Stability estimates in stationary inverse transport

Guillaume Bal and Alexandre Jollivet *

April 8, 2008

Abstract

We study the stability of the reconstruction of the scattering and absorption coefficients in a stationary linear transport equation from knowledge of the full albedo operator in dimension $n \geq 3$. The albedo operator is defined as the mapping from the incoming boundary conditions to the outgoing transport solution at the boundary of a compact and convex domain. The uniqueness of the reconstruction was proved in [3, 4] and partial stability estimates were obtained in [13] for spatially independent scattering coefficients. We generalize these results and prove an L^1-stability estimate for spatially dependent scattering coefficients.

1 Introduction

Let the spatial domain $X \subset \mathbb{R}^n$, $n \geq 2$, be a convex, open bounded subset with C^1 boundary ∂X, and let the velocity domain V be $\mathbb{S}^{n-1}$ or an open subset of $\mathbb{R}^n$ which satisfies $\inf_{v \in V} |v| > 0$. Let $\Gamma_\pm = \{(x, v) \in \partial X \times V; \pm n(x)v > 0\}$ where $n(x)$ denotes the outward normal vector to ∂X at $x \in \partial X$. The set Γ_- is the set of incoming boundary condition while Γ_+ is the set where we measure the outgoing solution to the following stationary linear Boltzmann transport equation in $X \times V$:

$$\begin{aligned} &v\nabla_x f(x,v) + \sigma(x,v)f(x,v) - \int_V k(x,v',v)f(x,v')dv' = 0 \quad \text{in } X \times V, \qquad (1.1)\\ &f_{|\Gamma_-} = f_-. \end{aligned}$$

Here, $f(x, v)$ models the density of particles at position $x \in X$ with velocity $v \in V$.

The albedo operator $\mathcal{A}$ is then defined by

$$\mathcal{A} : f_- \mapsto f_{|\Gamma_+}, \qquad (1.2)$$

*Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027, USA; gb2030@columbia.edu and aj2315@columbia.edu

where $f(x, v)$ is the solution to (1.1). The inverse transport problem consists of reconstructing the absorption coefficient $\sigma(x, v)$ and the scattering coefficient $k(x, v', v)$ from knowledge of $\mathcal{A}$. Stability estimates aim at controlling the variations in the reconstructed coefficients $\sigma(x, v)$ and $k(x, v', v)$ from variations in $\mathcal{A}$ in suitable metrics.

The forward transport equation has been analyzed in e.g. [5, 6, 8]. The inverse transport problem has been addressed in e.g. [3, 4, 9, 10] with stability estimates obtained in [9, 13]. For the two-dimensional case, in which proofs of uniqueness of the scattering coefficient are available only when it is sufficiently small or independent of the spatial variable, we refer the reader to e.g. [1, 11, 12].

To obtain our stability estimates, we follow a methodology based on the decomposition of the albedo operator into singular components [3, 4] and the use of appropriate functions on $\Gamma_\pm$ with decreasing support [13]. In dimensions $n \geq 3$ the contribution due to single scattering is more singular than the contribution due to higher orders of scattering. As a consequence, the single scattering in a direction v' generated by a delta function $f_- = \delta_{x_0}(x)\delta(v - v_0)$ is a one-dimensional curve on ∂X. In order to obtain general stability estimates for the scattering coefficient, one way to proceed is to construct test functions whose support converges to that specific curve. It turns out that it is simpler to work in a geometry in which this curve becomes a straight line.

We now briefly introduce that geometry and refer the reader to section 2 below for a formal presentation. Let R be a positive real constant such that X is included in the ball $B(R)$ of radius R centered at $x = 0$. On $(B(R)\backslash X) \times V$, the absorption and scattering coefficients vanish and we may solve the equation $v\nabla_x f = 0$. This allows us to map back the incoming conditions f_- on Γ_- as incoming conditions, which we shall still denote by f_-, on F_- and map forward the outgoing solution $f_{|\Gamma_+}$ to an outgoing solution f_+ on F_+, where we have defined

$$F_\pm := \{(x \pm R\hat{v}, v) \in \mathbb{R}^n \times V \text{ for } (x, v) \in \mathbb{R}^n \times V \text{ s.t. } vx = 0, \, |x| < R\}. \qquad (1.3)$$

In other words, $F_\pm$ is the union for each $v \in V$ of the spatial points on a disc of radius R in a plane orthogonal to v and tangent to the sphere of radius R.

The incoming boundary condition is thus now defined on F_- while measurements occur on F_+ and we may define the albedo operator still called $\mathcal{A}$ as an operator mapping f_- defined on F_- to the outgoing solution f_+ on F_+. We may now verify that the single scattering in a direction v' generated by a delta function $f_- = \delta(x - x_0)\delta(v - v_0)$ for $(x_0, v_0) \in F_-$ is a one-dimensional segment in F_+; see Fig. 1. Note also that the geometry we consider here may be more practical than the geometry based on $\Gamma_\pm$. Indeed, we assume that the incoming conditions are generated on a plane for each direction of incidence, and, more importantly, that our measurements are acquired on a plane for each outgoing direction. This is how the collimators used in Computerized Tomography [7] are currently set up.

Under appropriate assumptions on the coefficients, we aim to show that $\mathcal{A}$ is a well posed operator from $L^1(F_-)$ to $L^1(F_+)$. We shall then obtain a stability esti-

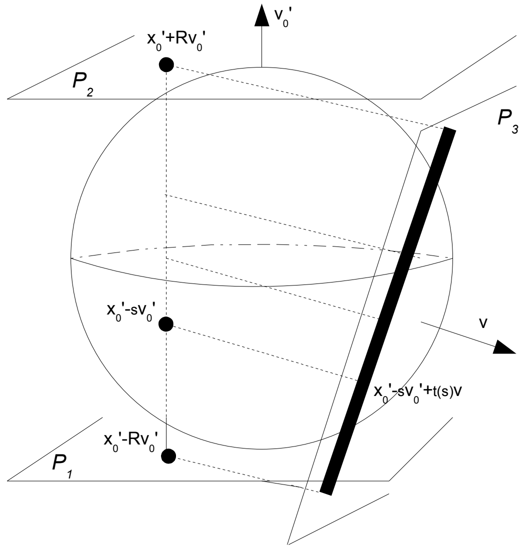

Figure 1: Geometry of the ballistic and single scattering components in dimension $n = 3$. The source term is non-zero in the vicinity (in F_-) of $x'_0 - Rv'_0$ in $P_1 = -Rv'_0 + \Pi_{v'_0}(R)$. The ballistic part is non-zero in the vicinity (in F_+) of $x'_0 + Rv'_0$ in $P_2 = Rv'_0 + \Pi_{v'_0}(R)$. The thick "line" represents the support of the single scattering contribution in the vicinity (in F_+) of the segment $\{x'_0 - sv'_0 + t(s)v; s \in (-R, R)\} \subset P_3 = Rv + \Pi_v(R)$. See text for the notation.

mate for the reconstruction $\sigma(x)$ (or $\sigma(x, |v|)$) and $k(x, v, v')$ with respect to the norm $\mathcal{L}(L^1(F_-), L^1(F_+))$ of $\mathcal{A}$.

