*) = \psi_2^*\psi_3\psi_4^* d\psi_1 + \cdots + \psi_1\psi_2^*\psi_3 d\psi_4^* + \psi_1\psi_2^* d\psi_3 d\psi_4^* + \cdots + \psi_3\psi_4^* d\psi_1 d\psi_2^*,$$

that m_4 solves the following equation

$$\begin{aligned}
\frac{\partial m_4}{\partial z} &= \frac{i}{\kappa}(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}} + \nabla_{\boldsymbol{\xi}}\cdot\nabla_{\mathbf{t}})m_4(z) - \mathcal{K}m_4(z),\\
\mathcal{K}(\mathbf{x},\mathbf{y},\boldsymbol{\xi},\mathbf{t}) &= K_{11}(\mathbf{0}) + K_{22}(\mathbf{0}) - K_{12}(\mathbf{y}) - K_{12}(\mathbf{t})\\
&\quad + K_{11}(\frac{\mathbf{x}-\boldsymbol{\xi}}{\varepsilon} + \frac{\mathbf{y}-\mathbf{t}}{2}) - K_{12}(\frac{\mathbf{x}-\boldsymbol{\xi}}{\varepsilon} + \frac{\mathbf{y}+\mathbf{t}}{2})\\
&\quad - K_{12}(\frac{\mathbf{x}-\boldsymbol{\xi}}{\varepsilon} - \frac{\mathbf{y}+\mathbf{t}}{2}) + K_{22}(\frac{\mathbf{x}-\boldsymbol{\xi}}{\varepsilon} - \frac{\mathbf{y}-\mathbf{t}}{2}).
\end{aligned} \tag{70}$$

Let us now introduce the second moment of $\mathcal{W}_{12}$:

$$\mathcal{W}(z,\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},\kappa) = \mathcal{W}_{12}(z,\mathbf{x},\mathbf{p},\kappa)\mathcal{W}_{12}(z,\boldsymbol{\xi},\mathbf{q},\kappa). \tag{71}$$

We verify that

$$m_4(z,\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\kappa) = \int_{\mathbb{R}^{2d}} e^{i\mathbf{p}\cdot\mathbf{y}+i\mathbf{q}\cdot\mathbf{t}}\mathbb{E}\{\mathcal{W}\}(z,\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{t},\kappa)d\mathbf{p}d\mathbf{q}, \tag{72}$$

so that $\mathbb{E}\{\mathcal{W}\}$ solves the following equation

$$\frac{\partial \mathcal{W}}{\partial z} + \frac{1}{\kappa}(\mathbf{p}\cdot\nabla_{\mathbf{x}} + \mathbf{q}\cdot\nabla_{\boldsymbol{\xi}})\mathcal{W} + \kappa^2(K_{11}(\mathbf{0}) + K_{22}(\mathbf{0}))\mathcal{W} = \kappa^2\mathcal{L}_2\mathcal{W} + \kappa^2\mathcal{L}_{12}\mathcal{W}, \tag{73}$$

where

$$\begin{aligned}
\mathcal{L}_2\mathcal{W} &= \int_{\mathbb{R}^{2d}} \big(\hat{K}_{12}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{q}-\mathbf{q}') + \hat{K}_{12}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{q}-\mathbf{q}')\big)\mathcal{W}(\mathbf{p}',\mathbf{q}')d\mathbf{p}'d\mathbf{q}' \\
\mathcal{L}_{12}\mathcal{W} &= \int_{\mathbb{R}^d} e^{i\frac{\mathbf{x}-\boldsymbol{\xi}}{\varepsilon}\cdot\mathbf{u}}\Big(\hat{K}_{12}(\mathbf{u})\big(\mathcal{W}(\mathbf{p}-\frac{\mathbf{u}}{2},\mathbf{q}-\frac{\mathbf{u}}{2}) + \mathcal{W}(\mathbf{p}+\frac{\mathbf{u}}{2},\mathbf{q}+\frac{\mathbf{u}}{2})\big) \\
&\qquad -\hat{K}_{11}(\mathbf{u})\mathcal{W}(\mathbf{p}-\frac{\mathbf{u}}{2},\mathbf{q}+\frac{\mathbf{u}}{2}) - \hat{K}_{22}(\mathbf{u})\mathcal{W}(\mathbf{p}+\frac{\mathbf{u}}{2},\mathbf{q}-\frac{\mathbf{u}}{2})\Big)d\mathbf{u}.
\end{aligned} \tag{74}$$

We thus obtain that both

$$\mathcal{U}_\varepsilon(z,\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},\kappa;\mathbf{k}) = \mathbb{E}\{U_\varepsilon(z,\mathbf{x},\mathbf{p},\kappa;\mathbf{k})U_\varepsilon(z,\boldsymbol{\xi},\mathbf{q},\kappa;\mathbf{k})\} \tag{75}$$

where U_ε is defined in (57), and

$$\mathcal{W}_\varepsilon(z,\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},\kappa) = \mathbb{E}\{W_\varepsilon(z,\mathbf{x},\mathbf{p},\kappa)W_\varepsilon(z,\boldsymbol{\xi},\mathbf{q},\kappa)\} \tag{76}$$