The rest of the paper is structured as follows. Because our geometry is not standard, we present a detailed analysis of the linear transport equation and of the singular decomposition of the albedo operator in section 2. Most of the material in that section is similar to that in [4]. One of the main physical constraints in the existence of solutions to (1.1) is that the system be "subcritical", in the sense that the "production" of particles by the scattering term involving the scattering coefficient $k(x, v, v')$ has to be compensated by the absorption of particles and the leakage of particles at the domain's boundary. Although this may be seen implicitly in [4], we state explicitly that the decomposition of the albedo operator used in the stability estimates holds as soon as the forward transport problem is well-posed in a reasonable way.

The stability results are stated in section 3. Under additional continuity assumptions on the absorption and scattering coefficients, we obtain that (i) the exponential of line integrals of the absorption coefficient and (ii) the scattering coefficient multiplied by the exponential of the integral of the absorption coefficient on a broken line are both stably determined by $\mathcal{A}$ in $\mathcal{L}(L^1(F_-), L^1(F_+))$; see Theorem 3.2. Under additional regularity hypotheses on the absorption coefficient, we obtain a stability result for the absorption coefficient in some Sobolev space H^s and for the scattering coefficient in the L^1 norm. The stability results in the geometry of (1.1) are presented in section 4. The proof of the stability results and the construction of the appropriate test functions are presented in section 5. Several proofs on the decomposition of the albedo operator and the uniqueness of the transport equation have been postponed to sections 6 and 7, respectively.

2 Transport equation and albedo operator

We now state our main results on the stationary linear transport equation and the corresponding albedo operator.

Let R be a positive real constant and let $n \in \mathbb{N}$, $n \geq 2$. Let V be $\mathbb{S}^{n-1}$ or an open subset of $\mathbb{R}^n$ which satisfies $\mathrm{v}_0 := \inf_{v\in V} |v| > 0$. For $v \in V$, we define $\hat{v} := \frac{v}{|v|}$. Then, we consider the open subset O of $\mathbb{R}^n \times V$ defined by

$$O := \{(x,v) \in \mathbb{R}^n \times V \mid |x\hat{v}| < R,\ |x - (x\hat{v})\hat{v}| < R\}, \tag{2.1}$$

and let F be the set

$$F := \{(x,v) \in \mathbb{R}^n \times V \mid x \in \Pi_v(R)\}, \tag{2.2}$$

where

$$\Pi_v(R) := \{x \in \mathbb{R}^n \mid xv = 0,\ |x| < R\}, \tag{2.3}$$

for all $v \in V$. For all $v \in V$ we also consider

$$R\hat{v} + \Pi_v(R) := \{R\hat{v} + x \mid x \in \Pi_v(R)\}. \tag{2.4}$$

When $V = \mathbb{S}^{n-1}$, then F is an open subset of $T\mathbb{S}^{n-1} := \{(x,v) \in \mathbb{R}^n \times \mathbb{S}^{n-1} \mid vx = 0\}$, the tangent space to the unit sphere. When V is an open subset of $\mathbb{R}^n$ (which satisfies $\mathrm{v}_0 = \inf_{v\in V} |v| > 0$) then F is an open subset of the $2n-1$ dimensional manifold $\{(x,v) \in \mathbb{R}^n \times V \mid vx = 0\}$. We also define $F_\pm$ by

$$F_\pm := \{(x \pm R\hat{v}, v) \in \mathbb{R}^n \times V | (x,v) \in F\}, \tag{2.5}$$

and recall that F_- is the set of incoming conditions for the transport equation while F_+ is the set in which measurements are performed.

We consider the space $L^1(O)$ with the usual norm

$$\|f\|_O := \int_O |f(x,v)| dx dv, \text{ for } f \in L^1(O). \tag{2.6}$$

We also consider the space $L^1(F)$ defined as the completed Banach space of the vector space of compactly supported continuous functions on F for the norm

$$\|f\|_F := \int_V \int_{\Pi_v(R)} |v||f(x,v)| dx dv,\ f \in L^1(F), \tag{2.7}$$

and similarly the spaces $L^1(F_\pm)$ defined as the completed Banach space of the vector space of compactly supported continuous functions on $F_\pm$ for the norm

$$\|f_\pm\|_{F_\pm} := \int_V \int_{\Pi_v(R)} |v||f_\pm(x \pm R\hat{v}, v)|dxdv,\ f_\pm \in L^1(F_\pm). \tag{2.8}$$

We assume that:

$$\begin{aligned}
&0 \le \sigma \in L^\infty(\mathbb{R}^n \times V),\\
&k(x, v', v) \text{ is a measurable function on } \mathbb{R}^n \times V \times V,\\
&\sigma(x, v) = k(x, v', v) = 0 \text{ for } (x, v', v) \in \mathbb{R}^n \times V \times V,\ |x| > R,\\
&0 \le k(x, v', .) \in L^1(V) \text{ for a.e. } (x, v') \in \mathbb{R}^n \times V \text{ and}\\
&\sigma_p(x, v') = \textstyle\int_V k(x, v', v)dv \text{ belongs to } L^\infty(\mathbb{R}^n \times V).
\end{aligned} \tag{2.9}$$

Under these conditions, we consider the stationary linear Boltzmann transport equation

$$\begin{aligned}
&v\nabla_x f(x, v) + \sigma(x, v)f(x, v) - \int_V k(x, v', v)f(x, v')dv' = 0 \text{ in } O,\\
&f_{|F_-} = f_-.
\end{aligned} \tag{2.10}$$

Throughout the paper, for $m \in \mathbb{N}$ and for any subset U of $\mathbb{R}^m$ we denote by χ_U the characteristic function defined by $\chi_U(x) = 0$ if $x \notin U$ and $\chi_U(x) = 1$ if $x \in U$.

We now analyze the well-posedness of (2.10). The following change of variables is useful.

Lemma 2.1. *For $f \in L^1(O)$, we have:*

$$\int_O f(x, v)dxdv = \int_V \int_{\Pi_v(R)} \int_{-R}^{R} f(y \pm t\hat{v}, v)dtdydv. \tag{2.11}$$

Proof. First using (2.1) we have

$$\int_O f(x, v)dxdv = \int_V \int_{\mathbb{R}^n} \chi_O(x, v)f(x, v)dxdv.$$

Then for a.e. $v \in V$ using the change of variables $\{x \in \mathbb{R}^n \mid |\hat{v}.x| < R,\ |x - (\hat{v}x)\hat{v}| < R\} \to \Pi_v(R) \times (-R, R)$, $x \mapsto (x - (\hat{v}x)\hat{v}, \pm\hat{v}x)$, we obtain (2.11). □

We introduce the following notation:

$$\begin{aligned}
&T_0 f = -v\nabla_x f \text{ in the distributional sense}, \quad A_1 f = -\sigma f,\\
&A_2 f = \int_V k(x, v', v)f(x, v')dv',\ T_1 = T_0 + A_1,\ T = T_0 + A_1 + A_2 = T_1 + A_2,
\end{aligned} \tag{2.12}$$

and the Banach spaces

$$\begin{aligned}
\mathcal{W} &:= \{f \in L^1(O, |v|dxdv); T_0 f \in L^1(O)\}, & \|f\|_{\mathcal{W}} &= \|T_0 f\|_O + \||v|f\|_O,\\
\tilde{\mathcal{W}} &:= \{f \in L^1(O); T_0 f \in L^1(O)\}, & \|f\|_{\tilde{\mathcal{W}}} &= \|T_0 f\|_O + \|f\|_O.
\end{aligned}$$