where W_ε is defined in (58), satisfy the same radiative transfer equation (73). There is however a fundamental difference between the two latter terms, namely that $\mathcal{W}_\varepsilon$ is bounded in $L^2(\mathbb{R}^{4d})$ at fixed κ, whereas $\mathcal{U}_\varepsilon$ is not bounded in the same norm at κ and $\mathbf{k}$ fixed. Indeed, $W_\varepsilon(z=0)$ is bounded in $L^2(\mathbb{R}^{2d})$, which is not the case for $U_\varepsilon(z=0)$. The results in [3, section 3] show that $\mathbb{E}\{W_\varepsilon\}(z)$ and $\mathcal{W}_\varepsilon(z)$, are then bounded in $L^2(\mathbb{R}^{2d})$ and $L^2(\mathbb{R}^{4d})$ respectively, uniformly in $z \geq 0$. More precisely, we have

$$W_\varepsilon(0,\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{y}}f(\mathbf{y})\chi(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\chi(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})d\mathbf{y}. \tag{77}$$

For $f(\mathbf{x})$ and $\chi(\mathbf{x})$ sufficiently smooth, Theorem 4.1 of [3] allows us to conclude that

$$\|\mathcal{W}_\varepsilon - \mathbb{E}\{W_\varepsilon(z,\mathbf{x},\mathbf{p},\kappa)\}\mathbb{E}\{W_\varepsilon(z,\boldsymbol{\xi},\mathbf{q},\kappa)\}\|_{L^2(\mathbb{R}^{4d})} \leq C\varepsilon^{d/2}, \tag{78}$$

uniformly on compact sets in z. This comes merely from the observation that $\mathcal{L}_{12}$ defined in (74) converges to zero as an operator on L^2. Moreover, (77) implies that $W_\varepsilon(z=0,\mathbf{x},\mathbf{k},\kappa)$ converges strongly to $\hat{f}(\mathbf{k})\chi^2(\mathbf{x})$ as $\varepsilon\to 0$ by Lemma 3.1. This implies that $\mathbb{E}\{W_\varepsilon(z,\mathbf{x},\mathbf{k},\kappa)\}$ converges strongly in $L^2(\mathbb{R}^{2d})$ and uniformly in z and κ on compact intervals to $\bar{W}(z,\mathbf{x},\mathbf{k},\kappa)$ solution to (61) as $\varepsilon\to 0$ (since the L^2 norm is preserved by (61)).

For a test function $\lambda\in L^2(\mathbb{R}^{2d})$, the above convergence implies that

$$\mathbb{E}\{\big((W_\varepsilon,\lambda) - (\mathbb{E}\{W_\varepsilon\},\lambda)\big)^2\} \leq C\varepsilon^{d/2}\|\lambda\|^2_{L^2(\mathbb{R}^{2d})}. \tag{79}$$

We deduce that (W_ε,λ) converges in probability to the deterministic number $(\bar{W},\lambda)$ as $\varepsilon\to 0$. We have thus obtained the (weak) stability of W_ε. Then we can pass to the limit $\varepsilon\to 0$ in (59) and obtain (60). This concludes the proof of Theorem 4.1.

5 Decoherence in time reversal

The two preceding sections were concerned with the derivation of the radiative transfer equations modeling time reversal when the medium during the backward propagation phase differs from the medium during the forward propagation stage. In both regimes, we observe that the main quantity governing refocusing is the ratio of the cross-correlation terms R_{12} and K_{12} to the auto-correlations R_{mm} and K_{mm}, $m = 1, 2$. When that ratio is large, time reversal refocusing works as if both media were the same. When the cross-correlation is small, the coherent effects that produce strong refocusing in time reversal are no longer present.

Let us focus on the two-media effect in the Itô-Schrödinger regime first. We recast (60) in the Fourier domain and obtain

$$\hat{\psi}^B(\mathbf{k}, \kappa; \mathbf{x}_0) = \bar{W}(L, \mathbf{x}_0, \mathbf{k}, \kappa)\hat{\psi}_0(\mathbf{k}, \kappa). \tag{80}$$

Therefore, the medium acts as a filter between the original signal $\psi_0(\mathbf{k}, \kappa)$ and the refocused signal $\hat{\psi}^B(\mathbf{k}, \kappa; \mathbf{x}_0)$. The back-propagated signal is all the tighter around $\mathbf{x}_0$ that the filter is close to a constant (in $\mathbf{k}$) non-zero value. Since $\bar{W}$ satisfies a radiative transfer equation, the regularity of $\bar{W}$ is increased by the scattering term on the right hand side in (61), as is discussed in detail in [7]. Indeed, multiple scattering has a regularizing effect. As the change in the propagating media increases, the cross correlation K_{12} decreases. This weakens the scattering term in (61), hence diminishes the regularizing effect of (61) and the re-focusing properties of the time reversed signal. Let us assume that K_{12} is real-valued to simplify the presentation. The weakened refocusing can be quantified by recasting the radiative transfer equation (61) as

$$\begin{aligned} &\frac{\partial W}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}} W + \kappa^2\sigma_a W = \kappa^2 \int_{\mathbb{R}^d} \hat{K}_{12}(\mathbf{p}-\mathbf{k})(W(\mathbf{p}) - W(\mathbf{k}))d\mathbf{p} \\ &W(0, \mathbf{x}, \mathbf{k}, \kappa) = \hat{f}(\mathbf{k})\chi^2(\mathbf{x}), \end{aligned} \tag{81}$$

where we have defined the apparent absorption coefficient

$$\sigma_a = \frac{K_{11}(\mathbf{0}) + K_{22}(\mathbf{0})}{2} - \int_{\mathbb{R}^d} \hat{K}_{12}(\mathbf{p}-\mathbf{k})d\mathbf{p}. \tag{82}$$

As the media decorrelate, the absorption coefficient σ_a increases up to the value $\frac{1}{2}(K_{11}(\mathbf{0}) + K_{22}(\mathbf{0}))$ when the two media become completely uncorrelated. The right-hand side in (81) then vanishes and the back-propagated signal is the poorly refocused signal one would obtain in a homogeneous medium with constant wave speed $c = c_0$, albeit with a decreased amplitude by a factor $e^{-\kappa^2 L(K_{11}(\mathbf{0})+K_{22}(\mathbf{0}))}$.

Similarly, a signal that is back-propagated in a homogeneous medium would be modeled by $V_2 \equiv 0$, which implies that $K_{12} = K_{22} = 0$. So the back-propagated signal would similarly be, up to a factor $e^{-\kappa^2 L K_{11}(\mathbf{0})}$, the poorly refocused signal one would obtain in a homogeneous medium. Unless we have a sufficiently accurate knowledge of the underlying medium, back-propagating a recorded signal in a homogeneous medium, for instance on a computer, will not tightly refocus at the original location of the source term.

The situation is somewhat richer in the paraxial regime. The radiative transfer equation takes the form

$$\begin{aligned} &\frac{\partial W}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}} W + \kappa^2(\sigma_a(\mathbf{k}) + i\Pi(\mathbf{k}))W = \kappa^2 \int_{\mathbb{R}^d} \hat{R}_{12}(\frac{\mathbf{p}^2-\mathbf{k}^2}{2}, \mathbf{p}-\mathbf{k})(W(\mathbf{p}) - W(\mathbf{k}))d\mathbf{p} \\ &W(0, \mathbf{x}, \mathbf{k}, \kappa) = \hat{f}(\mathbf{k})\chi^2(\mathbf{x}), \end{aligned} \tag{83}$$

where we have defined

$$\begin{aligned}\sigma_a(\mathbf{k}) &= \int_{\mathbb{R}^d}\Big[\frac{1}{2}\big(\hat R_{11}(\frac{\mathbf{p}^2-\mathbf{k}^2}{2},\mathbf{p}-\mathbf{k})+\hat R_{22}(\frac{\mathbf{p}^2-\mathbf{k}^2}{2},\mathbf{p}-\mathbf{k})\big)-\hat R_{12}(\frac{\mathbf{p}^2-\mathbf{k}^2}{2},\mathbf{p}-\mathbf{k})\Big]d\mathbf{p},\\ \Pi(\mathbf{k}) &= \int_{\mathbb{R}^d}\text{p.v.}\int_{\mathbb{R}}\frac{\hat R_{22}(\omega,\mathbf{k}-\mathbf{p})-\hat R_{11}(\omega,\mathbf{k}-\mathbf{p})}{\omega-\frac{|\mathbf{p}|^2-|\mathbf{k}|^2}{2}}\frac{d\omega d\mathbf{p}}{(2\pi)^{d+1}}.\end{aligned} \tag{84}$$

Still assuming that $\hat R_{12}$ is real-valued, we obtain that $\sigma_a(\mathbf{k})$ is an apparent non-negative absorption coefficient and $i\Pi(\mathbf{k})$ is a purely imaginary modulation term.

We have seen the role of the absorption σ_a in the Itô-Schrödinger regime. The role of the new modulation term $i\Pi(\mathbf{k})$ is somewhat different. It also reduces the strength of the right hand side in (83) but only in the time domain, when we integrate over all frequencies. Let us assume for instance that $\Pi(\mathbf{k})$ is constant. Then we verify that $W(z)=e^{i\kappa^2\Pi z}U(z)$, where $U(z)$ satisfies the same equation (83) with Π replaced by zero. Consequently, Π has a tendency to modulate the filter $\bar W(z)$ that appears in (80). The modulation is independent of the wave vector $\mathbf{k}$ or the position $\mathbf{x}_0$. However, it depends on the longitudinal length z and on the reduced wave number κ. Therefore, in the time dependent time reversal experiments, where the refocused signal $p^B(0,\mathbf{x},t)$ is given by (2) with ψ replaced by ψ^B, that is, as an average over reduced wave numbers κ (after an appropriate re-scaling), the modulation factor Π will imply that the back-propagated signal is given by

$$\hat p^B(0,\boldsymbol{\xi},t)\approx\int_{\mathbb{R}}e^{-i\kappa c_0 t}\hat\psi^B(0,\boldsymbol{\xi},\kappa)c_0 d\kappa=\int_{\mathbb{R}}e^{-i\kappa c_0 t}e^{i\kappa^2\Pi L}\bar W_0(L,\mathbf{x}_0,\mathbf{k},\kappa)\hat\psi_0(\mathbf{k},\kappa)c_0 d\kappa, \tag{85}$$

where $\bar W_0$ is the filter obtained when $\Pi=0$. Obviously, the magnitude of the above oscillatory integral decreases as Π increases. The interpretation of the modulation term Π is thus the following. Although it does not modify the intensity of the filter $\bar W(L,\mathbf{x},\mathbf{k},\kappa)$ at a fixed frequency, it introduces a modulation of order $e^{i\kappa^2\Pi L}$ that significantly reduces the back-propagated signal recorded in the time domain.