We consider the space $L(F_\pm)$ defined as the completed Banach space of the vector space of compactly supported continuous functions on $F_\pm$ for the norm

$$\|f_\pm\|_{L(F_\pm)} := \int_V \int_{\Pi_v(R)} |f_\pm(x' \pm R\hat{v}, v)| dx'dv, \text{ for } f_\pm \in L(F_\pm). \tag{2.13}$$

Note that $\mathcal{W} \subseteq \tilde{\mathcal{W}}$ and $L^1(F_\pm) \subseteq L(F_\pm)$. The spaces $\tilde{\mathcal{W}}$ and $L(F_\pm)$ are used only to define the unbounded operators $\mathbf{T}$ and $\mathbf{T_1}$ below. We obtain the following trace result.

Lemma 2.2. *We have*

$$\|f_{|F_\pm}\|_{F_\pm} \le C\|f\|_{\mathcal{W}}, \tag{2.14}$$

for $f \in \mathcal{W}$*, where* $C = \max((2R)^{-1}, 1)$ *and*

$$\|f_{|F_\pm}\|_{L(F_\pm)} \le C'\|f\|_{\tilde{\mathcal{W}}}, \tag{2.15}$$

for $f \in \tilde{\mathcal{W}}$*, where* $C' = \max((2R)^{-1}, \mathrm{v}_0^{-1})$*.*

Proof. Let f be a C^1 function in $\mathbb{R}^n \times V$ with compact support. Then from (2.11), it follows that

$$\|f\|_{\mathcal{W}} = \int_V \int_{\Pi_v(R)} \int_{-R}^{R} (|v||\frac{d}{dt} f(x' \pm t\hat{v}, v)| + |v||f(x' \pm t\hat{v}, v)|) dt dx' dv. \tag{2.16}$$

Let $v \in V$ and $x' \in \mathbb{R}^n$ such that $vx = 0$. Note that $f(x' \mp R\hat{v}, v) = f(x' \pm t\hat{v}, v) - \int_{-R}^{t} \frac{d}{ds} f(x' \pm s\hat{v}, v) ds$ for all $t \in (-R, R)$. Hence $|f(x' \mp R\hat{v}, v)| \le |f(x' \pm t\hat{v}, v)| + \int_{-R}^{R} |\frac{d}{ds} f(x' \pm s\hat{v}, v)| ds$. Upon integrating the latter equality, we obtain

$$|f(x' \mp R\hat{v}, v)| \le \frac{1}{2R} \int_{-R}^{R} |f(x' \pm t\hat{v}, v)| dt + \int_{-R}^{R} |\frac{d}{ds} f(x' \pm s\hat{v}, v)| ds. \tag{2.17}$$

Combining (2.16) et (2.17), we obtain (2.14). The proof of (2.15) is similar. □

For a continuous function f_- on F_-, we define the following extension of f_- in O:

$$Jf_-(x, v) = e^{-|v|^{-1} \int_0^{R + x\hat{v}} \sigma(x - s\hat{v}, v) ds} f_-(x - (x\hat{v} + R)\hat{v}, v), \ (x, v) \in O. \tag{2.18}$$

Lemma 2.3. *For* $f_- \in L^1(F_-)$ *with* $C = 2R(1 + \mathrm{v}_0^{-1}\|\sigma\|_\infty)$*, we have:*

$$\|Jf_-\|_{\mathcal{W}} \le C\|f_-\|_{F_-}. \tag{2.19}$$

Proof. Let f_- be a compactly supported continous function on F_-. From (2.11) and (2.18) it follows that

$$\||v|Jf_-\|_O = \int_V \int_{\Pi_v(R)} |v||f_-(x' - R\hat{v}, v)| \int_{-R}^{R} e^{-|v|^{-1} \int_{-R}^{t} \sigma(x' + s\hat{v}, v) ds} dt dx' dv \le 2R\|f_-\|_{F_-}. \tag{2.20}$$

One can check that Jf_- satisfies $T_0 Jf_- = -A_1 Jf_-$ in the distributional sense. Therefore using also (2.20) we obtain $\|T_0 Jf_-\|_O + \||v|Jf_-\|_O \le (1 + \|A_1|v|^{-1}\|) \times \||v|Jf_-\|_O = 2R(1 + \mathrm{v}_0^{-1}\|\sigma\|_\infty)\|f_-\|_{F_-}$, which proves the lemma. □

2.1 Existence theory for the albedo operator

We consider the following unbounded operators:

$$\mathbf{T_1}f = T_1 f,\ \mathbf{T}f = Tf,\ D(\mathbf{T_1}) = D(\mathbf{T}) = \{f \in \tilde{\mathcal{W}}\ ;\ f_{|F_-} = 0\}. \tag{2.21}$$

The operator $\mathbf{T_1} : D(\mathbf{T_1}) \to L^1(O)$ is close, one-to-one, onto, and its inverse $\mathbf{T_1}^{-1}$ is given for all $f \in L^1(O)$ by

$$\mathbf{T_1}^{-1}f(x,v) = -|v|^{-1}\int_0^{R+x\hat{v}} e^{-|v|^{-1}\int_0^t \sigma(x-s\hat{v},v)ds} f(x-t\hat{v},v)dt,\ (x,v) \in O. \tag{2.22}$$

Lemma 2.4. *The following statements hold:*

i. The bounded operator $|v|\mathbf{T_1}^{-1}$ in $L^1(O)$ has norm less or equal to $2R$ and the bounded operator $A_2|v|^{-1}$ in $L^1(O)$ has norm less than $\||v|^{-1}\sigma_p(x,v)\|_{L^\infty(O)}$.

ii. Under the hypothesis

$$\sigma - \sigma_p \geq 0, \tag{2.23}$$

the bounded operator $A_2\mathbf{T_1}^{-1}$ in $L^1(O)$ has norm less than $1 - e^{-2R\mathrm{v}_0^{-1}\|\sigma_p\|_\infty}$.

iii. Assume either condition (2.23) or

$$2R\||v|^{-1}\sigma_p(x,v)\|_{L^\infty(O)} < 1. \tag{2.24}$$

Then $I + A_2\mathbf{T_1}^{-1}$ is invertible in $L^1(O)$.

Lemma 2.4 is proved in section 7. We denote by K the bounded operator in $L^1(O, |v|dxdv)$ defined by $K = \mathbf{T_1}^{-1}A_2$:

$$Kf(x,v) = -|v|^{-1}\int_0^{R+x\hat{v}} e^{-|v|^{-1}\int_0^t \sigma(x-s\hat{v},v)ds}(A_2 f)(x-t\hat{v},v)dt,\ (x,v) \in X \times V,$$

for all $f \in L^1(O, |v|dxdv)$. The operator K also defines a bounded operator in $L^1(O)$. This allows us to recast the stationary linear Boltzmann transport equation as the following integral equation:

$$(I + K)f = Jf_-. \tag{2.25}$$

The existence theory for the above integral equation is addressed in the following result.