Let us conclude with a remark on the comparison between the radiative transfer equations in the paraxial and Itô-Schrödinger regimes. The latter regime should be seen as a limit of the former regime as the oscillations in the z direction become faster and faster. Indeed, the fast oscillations in the variable z imply a decorrelation in the term $R(\mathbf{x},z)$, which converges to $K(\mathbf{k})\delta(z)$. This in turn is consistent with $\hat R(\omega,\mathbf{p})$ converging to $\hat K(\mathbf{p})$. It remains to observe that the Hilbert transform (the principal value integral in (84)) of a constant function vanishes to conclude that $\Pi(\mathbf{k})$ vanishes in the limit of fast oscillations in the z direction. This implies that the oscillatory integral obtained in (85) can only be observed in media where the oscillations in the z variable have a sufficiently large correlation length.

All the effects mentioned in this section are in agreement with the radiative transfer and diffusion numerical simulations performed in [8] in the so-called weak-coupling regime, which is the limit $L_x \approx L_z$ of the two regimes considered in this paper and for which no rigorous mathematical derivation is available.

6 Conclusions

When the medium is fixed during the forward and backward stages of a time reversal experiment, the refocusing of the back-propagated pulse is characterized in many high frequency regimes by a radiative transfer equation. The solution to the radiative transfer equation acts as a transfer function and indicates how the shape of the original source term is modified by the time reversal experiment. We have shown in this paper that this picture remains valid when the two media during the forward and backward stages differ. We have also described how the constitutive parameters of the radiative transfer equation change as the back-propagation medium is modified. Moreover, these parameters only depend on the correlation function of the two media. Finally, we have observed that the refocused signal was essentially independent of the realization of the random medium. More precisely we have shown that the back-propagated signal converges weakly and in probability to a deterministic function in the high frequency limit. This results from a similar convergence property for the properly regularized Wigner transform of two fields propagating in two different media.

As the two media are increasingly decorrelated, the refocusing of the back-propagated pulse degrades. Two mechanisms are responsible for this degradation. The first mechanism consists of a purely absorbing term indicating that wave mixing by scattering is less efficient as the two media become less correlated. This effect, though frequency-dependent, can be observed at all frequencies, hence also in the time domain. The second mechanism, which is absent in the Itô-Schrödinger regime, is a phase modulation phenomenon in the frequency domain. The signal at frequency c_0k is modified by a phase proportional to k^2, which has an important cancellation effect in the time domain after Fourier transforms are performed.

A The proof of Lemma 3.3

Given a test function $\lambda(z,\mathbf{x},\mathbf{k})\in C^1([0,Z];\mathcal{S})$ we define the following approximation

$$\lambda_\varepsilon(z,\mathbf{x},\mathbf{k},\hat V)=\lambda(z,\mathbf{x},\mathbf{k})+\sqrt{\varepsilon}\lambda_1^\varepsilon(z,\mathbf{x},\mathbf{k},\hat V)+\varepsilon\lambda_2^\varepsilon(z,\mathbf{x},\mathbf{k},\hat V) \tag{86}$$

with $\lambda_{1,2}^\varepsilon(z)$ bounded in $L^\infty(\mathcal{V};L^2(\mathbb{R}^{2d}))$ uniformly in $z\in[0,Z]$. The functions $\lambda_{1,2}^\varepsilon$ will be chosen in such a way that

$$\|G_{\lambda_\varepsilon}^\varepsilon(z)-G_\lambda(z)\|_{L^\infty(\mathcal{V})}\le C_\lambda\sqrt{\varepsilon} \tag{87}$$

for all times $z\in[0,Z]$. Here the functional G^ε is defined by (37) and the functional G by (34).

The functions λ_1^ε and λ_2^ε are constructed as follows. Let $\lambda_1(z,\mathbf{x},\boldsymbol{\xi},\mathbf{k},\hat V)$ be the mean-zero solution of the Poisson equation

$$\mathbf{k}\cdot\nabla_{\boldsymbol{\xi}}\lambda_1+Q\lambda_1=\mathcal{K}\lambda. \tag{88}$$

It is given explicitly by

$$\begin{aligned}\lambda_1(z,\mathbf{x},\boldsymbol{\xi},\mathbf{k},\hat V)=&-\frac{1}{i}\int_0^\infty dre^{rQ}\int_{\mathbb{R}^d}\frac{d\hat V_1(\mathbf{p})}{(2\pi)^d}e^{ir(\mathbf{k}\cdot\mathbf{p})+i(\boldsymbol{\xi}\cdot\mathbf{p})}\lambda(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})\\&+\frac{1}{i}\int_0^\infty dre^{rQ}\int_{\mathbb{R}^d}\frac{d\hat V_2(\mathbf{p})}{(2\pi)^d}e^{ir(\mathbf{k}\cdot\mathbf{p})+i(\boldsymbol{\xi}\cdot\mathbf{p})}\lambda(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}).\end{aligned} \tag{89}$$