Proposition 2.5. *The following statements hold:*

i. The conditions (2.26) and (2.27) below are equivalent.

The bounded operator $I + K$ in $L^1(O)$ admits a bounded inverse in $L^1(O)$. (2.26)

The bounded operator $I + A_2\mathbf{T_1}^{-1}$ in $L^1(O)$ admits a bounded inverse in $L^1(O)$. (2.27) *ii. Assume either* (2.23) *or* (2.24)*. Then condition* (2.26) *is satisfied.*

iii. If (2.26) *is satisfied then*

$$\begin{aligned}&\textit{the bounded operator } I+K \textit{ in } L^1(O,|v|dxdv)\\ &\textit{admits a bounded inverse in } L^1(O,|v|dxdv).\end{aligned} \tag{2.28}$$

Proposition 2.5 is proved in Section 7. The following proposition deals with the existence of the albedo operator.

Proposition 2.6. *Assume* (2.28)*. Then*

i. the integral equation (2.25) *is uniquely solvable for all* $f_- \in L^1(F_-)$ *and* $f \in \mathcal{W}$ *;*

ii. the albedo operator $\mathcal{A} : f_- \mapsto f_+ = f_{|F_+}$ *is a bounded operator* $\mathcal{A} : L^1(F_-) \to L^1(F_+)$*.*

Proposition 2.6 is proved in Section 7.

2.2 Singular decomposition of the albedo operator

We assume that condition (2.28) is satisfied. Let us consider the operator $\mathcal{R} : L^1(O,|v|dxdv) \to L^1(F_+)$, defined by

$$\psi \mapsto \mathcal{R}\psi := \left(K^2\psi\right)_{|F_+}, \tag{2.29}$$

for $\psi \in L^1(O,|v|dxdv)$. Using the equality (in the distributional sense) $T_0Kf = -A_1f - A_2f$ for $f \in L^1(O,|v|dxdv)$ and the boundedness of the operators A_1 and A_2 from $L^1(O,|v|dxdv)$ to $L^1(O,dxdv)$ and using (2.14), we obtain that $\mathcal{R}$ is a well defined and bounded operator from $L^1(O,|v|dxdv)$ to $L^1(F_+)$. We shall use the following lemma for the kernel distribution of $\mathcal{R}$.

Lemma 2.7. *We have the following decomposition:*

$$\mathcal{R}\psi(x,v) = \int_O \beta(x,v,x',v')\psi(x',v')dx'dv', \tag{2.30}$$

for a.e. $(x,v) \in F_+$ *and for any* $\psi \in L^1(O,|v|dxdv)$*, where*

$$0 \le |v'|^{-1}\beta \in L^\infty(O, L^1(F_+)). \tag{2.31}$$

In addition if $k \in L^\infty(\mathbb{R}^n \times V \times V)$*, then for any* $\varepsilon' > 0$*,* $\delta > 0$*, and any* $1 < p < 1 + \frac{1}{n-1}$ *there exists some nonnegative constant* $C(\varepsilon',\delta,p)$ *such that*

$$\begin{aligned}&\left\|\int_V \int_{R\hat v + \Pi_v(R)} \phi(x,v)\beta(x,v,x',v')|v|dxdv\right\|_{L^\infty(O_{x',v'})}\\ &\le C(\varepsilon',\delta,p)\left(\int_V \int_{R\hat v + \Pi_v(R)} |\phi(x,v)|^{p'}dxdv\right)^{\frac{1}{p'}} + \varepsilon'\|\phi\|_{L^\infty(F_+)},\end{aligned} \tag{2.32}$$

for any continuous compactly supported function ϕ on F_+ such that $\mathrm{supp}\phi \subset \{(x,v) \in F_+ \mid |v| < \delta^{-1}\}$*, and where $p'^{-1} + p^{-1} = 1$.*

Lemma 2.7 is proved in Section 6. The last inequality shows that the kernel of the second scattering operator $\mathcal{R}$ is more regular than is indicated in (2.31). When V is bounded, then we can choose $\varepsilon' = 0$ in (2.32), in which case we obtain that $|v'|^{-1}\beta \in L^\infty(O, L^p(F_+))$ for $1 < p < \frac{n}{n-1}$. This regularity is sufficient (while that described in (2.31) is not) to show that multiple scattering contributions do not interfere with our stability estimates. Taking account of Lemma 2.7, we have the following decomposition for the albedo operator.

Lemma 2.8. *Under condition* (2.28)*, the following equality in the distributional sense is valid*

$$\begin{aligned}\mathcal{A}\phi_-(x,v) &= \int_V \int_{\Pi_{v'}(R)} \alpha(x,v,x',v')\phi_-(x' - R\hat{v'}, v')dx'dv' \\ &+ \int_O \beta(x,v,x',v')((I+K)^{-1}J\phi_-)(x',v')dx'dv',\end{aligned} \tag{2.33}$$

for a.e. $(x,v) \in F_+$ and for any C^1 compactly supported function ϕ_- on F_-, where

$$\alpha(x,v,x',v') = \alpha_1(x,v,x',v') + \alpha_2(x,v,x',v'), \tag{2.34}$$

$$\alpha_1(x,v,x',v') = e^{-|v|^{-1}\int_0^{2R}\sigma(x-s\hat{v},v)ds}\delta_v(v')\delta_{x-(x\hat{v'})\hat{v'}}(x'), \tag{2.35}$$

$$\begin{aligned}\alpha_2(x,v,x',v') &= |v|^{-1}\int_0^{2R} e^{-|v|^{-1}\int_0^t \sigma(x-s\hat{v},v)ds - |v'|^{-1}\int_0^{R+(x-t\hat{v})\hat{v'}}\sigma(x-t\hat{v}-s\hat{v'},v')ds} \\ &\times k(x-t\hat{v},v',v)\delta_{x-t\hat{v}-((x-t\hat{v})\hat{v'})\hat{v'}}(x')dt,\end{aligned} \tag{2.36}$$

for a.e. $(x,v) \in F_+$ and $(x',v') \in F$, and where β is given by (2.30)*.*

Lemma 2.8 is proved in Section 6. The above decomposition is similar to that obtained in [3, 4] except that the multiple scattering contribution is written here in terms of the distribution kernel of $\mathcal{R}$ rather than that of $\mathcal{R}(I+K)^{-1}J$.

3 Stability estimates

In this section, we give stability estimates for the reconstruction of the absorption and scattering coefficient from the albedo operator following the approach in [13].

We assume that conditions (2.9) and (2.28) are satisfied and that there exists a convex open subset X of $\mathbb{R}^n$ with C^1 boundary ∂X such that $\bar{X} \subset B(0,R) := \{x \in \mathbb{R}^n \mid |x| < R\}$ and

$$\begin{aligned}&\text{the function } 0 \le \sigma_{|X\times V} \text{ is continous and bounded in } X \times V, \\ &\text{the function } 0 \le k_{|X\times V\times V} \text{ is continous and bounded in } X \times V \times V, \\ &\sigma(x,v) = k(x,v,v') = 0 \text{ for } x \notin \bar{X},\ (v,v') \in V \times V.\end{aligned} \tag{3.1}$$

Let $(\tilde\sigma, \tilde k)$ be a pair of absorption and scattering coefficients that also satisfy (2.9), (2.28), and (3.1). We denote by a superscript $\tilde{}$ any object (such as the albedo operator $\tilde{\mathcal A}$ or the distribution kernels $\tilde\alpha_i$, $i = 1, 2$) associated to $(\tilde\sigma, \tilde k)$.

Let $(x'_0, v'_0) \in F$ such that the intersection of X and the straight line $\{tv'_0 + x'_0 \mid t \in \mathbb{R}\}$ is not empty. The point $(x'_0 - R\hat{v}'_0, v'_0) \in F_-$ models the incoming condition and is fixed in the analysis that follows. For $\varepsilon > 0$ let $f_\varepsilon \in C_0^\infty(F_-)$ such that $\|f_\varepsilon\|_{F_-} = 1$, $f_\varepsilon \geq 0$ and $\mathrm{supp} f_\varepsilon \subset \{(x' - R\hat{v}', v') \in F_- \mid |v' - v'_0| + |x' - x'_0| < \varepsilon\}$. Hence $|v'|f_\varepsilon$ is a smooth approximation of the delta function on F_- at $(x'_0 - R\hat{v}'_0, v'_0)$ as $\varepsilon \to 0^+$ and is thus an admissible incoming condition in $L^1(F_-)$. The support of f_ε is represented in Fig. 1.

For a.e. $(x, v) \in F$, $t \in \mathbb{R}$ and $v' \in V$ let $E(x, t, v, v') \geq 0$ be defined by

$$E(x,t,v,v') := e^{-|v|^{-1}\int_{-R}^{t}\sigma(x-s\hat v,v)ds-|v'|^{-1}\int_0^{R+(x-t\hat v)\hat v'}\sigma(x-t\hat v-s\hat v',v')ds}. \tag{3.2}$$

Replacing σ by $\tilde\sigma$ in (3.2) we also define $\tilde E(x, t, v, v')$ for a.e. $(x, v) \in F$, $t \in \mathbb{R}$ and $v' \in V$.

Let $\delta > 0$ and let ϕ be any compactly supported continuous function on F_+ such that $\|\phi\|_\infty \leq 1$ and

$$\mathrm{supp}\phi \subseteq \{(x, v) \in F_+ \mid |v| < \delta^{-1}\}. \tag{3.3}$$

Then using (2.33) and (3.2) we obtain for $\varepsilon > 0$ that

$$\int_V \int_{R\hat v+\Pi_v(R)} |v|\phi(x,v)(\mathcal A-\tilde{\mathcal A})f_\varepsilon(x,v)dxdv = I_1(\phi,\varepsilon)+I_2(\phi,\varepsilon)+I_3(\phi,\varepsilon), \tag{3.4}$$

where

$$\begin{aligned}
I_1(\phi,\varepsilon) &= \int_V\int_{\Pi_v(R)} |v|\phi(x+R\hat v,v)\Big(e^{-|v|^{-1}\int_{-R}^{R}\sigma(x-s\hat v,v)ds} \\
&\quad -e^{-|v|^{-1}\int_{-R}^{R}\tilde\sigma(x-s\hat v,v)ds}\Big) f_\varepsilon(x-R\hat v,v)dxdv \qquad (3.5)\\
I_2(\phi,\varepsilon) &= \int_{V\times V}\int_{\Pi_v(R)} \phi(x+R\hat v,v)\int_{-R}^{R}(k(x-t\hat v,v',v)E(x,t,v,v') \\
&\quad -\tilde k(x-t\hat v,v',v)\tilde E(x,t,v,v')\Big) \qquad (3.6)\\
&\quad f_\varepsilon(x-t\hat v-(x-t\hat v)\hat v'-R\hat v',v')dtdxdvdv',\\
I_3(\phi,\varepsilon) &= I_3^1(\phi,\varepsilon)-I_3^2(\phi,\varepsilon), \qquad (3.7)
\end{aligned}$$

and where

$$I_3^1(\phi,\varepsilon) = \int_V\int_{R\hat v+\Pi_v(R)}|v|\phi(x,v)\int_O \beta(x,v,x',v')((I+K)^{-1}Jf_\varepsilon)(x',v')dx'dv'dxdv, \tag{3.8}$$

$$I_3^2(\phi,\varepsilon) = \int_V\int_{R\hat v+\Pi_v(R)}|v|\phi(x,v)\int_O \tilde\beta(x,v,x',v')((I+\tilde K)^{-1}\tilde Jf_\varepsilon)(x',v')dx'dv'dxdv. \tag{3.9}$$

In addition using the estimate $\|\phi\|_\infty \le 1$, item ii of Proposition 2.6 and the definition of f_ε, we obtain

$$\left|\int_V \int_{R\hat v+\Pi_v(R)} |v|\phi(x,v)(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon(x,v)dxdv\right| \le \|(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon\|_{F_+}$$
$$\le \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(F_-),L^1(F_+))}\|f_\varepsilon\|_{F_-} = \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(F_-),L^1(F_+))}. \quad (3.10)$$

3.1 First stability estimate

We now prove a stability estimate under conditions (2.9), (2.28), and (3.1). Taking (3.5)–(3.7) into account, we obtain the following preparatory lemma:

Lemma 3.1. *Assume that (σ,k) and $(\tilde\sigma,\tilde k)$ satisfy conditions (2.9), (2.28), and (3.1). Then the following limits and estimate hold:*

$$I_1(\phi,\varepsilon) \underset{\varepsilon\to 0^+}{\longrightarrow} \phi(x_0'+R\hat{v_0'},v_0') \times\left(e^{-|v_0'|^{-1}\int_{-R}^R\sigma(x_0'-s\hat{v_0'},v_0')ds} - e^{-|v_0'|^{-1}\int_{-R}^R\tilde\sigma(x_0'-s\hat{v_0'},v_0')ds}\right), \quad (3.11)$$
$$I_2(\phi,\varepsilon) \underset{\varepsilon\to 0^+}{\longrightarrow} I_2^1(\phi)+I_2^2(\phi), \quad (3.12)$$

for any compactly supported continuous function ϕ on F_+, where

$$I_2^1(\phi) = \frac{1}{|v_0'|}\int_V\int_{-R}^R (k-\tilde k)(x_0'+t'\hat{v_0'},v_0',v) \times\left[\phi(x+R\hat v,v)E(x,t,v,v_0')\right]_{\substack{t=t(x_0',v_0',t',v)\\ x=x(x_0',v_0',t',v)}} dt'dv, \quad (3.13)$$

$$I_2^2(\phi) = \frac{1}{|v_0'|}\int_V\int_{-R}^R \tilde k(x_0'+t'\hat{v_0'},v_0',v) \times\left[\phi(x+R\hat v,v)(E-\tilde E)(x,t,v,v_0')\right]_{\substack{t=t(x_0',v_0',t',v)\\ x=x(x_0',v_0',t',v)}} dt'dv, \quad (3.14)$$