Then we let $\lambda_1^\varepsilon(z,\mathbf{x},\mathbf{k},\hat V)=\lambda_1(z,\mathbf{x},\mathbf{x}/\varepsilon,\mathbf{k},\hat V)$. Furthermore, the second order corrector is given by $\lambda_2^\varepsilon(z,\mathbf{x},\mathbf{k},\hat V)=\lambda_2(z,\mathbf{x},\mathbf{x}/\varepsilon,\mathbf{k},\hat V)$ where $\lambda_2(z,\mathbf{x},\boldsymbol{\xi},\mathbf{k},\hat V)$ is the mean-zero solution of

$$\mathbf{k}\cdot\nabla_{\boldsymbol{\xi}}\lambda_2+Q\lambda_2=\mathcal{K}\lambda_1-\mathbb{E}\{\mathcal{K}\lambda_1\}. \tag{90}$$

A mean-zero solution of (90) exists according to the Fredholm alternative, as the operator Q has a spectral gap. A straightforward calculation presented below shows that

$$\mathbb{E}\left\{\mathcal{K}\lambda_1\right\}=-\mathcal{L}^*\lambda. \tag{91}$$

Hence the second corrector is given by

$$\lambda_2(z,\mathbf{x},\boldsymbol{\xi},\mathbf{k},\hat V)=-\int_0^\infty dre^{rQ}\left[\mathcal{L}^*\lambda(z,\mathbf{x},\mathbf{k})+[\mathcal{K}\lambda_1](z,\mathbf{x},\boldsymbol{\xi}+r\mathbf{k},\mathbf{k},\hat V)\right].$$

The above computation and straightforward estimates, as in [6], show that

$$\frac{d}{dh}\mathbb{E}^{\tilde{P}_\varepsilon}_{W,\hat{V},z}\left\{\langle W,\lambda_\varepsilon\rangle\right\}(z+h)\bigg|_{h=0} = \left\langle W, \left(\frac{\partial}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\right)\lambda + \mathcal{L}^*\lambda\right\rangle + \sqrt{\varepsilon}\langle W,\zeta_\varepsilon^\lambda\rangle$$

where $\|\zeta_\varepsilon^\lambda\|_2 \le C$, with a deterministic constant $C>0$. It follows that $G^\varepsilon_{\lambda_\varepsilon}$ given by

$$G^\varepsilon_{\lambda_\varepsilon}(t) = \langle W(t),\lambda_\varepsilon\rangle - \int_0^t ds\left\langle W,\left(\frac{\partial}{\partial s}+\mathbf{k}\cdot\nabla_{\mathbf{x}}+\mathcal{L}^*\right)\lambda\right\rangle(s) - \sqrt{\varepsilon}\int_0^t ds\langle W,\zeta_\varepsilon^\lambda\rangle(s) \qquad (92)$$

is a martingale with respect to the measure $\tilde{P}_\varepsilon$ defined on $D([0,Z];X\times\mathcal{V})$, the space of right-continuous paths with left-side limits. In order to show that (91) holds let us compute

$$\begin{aligned}\mathbb{E}\left\{-\mathcal{K}\lambda_1\right\} &= \mathbb{E}\left\{-\frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\lambda_1(\mathbf{x},\boldsymbol{\xi},\mathbf{k}-\frac{\mathbf{p}}{2}) + \frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\lambda_1(\mathbf{x},\boldsymbol{\xi},\mathbf{k}+\frac{\mathbf{p}}{2})\right\}\\ &= I_1+I_2+II_1+II_2.\end{aligned}$$

We compute the four terms above separately:

$$I = \mathbb{E}\left\{-\frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\lambda_1(\mathbf{x},\boldsymbol{\xi},\mathbf{k}-\frac{\mathbf{p}}{2})\right\} = I_1+I_2$$

with

$$\begin{aligned}I_1 &= -\mathbb{E}\left\{\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\int_0^\infty dr e^{rQ}\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{q})}{(2\pi)^d}e^{ir((\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{q})+i(\boldsymbol{\xi}\cdot\mathbf{q})}\lambda(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2})\right\}\\ &= -\int_0^\infty dr\int\tilde{R}_{11}(r,\mathbf{p})e^{-ir((\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p})}\lambda(z,\mathbf{x},\mathbf{k})\frac{d\mathbf{p}}{(2\pi)^d}\\ &= -\int\frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}}\hat{R}_{11}(\omega,\mathbf{p})\lambda(z,\mathbf{x},\mathbf{k})\int_0^\infty dr\exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}.\end{aligned}$$

The second term is

$$\begin{aligned}I_2 &= \mathbb{E}\left\{\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\int_0^\infty dr e^{rQ}\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{q})}{(2\pi)^d}e^{ir((\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{q})+i(\boldsymbol{\xi}\cdot\mathbf{q})}\lambda(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}+\frac{\mathbf{q}}{2})\right\}\\ &= \int_0^\infty dr\int\tilde{R}_{12}(r,\mathbf{p})e^{-ir((\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p})}\lambda(z,\mathbf{x},\mathbf{k}-\mathbf{p})\frac{d\mathbf{p}}{(2\pi)^d}\\ &\int\frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}}\hat{R}_{12}(\omega,\mathbf{p})\lambda(z,\mathbf{x},\mathbf{k}-\mathbf{p})\int_0^\infty dr\exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}.\end{aligned}$$