where E and $\tilde E$ are defined by (3.2) and

$$(t(x_0',v_0',t',v),x(x_0',v_0',t',v)) = \left(-(x_0'+t'\hat{v_0'})\hat v, x_0'+t'\hat{v_0'}-((x_0'+t'\hat{v_0'})\hat v)\hat v\right), \quad (3.15)$$

for $t'\in\mathbb{R}$. In addition, for all $\varepsilon'>0$, $\delta>0$ and for all $1<p<1+\frac{1}{n-1}$ there exists some nonnegative real valued constant $C(\varepsilon',\delta,p)$ such that

$$\sup_{\varepsilon>0}|I_3(\phi,\varepsilon)| \le C\left(C(\varepsilon',\delta,p)\left(\int_V\int_{\Pi_v(R)}\chi_{\mathrm{supp}\phi}(x+R\hat v,v)dxdv\right)^{\frac{1}{p'}}+\varepsilon'\right), \quad (3.16)$$

for any compactly supported continuous function ϕ on F_+, which satisfies $\|\phi\|_\infty \le 1$ and (3.3) *for $\delta > 0$, where ${p'}^{-1} + p^{-1} = 1$ and*

$$C := 2R\|(I+K)^{-1}\|_{\mathcal{L}(L^1(O,|v|dxdv))} + 2R\|(I+\tilde{K})^{-1}\|_{\mathcal{L}(L^1(O,|v|dxdv))}. \qquad (3.17)$$

Lemma 3.1 is proved in Section 5.

Taking account of Lemma 3.1 and (3.10), and choosing an appropriate sequence of functions "ϕ", we obtain the main result of the paper:

Theorem 3.2. *Assume that $n \ge 3$ and that (σ, k) and $(\tilde{\sigma}, \tilde{k})$ satisfy conditions* (2.9), (2.28)*, and* (3.1)*. Then the following estimates are valid:*

$$\left| e^{-|v_0'|^{-1}\int_{-R}^{R}\sigma(x_0'-s\hat{v_0'},v_0')ds} - e^{-|v_0'|^{-1}\int_{-R}^{R}\tilde{\sigma}(x_0'-s\hat{v_0'},v_0')ds} \right| \le \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(F_-),L^1(F_+))}; \qquad (3.18)$$

$$\begin{aligned} & |v_0'|^{-1}\int_V\int_{-R}^{R}\left|(k-\tilde{k})(x_0'+t'\hat{v_0'},v_0',v)\right| [E(x,t,v,v_0')]_{\substack{t=t(x_0',v_0',t',v)\\ x=x(x_0',v_0',t',v)}} dt'dv \\ \le\ & 2R|v_0'|^{-1}\|\tilde{\sigma}_p(x_0'+t'\hat{v_0'},v_0')\|_{L^\infty(\mathbb{R}_{t'})} \sup_{\substack{(x,v)\in F\\ t\in\mathbb{R}}} \left|(E-\tilde{E})(x,t,v,v_0')\right| \\ & + \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(F_-),L^1(F_+))}, \end{aligned} \qquad (3.19)$$

where E and $\tilde{E}$ are defined by (3.2)*, and where $(t(x_0',v_0',t',v), x(x_0',v_0',t',v))$ is defined by* (3.15) *for $t' \in \mathbb{R}$ and $v \in V$.*

Theorem 3.2 is proved in Section 5.

Remark 3.3. *Under condition* (2.26)*, we can obtain similar estimates to those in Theorem 3.2 for the albedo operator defined on $L(F_-)$ with values in $L(F_+)$. Note that $L(F_\pm) = L^1(F_\pm)$ when V is bounded.*

3.2 Stability results under additional regularity assumptions

The second inequality in Theorem 3.2 provides an L^1 stability result for $k(x,v',v)$ provided that $\sigma(x,v)$ is known. The first inequality in Theorem 3.2 shows that the Radon transform of $\sigma(x,v)$ is stably determined by the albedo operator. Because the inverse Radon transform is an unbounded operation, additional constraints, including regularity constraints, on σ are necessary to obtain a stable reconstruction. We assume here that

$$\left\{\frac{v}{|v|} \mid v \in V\right\} = \mathbb{S}^{n-1},\ \mathrm{V}_0 := \sup_{v\in V}|v| < \infty, \qquad (3.20)$$

and that the absorption coefficient σ does not depend on the velocity variable, i.e. $\sigma(x,v)=\sigma(x)$, $x\in\mathbb{R}^n$; see also remark 3.5 below. Then let

$$\mathcal{M} := \big\{(\sigma(x),k(x,v',v)) \in L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n\times V\times V) \mid (\sigma,k) \text{ satisfies (3.1)}, \\ \text{(2.9) and (2.28), and } \sigma_{|X}\in H^{\frac{n}{2}+\tilde r}(X), \|\sigma\|_{H^{\frac{n}{2}+\tilde r}(X)}\le M, \|\sigma_p\|_\infty\le M\big\}, \tag{3.21}$$

for some $\tilde r>0$ and $M>0$. Using Theorem 3.2 for any $(x_0',v_0')\in F$ such that the intersection of X and the straight line $\{tv_0'+x_0' \mid t\in\mathbb{R}\}$ is not empty, we obtain the following theorem.

Theorem 3.4. *Assume that $n\ge 3$. Under condition (3.20), for any $(\sigma,k)\in\mathcal{M}$ and $(\tilde\sigma,\tilde k)\in\mathcal{M}$ the following stability estimates are valid:*

$$\|\sigma-\tilde\sigma\|_{H^s(X)} \le C_1\|\mathcal{A}-\tilde{\mathcal{A}}\|^\theta_{\mathcal{L}(L^1(F_-),L^1(F_+))}, \tag{3.22}$$

where $-\frac{1}{2}\le s<\frac{n}{2}+\tilde r$, $\theta=\frac{n+2(\tilde r-s)}{n+1+2\tilde r}$, and $C_1=C_1(R,X,\mathrm{v}_0,\mathrm{V}_0,M,s,\tilde r)$;

$$\begin{aligned}&\int_V\int_{-R}^{R}\Big|k(x_0'+t'\hat{v_0'},v_0',v)-\tilde k(x_0'+t'\hat{v_0'},v_0',v)\Big|dt'dv\\ \le\;& C_2\|\mathcal{A}-\tilde{\mathcal{A}}\|^\theta_{\mathcal{L}(L^1(F_-),L^1(F_+))}\Big(1+\|\mathcal{A}-\tilde{\mathcal{A}}\|^{1-\theta}_{\mathcal{L}(L^1(F_-),L^1(F_+))}\Big),\end{aligned} \tag{3.23}$$

for $(x_0',v_0')\in F$ such that $x_0'+t'v_0'\in X$ for some $t'\in\mathbb{R}$, and where $\theta=\frac{2(\tilde r-r)}{n+1+2\tilde r}$, $0<r<\tilde r$, and $C_2=C_2(R,X,\mathrm{v}_0,\mathrm{V}_0,M,r,\tilde r)$; in addition,

$$\begin{aligned}&\|k-\tilde k\|_{L^1(\mathbb{R}^n\times V\times V)}\\ \le\;& C_3\|\mathcal{A}-\tilde{\mathcal{A}}\|^\theta_{\mathcal{L}(L^1(F_-),L^1(F_+))}\Big(1+\|\mathcal{A}-\tilde{\mathcal{A}}\|^{1-\theta}_{\mathcal{L}(L^1(F_-),L^1(F_+))}\Big),\end{aligned} \tag{3.24}$$

where $\theta=\frac{2(\tilde r-r)}{n+1+2\tilde r}$, $0<r<\tilde r$, and $C_3=C_3(R,X,\mathrm{v}_0,\mathrm{V}_0,M,r,\tilde r)$.