The term II is given by

$$II = \frac{1}{i}\mathbb{E}\left\{\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\lambda_1(\mathbf{x},\boldsymbol{\xi},\mathbf{k}+\frac{\mathbf{p}}{2})\right\} = II_1+II_2$$

with

$$
\begin{aligned}
II_1 &= \mathbb{E}\left\{\int_{\mathbb{R}^d} \frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d} e^{i\mathbf{p}\cdot\boldsymbol{\xi}} \int_0^\infty dr e^{rQ} \int_{\mathbb{R}^d} \frac{d\hat{V}_1(\mathbf{q})}{(2\pi)^d} e^{ir((\mathbf{k}+\mathbf{p}/2)\cdot\mathbf{q})+i(\boldsymbol{\xi}\cdot\mathbf{q})} \lambda(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2})\right\}\\
&= \int_0^\infty dr \int_{\mathbb{R}^d} \tilde{R}_{21}(r,\mathbf{p}) e^{-ir((\mathbf{k}+\mathbf{p}/2)\cdot\mathbf{p})} \lambda(t,\mathbf{x},\mathbf{k}+\mathbf{p}) \frac{d\mathbf{p}}{(2\pi)^d}\\
&= \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{21}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}+\mathbf{p}) \int_0^\infty dr \exp\{ir[\omega-(\mathbf{k}+\mathbf{p}/2)\cdot\mathbf{p}]\}\\
&= \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{12}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}-\mathbf{p}) \int_0^\infty dr \exp\{ir[-\omega+(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}\\
&= \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{12}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}-\mathbf{p}) \int_{-\infty}^0 dr \exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}
\end{aligned}
$$

and

$$
\begin{aligned}
II_2 &= -\mathbb{E}\left\{\int_{\mathbb{R}^d} \frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d} e^{i\mathbf{p}\cdot\boldsymbol{\xi}} \int_0^\infty dr e^{rQ} \int_{\mathbb{R}^d} \frac{d\hat{V}_2(\mathbf{q})}{(2\pi)^d} e^{ir((\mathbf{k}+\mathbf{p}/2)\cdot\mathbf{q})+i(\boldsymbol{\xi}\cdot\mathbf{q})} \lambda(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}+\frac{\mathbf{q}}{2})\right\}\\
&= -\int_0^\infty dr \int_{\mathbb{R}^d} \tilde{R}_{22}(r,\mathbf{p}) e^{-ir((\mathbf{k}+\mathbf{p}/2)\cdot\mathbf{p})} \lambda(z,\mathbf{x},\mathbf{k}) \frac{d\mathbf{p}}{(2\pi)^d}\\
&= -\int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{22}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}) \int_0^\infty dr \exp\{ir[\omega-(\mathbf{k}+\mathbf{p}/2)\cdot\mathbf{p}]\}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
I_2 + II_1 &= \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{12}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}-\mathbf{p}) \int_{-\infty}^\infty dr \exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}\\
&= \int \hat{R}_{12}((\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p},\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}-\mathbf{p}) \frac{d\mathbf{p}}{(2\pi)^d} = \int \hat{R}_{12}(\frac{\mathbf{k}^2-\mathbf{p}^2}{2},\mathbf{k}-\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{p}) \frac{d\mathbf{p}}{(2\pi)^d}.
\end{aligned}
$$

Furthermore, we also have

$$
\begin{aligned}
-[I_1 + II_2] &= \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{11}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}) \int_0^\infty dr \exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}\\
&+ \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{22}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}) \int_0^\infty dr \exp\{ir[\omega-(\mathbf{k}+\mathbf{p}/2)\cdot\mathbf{p}]\}\\
&= \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{11}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}) \int_0^\infty dr \exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}\\
&+ \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \hat{R}_{22}(\omega,\mathbf{p}) \lambda(z,\mathbf{x},\mathbf{k}) \int_{-\infty}^0 dr \exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}\\
&= \int \frac{\hat{R}_{11}((\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p},\mathbf{p}) + \hat{R}_{22}((\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p},\mathbf{p})}{2} \lambda(z,\mathbf{x},\mathbf{k}) \frac{d\mathbf{p}}{(2\pi)^d}\\
&+ \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \frac{\hat{R}_{11}(\omega,\mathbf{p}) - \hat{R}_{22}(\omega,\mathbf{p})}{2} \lambda(z,\mathbf{x},\mathbf{k}) \int_{-\infty}^\infty dr \exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}\mathrm{sgn}(r) = A + B
\end{aligned}
$$

with

$$
A = \int \frac{\hat{R}_{11}(\frac{\mathbf{k}^2-\mathbf{p}^2}{2},\mathbf{k}-\mathbf{p}) + \hat{R}_{22}(\frac{\mathbf{k}^2-\mathbf{p}^2}{2},\mathbf{k}-\mathbf{p})}{2} \lambda(z,\mathbf{x},\mathbf{k}) \frac{d\mathbf{p}}{(2\pi)^d}
$$

and

$$
\begin{aligned}
B &= \int \frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}} \frac{\hat{R}_{11}(\omega,\mathbf{p})-\hat{R}_{22}(\omega,\mathbf{p})}{2} \int_{-\infty}^{\infty} dr \exp\{ir[\omega-(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}]\}\mathrm{sgn}(r)\lambda(z,\mathbf{x},\mathbf{k}) \\
&= \int_{-\infty}^{\infty} dr \int \frac{d\mathbf{p}}{(2\pi)^d} \frac{\tilde{R}_{11}(r,\mathbf{p})-\tilde{R}_{22}(r,\mathbf{p})}{2} \exp\{-ir(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{p}\}\mathrm{sgn}(r)\lambda(z,\mathbf{x},\mathbf{k}).
\end{aligned}
$$

Hence (91) indeed holds and the proof of Lemma 3.3 is complete.

B The proof of Lemma 3.4

The proof is very similar to what is presented in [6]. We highlight the main differences here and refer the reader to that work for additional details. We let $\mu(z,\mathbf{X},\mathbf{K}) \in \mathcal{S}(\mathbb{R}^{2d}\times\mathbb{R}^{2d})$ be a test function independent of $\hat{V}_{1,2}$, where $\mathbf{X}=(\mathbf{x}_1,\mathbf{x}_2)$, and $\mathbf{K}=(\mathbf{k}_1,\mathbf{k}_2)$. We define an approximation

$$
\mu_\varepsilon(z,\mathbf{X},\mathbf{K}) = \mu(z,\mathbf{X},\mathbf{K}) + \sqrt{\varepsilon}\mu_1(z,\mathbf{X},\mathbf{X}/\varepsilon,\mathbf{K}) + \varepsilon\mu_2(z,\mathbf{X},\mathbf{X}/\varepsilon,\mathbf{K}).
$$

We will use the notation $\mu_1^\varepsilon(z,\mathbf{X},\mathbf{K}) = \mu_1(z,\mathbf{X},\mathbf{X}/\varepsilon,\mathbf{K})$ and $\mu_2^\varepsilon(z,\mathbf{X},\mathbf{K}) = \mu_2(z,\mathbf{X},\mathbf{X}/\varepsilon,\mathbf{K})$. The functions μ_1 and μ_2 are to be determined. We now use (42) to get

$$
\begin{aligned}
D_\varepsilon := \frac{d}{dh}\Big|_{h=0} \mathbb{E}_{W,\hat{V},z}(\langle W\otimes W, \mu_\varepsilon(\hat{V})\rangle)(z+h) = \frac{1}{\varepsilon}\left\langle W\otimes W, \left(Q+\sum_{j=1}^2 \mathbf{k}^j\cdot\nabla_{\boldsymbol{\xi}^j}\right)\mu\right\rangle \qquad (93) \\
+\frac{1}{\sqrt{\varepsilon}}\left\langle W\otimes W, \left(Q+\sum_{j=1}^2 \mathbf{k}^j\cdot\nabla_{\boldsymbol{\xi}^j}\right)\mu_1 - \sum_{j=1}^2 \mathcal{K}_j\left[\hat{V},\boldsymbol{\xi}^j\right]\mu\right\rangle \\
+\left\langle W\otimes W, \left(Q+\sum_{j=1}^2 \mathbf{k}^j\cdot\nabla_{\boldsymbol{\xi}^j}\right)\mu_2 - \sum_{j=1}^2 \mathcal{K}_j\left[\hat{V},\boldsymbol{\xi}^j\right]\mu_1 + \frac{\partial\mu}{\partial z} + \sum_{j=1}^2 \mathbf{k}^j\cdot\nabla_{\mathbf{x}^j}\mu\right\rangle \\
+\sqrt{\varepsilon}\left\langle W\otimes W, -\sum_{j=1}^2 \mathcal{K}_j\left[\hat{V},\boldsymbol{\xi}^j\right]\mu_2 + \left(\frac{\partial}{\partial z}+\sum_{j=1}^2 \mathbf{k}^j\cdot\nabla_{\mathbf{x}^j}\right)(\mu_1+\sqrt{\varepsilon}\mu_2)\right\rangle.
\end{aligned}
$$

The above expression is evaluated at $\boldsymbol{\xi}_j = \mathbf{x}_j/\varepsilon$. The term of order ε^{-1} in D_ε vanishes since μ is independent of V and the fast variable $\boldsymbol{\xi}$. We cancel the term of order $\varepsilon^{-1/2}$ in the same way as in the proof of Lemma 3.3 by defining μ_1 as the unique mean-zero (in the variables $\hat{V}$ and $\boldsymbol{\xi}=(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$) solution of

$$
(Q+\sum_{j=1}^2 \mathbf{k}^j\cdot\nabla_{\boldsymbol{\xi}^j})\mu_1 - \sum_{j=1}^2 \mathcal{K}_j\left[\hat{V},\boldsymbol{\xi}^j\right]\mu = 0. \qquad (94)
$$

It is given explicitly by

$$\mu_1(\mathbf{X},\boldsymbol{\xi},\mathbf{K},\hat{V}) = \frac{1}{i}\int_0^\infty dr e^{rQ}\left\{\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{ir(\mathbf{k}_1\cdot\mathbf{p})+i(\boldsymbol{\xi}_1\cdot\mathbf{p})}\mu(\mathbf{k}_1-\frac{\mathbf{p}}{2},\mathbf{k}_2)\right.$$
$$\left.-\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d}e^{ir(\mathbf{k}_1\cdot\mathbf{p})+i(\boldsymbol{\xi}_1\cdot\mathbf{p})}\mu(\mathbf{k}_1+\frac{\mathbf{p}}{2},\mathbf{k}_2)\right\}$$
$$+\frac{1}{i}\int_0^\infty dr e^{rQ}\left\{\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{ir(\mathbf{k}_2\cdot\mathbf{p})+i(\boldsymbol{\xi}_2\cdot\mathbf{p})}\mu(\mathbf{k}_1,\mathbf{k}_2-\frac{\mathbf{p}}{2})\right.$$
$$\left.-\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d}e^{ir(\mathbf{k}_2\cdot\mathbf{p})+i(\boldsymbol{\xi}_2\cdot\mathbf{p})}\mu(\mathbf{k}_1,\mathbf{k}_2+\frac{\mathbf{p}}{2})\right\}.$$

Let us also define μ_2 as the mean zero with respect to π_V solution of

$$(Q+\sum_{j=1}^2\mathbf{k}^j\cdot\nabla_{\boldsymbol{\xi}^j})\mu_2-\sum_{j=1}^2\mathcal{K}_j\left[\hat{V},\boldsymbol{\xi}^j\right]\mu_1=-\overline{\sum_{j=1}^2\mathcal{K}_j\left[\hat{V},\boldsymbol{\xi}^j\right]\mu_1}, \tag{95}$$

where $\overline{f}=\int d\pi_V f$.