Theorem 3.4 is proved in Section 5.

Remark 3.5. *Theorem 3.4 can be extended to the case $\sigma=\sigma(x,|v|)$ and $V=\{v\in\mathbb{R}^n \mid 0<\lambda_1\le|v|\le\lambda_2<\infty\}$. In this case the class $\mathcal{M}$ is replaced by the class*

$$\mathcal{N} := \big\{(\sigma(x,|v|),k(x,v',v)) \in L^\infty(\mathbb{R}^n\times V)\times L^\infty(\mathbb{R}^n\times V\times V) \mid \\ (\sigma,k) \text{ satisfies (3.1), (2.9) and (2.28)}, \|\sigma_p\|_\infty\le M \text{ and for any } \lambda\in(\lambda_1,\lambda_2), \\ \sigma_{|X}(.,\lambda)\in H^{\frac{n}{2}+\tilde r}(X), \sup_{\lambda\in(\lambda_1,\lambda_2)}\|\sigma(.,\lambda)\|_{H^{\frac{n}{2}+\tilde r}(X)}\le M\big\}. \tag{3.25}$$

Then the left-hand side of (3.22) is replaced by $\sup_{\lambda\in(\lambda_1,\lambda_2)}\|\sigma(.,\lambda)\|_{H^{\frac{n}{2}+\tilde r}(X)}$ whereas the right-hand side of (3.22) and estimates (3.23)–(3.24) remain unchanged (see the proof of Theorem 3.4).

4 Stability in $\Gamma_\pm$

We now come back to the original geometry in (1.1) and present a similar stability result (Theorem 4.3 below) to Theorem 3.4. The case of a scattering coefficient $k(x,v',v) = k(v',v)$ that does not depend of the space variable x was studied in [13]. We now introduce the notation we need to state our stability result.

Recall that $X \subset \mathbb{R}^n$, $n \geq 2$, is an open bounded subset with C^1 boundary ∂X, and that V is $\mathbb{S}^{n-1}$ or an open subset of $\mathbb{R}^n$ which satisfies $\mathrm{v}_0 := \inf_{v\in V}|v| > 0$, and that the linear stationary Boltzmann transport equation in $X \times V$ takes the form

$$v\nabla_x f(x,v) + \sigma(x,v)f(x,v) - \int_V k(x,v',v)f(x,v')dv' = 0 \text{ in } X \times V, \tag{4.1}$$
$$f_{|\Gamma_-} = f_-.$$

We assume that (σ, k) is admissible if

$$\begin{array}{l} 0 \leq \sigma \in L^\infty(X \times V), \\ k(x,v',v) \text{ is a measurable function on } X \times V \times V, \text{ and} \\ 0 \leq k(x,v',.) \in L^1(V) \text{ for a.e. } (x,v') \in X \times V \\ \sigma_p(x,v') = \int_V k(x,v',v)dv \text{ belongs to } L^\infty(X \times V). \end{array} \tag{4.2}$$

For $(x,v) \in (X \times V) \cup \Gamma_\mp$, let $\tau_\pm(x,v)$ be the real number defined by $\tau_\pm(x,v) = \sup\{t > 0 \mid x \pm sv \in X \text{ for all } s \in (0,t)\}$. For $(x,v) \in X \times V$, let $\tau(x,v)$ be defined by

$$\tau(x,v) = \tau_+(x,v) + \tau_-(x,v).$$

For $(x,v) \in \Gamma_\mp$, we put $\tau(x,v) = \tau_\pm(x,v)$. We consider the measure $d\xi(x,v) = |n(x)v|d\mu(x)dv$ on $\Gamma_\pm$. We still use the notation T_0, T_1, T, A_1, and A_2 as in (2.12) and introduce the following Banach space

$$\begin{array}{l} \mathrm{W} := \{f \in L^1(X \times V); T_0 f \in L^1(X \times V),\ \tau^{-1}f \in L^1(X \times V)\}, \\ \|f\|_{\mathrm{W}} = \|T_0 f\|_{L^1(X\times V)} + \|\tau^{-1}f\|_{L^1(X\times V)}. \end{array}$$

We recall the following trace formula (see Theorem 2.1 of [4])

$$\|f_{|\Gamma_\pm}\|_{L^1(\Gamma_\pm,d\xi)} \leq \|f\|_{\mathrm{W}}, \text{ for } f \in \mathrm{W}. \tag{4.3}$$

Estimate (4.3) is the analog of the estimate (2.14) in the previous measurement setting. For a continuous function f_- on Γ_-, we define $\mathcal{J}f_-$ as the extension of f_- in $X \times V$ given by :

$$\mathcal{J}f_-(x,v) = e^{-\int_0^{\tau_-(x,v)}\sigma(x-sv,v)ds}f_-(x-\tau_-(x,v)v,v),\ (x,v) \in X \times V. \tag{4.4}$$

Note that $\mathcal{J}$ has the following trace property (see Proposition 2.1 of [4]):

$$\|\mathcal{J}f_-\|_{\mathrm{W}} \leq C\|f_-\|_{L^1(\Gamma_-,d\xi)}, \tag{4.5}$$

for $f_- \in L^1(\Gamma_-, d\xi)$, where $C = 1 + \mathrm{diam}(X)\mathrm{v}_0^{-1}\|\sigma\|_\infty$ and where $\mathrm{diam}(X) := \sup_{x,y\in X}|x-y|$. Estimate (4.5) is the analog of estimate (2.19) in the previous measurement setting.

4.1 Existence theory for the albedo operator

We denote by $\mathcal{K}$ the bounded operator of $L^1(X \times V, \tau^{-1}dxdv)$ defined by

$$\mathcal{K}f(x,v) = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma(x-sv,v)ds} \int_V k(x,v',v)f(x-tv,v')dv'dt,\ (x,v) \in X \times V.$$

for all $f \in L^1(X \times V, \tau^{-1}dxdv)$. We transform the stationary linear Boltzmann transport equation (4.1) into the following integral equation

$$(I + \mathcal{K})f = \mathcal{J}f_-. \tag{4.6}$$

We have the following proposition, which is the analog of Proposition 2.6.

Proposition 4.1. *Assume that*

the bounded operator $I + \mathcal{K}$ in $L^1(X \times V, \tau^{-1}dxdv)$ admits a bounded inverse in $L^1(X \times V, \tau^{-1}dxdv)$. (4.7)

Then

i. *the integral equation* (4.6) *is uniquely solvable for all $f_- \in L^1(\Gamma_-, d\xi)$, and $f \in \mathrm{W}$,*

ii. *the operator $\mathrm{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$, $f_- \to f_{|\Gamma_+}$, is a bounded operator. This operator is called the albedo operator* A *for* (4.1).