In order to finish the proof of Lemma 3.4 we have to compute

$$\mathcal{L}^*_{2,\varepsilon}\mu=-\mathbb{E}\left\{\sum_{j=1}^2\mathcal{K}_j\left[\hat{V},\boldsymbol{\xi}^j\right]\mu_1\right\}=-\mathbb{E}\left\{\mathcal{K}_1\left[\hat{V},\boldsymbol{\xi}^1\right]\mu_1\right\}+\mathbb{E}\left\{\mathcal{K}_2\left[\hat{V},\boldsymbol{\xi}^2\right]\mu_1\right\}=I_1+I_2 \tag{96}$$

and verify that

$$\|\mathcal{L}^*_{2,\varepsilon}-\mathcal{L}^*\otimes\mathcal{L}^*\|_{L^2\to L^2}\to 0 \tag{97}$$

as $\varepsilon\to 0$. This is done by a straightforward but tedious calculation. We present some of the details for the convenience of the reader. The first term in (96) is

$$I_1=\frac{1}{i}\mathbb{E}\left\{\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i(\mathbf{p}\cdot\boldsymbol{\xi}_1)}\mu_1(\mathbf{k}_1-\frac{\mathbf{p}}{2},\mathbf{k}_2)-\int_{\mathbb{R}^d}\frac{d\hat{V}_2(\mathbf{p})}{(2\pi)^d}e^{i(\mathbf{p}\cdot\boldsymbol{\xi}_1)}\mu_1(\mathbf{k}_1+\frac{\mathbf{p}}{2},\mathbf{k}_2)\right\}=I_{11}+I_{12}.$$

Now we further split

$$I_{11}=I_{1111}+I_{1121}+I_{1112}+I_{1122}$$

according to the four terms in the expression for μ_2. We compute the first and the third terms as they illustrate the general picture:

$$I_{1111}=\frac{1}{i}\mathbb{E}\left\{\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i(\mathbf{p}\cdot\boldsymbol{\xi}_1)}\frac{1}{i}\int_0^\infty dr e^{rQ}\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{q})}{(2\pi)^d}e^{ir((\mathbf{k}_1-\mathbf{p}/2)\cdot\mathbf{q})+i(\boldsymbol{\xi}_1\cdot\mathbf{q})}\mu(\mathbf{k}_1-\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2},\mathbf{k}_2)\right\}$$
$$=-\int_0^\infty dr\int\tilde{R}_{11}(r,\mathbf{p})e^{-ir((\mathbf{k}_1-\mathbf{p}/2)\cdot\mathbf{p})}\mu(\mathbf{k}_1,\mathbf{k}_2)\frac{d\mathbf{p}}{(2\pi)^d} \tag{98}$$
$$=-\int_0^\infty dr\int\hat{R}_{11}(\omega,\mathbf{p})e^{ir(\omega-(\mathbf{k}_1-\mathbf{p}/2)\cdot\mathbf{p})}\mu(\mathbf{k}_1,\mathbf{k}_2)\frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}},$$

and

$$I_{1112}=-\mathbb{E}\left\{\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{p})}{(2\pi)^d}e^{i(\mathbf{p}\cdot\boldsymbol{\xi}_1)}\int_0^\infty dr e^{rQ}\int_{\mathbb{R}^d}\frac{d\hat{V}_1(\mathbf{q})}{(2\pi)^d}e^{ir(\mathbf{k}_2\cdot\mathbf{q})+i(\boldsymbol{\xi}_2\cdot\mathbf{q})}\mu(\mathbf{k}_1-\frac{\mathbf{p}}{2},\mathbf{k}_2-\frac{\mathbf{q}}{2})\right\}$$
$$=-\int_0^\infty dr\int\hat{R}_{11}(\omega,\mathbf{p})e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)/\varepsilon}e^{ir(\omega-(\mathbf{k}_2\cdot\mathbf{p}))}\mu(\mathbf{k}_1-\frac{\mathbf{p}}{2},\mathbf{k}_2+\frac{\mathbf{p}}{2})\frac{d\mathbf{p}d\omega}{(2\pi)^{d+1}}. \tag{99}$$

The terms as in (98) combine exactly to be equal to $\mathcal{L}^* \otimes \mathcal{L}^*$. The terms as in (99) vanish as $\varepsilon \to 0$ in the L^2-sense – this is verified as in [6]. Notice that the a priori regularity of the Wigner measure in $L^2(\mathbb{R}^{2d})$ resulting from the mixture of states is crucial to obtain convergence to 0 in (99); see the difference between [5] and [6]. This completes the sketch of the proof of Lemma 3.4.

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