The above proposition can be proved by slightly modifying the proofs of Propositions 2.3 and 2.4 of [4].

Remark 4.2. i. *Assume that X is also convex. Let $f \in L^1(F_\pm)$ be such that* $\mathrm{supp} f \subseteq \{(x,v) \in F_\pm \mid x + tv \in X$ *for some* $t \in \mathbb{R}\}$, *where $F_\pm$ is defined by* (2.5) *and $R > \mathrm{diam}(X)$. Then we obtain that:*

$$\int_V \int_{\Pi_v(R)} f(x \pm R\hat{v}, v)|v|dxdv = \int_{\Gamma_\pm} f(\gamma_\pm(x,v),v)d\xi(x,v), \tag{4.8}$$

where $\gamma_\pm(x,v) = x - (x\hat{v})\hat{v} \pm R\hat{v}$ for any $(x,v) \in \Gamma_\pm$. Therefore, considering results on existence of the albedo operator A *obtained in [4] and our assumptions* (2.9), *equality* (4.8) *leads us to define the albedo operator $\mathcal{A}$ from $L^1(F_-)$ to $L^1(F_+)$.*

ii. *The condition* (4.7) *is satisfied under either of the following constraints:*

$$\|\tau\sigma_p\|_\infty < 1, \tag{4.9}$$

$$\sigma - \sigma_p \geq 0. \tag{4.10}$$

iii. Assume that

$$\text{the bounded operator } I+\mathcal{K} \text{ in } L^1(X\times V) \text{ admits a bounded inverse in } L^1(X\times V). \tag{4.11}$$

Then we can define the albedo operator from $L^1(\Gamma_-, d\tilde{\xi})$ *to* $L^1(\Gamma_+, d\tilde{\xi})$ *where* $d\tilde{\xi} = \min(\tau,\lambda)d\xi$ *and where* λ *is a positive constant. To prove this latter statement, we need trace results for the functions* $f \in \tilde{\mathrm{W}} := \{f \in L^1(X\times V) \mid T_0 f \in L^1(X\times V)\}$.

iv. Under (4.9) *and the condition* $\|\tau\sigma\| < \infty$*, the existence of the albedo operator* $\mathrm{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$ *is proved in [4] (Proposition 2.3) when* V *is an open subset of* $\mathbb{R}^n$ *(the condition* $\inf_{v\in V}|v| > 0$ *is not required).*

v. Under the condition $\sigma - \sigma_p \geq \nu > 0$*, the existence of the albedo operator* $\mathrm{A} : L^1(\Gamma_-, d\tilde{\xi}) \to L^1(\Gamma_+, d\tilde{\xi})$ *is proved in [4] (Proposition 2.4) when* V *is an open subset of* $\mathbb{R}^n$ *(the condition* $\inf_{v\in V}|v| > 0$ *is not required).*

Finally under (4.7), we also obtain a decomposition of the albedo operator A similar to that of $\mathcal{A}$ given in Lemma 2.8.

4.2 Stability estimates

We assume that X is convex and

$$\begin{array}{l}\text{the function } 0 \leq \sigma \text{ is continous and bounded on } X\times V,\\ \text{the function } 0 \leq k \text{ is continous and bounded on } X\times V\times V.\end{array} \tag{4.12}$$

Let $(\tilde{\sigma}, \tilde{k})$ be a pair of absorption and scattering coefficients that also satisfy (4.12), (4.2) and (4.7). Let $\tilde{\mathrm{A}}$ be the albedo operator from $L^1(\Gamma_-, d\xi)$ to $L^1(\Gamma_+, d\xi)$ associated to $(\tilde{\sigma}, \tilde{k})$. We can now obtain stability results similar to those in Lemma 3.1 and Theorem 3.2. Consider

$$\begin{aligned}\mathrm{M} \ := \ & \big\{(\sigma(x), k(x,v',v)) \in H^{\frac{n}{2}+\tilde{r}}(X)\times C(X\times V\times V) \mid \\ & (\sigma,k) \text{ satisfies (4.12), (4.7)}, \|\sigma\|_{H^{\frac{n}{2}+\tilde{r}}(X)} \leq M, \|\sigma_p\|_\infty \leq M\big\}\end{aligned} \tag{4.13}$$

for some $\tilde{r} > 0$ and $M > 0$. We obtain the following theorem.

Theorem 4.3. *Assume* $n \geq 3$. *Under conditions* (3.20)*, for any* $(\sigma,k) \in \mathrm{M}$ *and* $(\tilde{\sigma},\tilde{k}) \in \mathrm{M}$*, the following stability estimates are valid:*

$$\|\sigma - \tilde{\sigma}\|_{H^s(X)} \leq C_1 \|\mathrm{A}-\tilde{\mathrm{A}}\|^{\theta}_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(\Gamma_+,d\xi))}, \tag{4.14}$$

where $-\frac{1}{2} \leq s < \frac{n}{2}+\tilde{r}$, $\theta = \frac{n+2(\tilde{r}-s)}{n+1+2\tilde{r}}$*, and* $C_1 = C_1(X, \mathrm{v}_0, \mathrm{V}_0, M, s, \tilde{r})$*;*

$$\begin{aligned}&\int_V \int_0^{\tau_+(x_0',v_0')} \left| k(x_0'+t'\hat{v_0'}, v_0', v) - \tilde{k}(x_0'+t'\hat{v_0'}, v_0', v)\right| dt'dv \\ \leq\ & C_2 \|\mathrm{A}-\tilde{\mathrm{A}}\|^{\theta}_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(\Gamma_+,d\xi))}\left(1+\|\mathrm{A}-\tilde{\mathrm{A}}\|^{1-\theta}_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(\Gamma_+,d\xi))}\right),\end{aligned} \tag{4.15}$$

for $(x_0', v_0') \in \Gamma_-$ *and where* $\theta = \frac{2(\tilde r - r)}{n+1+2\tilde r}$, $0 < r < \tilde r$, *and* $C_2 = C_2(X, \mathrm{v}_0, \mathrm{V}_0, M, r, \tilde r)$. *As a consequence, we have*

$$\begin{aligned} & \|k - \tilde k\|_{L^1(X\times V\times V)} \\ \le\ & C_3 \|\mathrm{A} - \tilde{\mathrm{A}}\|^{\theta}_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(\Gamma_+,d\xi))} \Big(1 + \|\mathrm{A} - \tilde{\mathrm{A}}\|^{1-\theta}_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(\Gamma_+,d\xi))}\Big), \end{aligned} \tag{4.16}$$

where $\theta = \frac{2(\tilde r - r)}{n+1+2\tilde r}$, $0 < r < \tilde r$, *and* $C_3 = C_3(X, \mathrm{v}_0, \mathrm{V}_0, M, r, \tilde r)$.

The proof of Theorem 4.3 is similar to that of Theorem 3.4.

Remark 4.4. *Theorem 4.3 can also be extended to the case* $\sigma = \sigma(x, |v|)$ *and* $V = \{v \in \mathbb{R}^n \mid 0 < \lambda_1 \le |v| \le \lambda_2 < \infty\}$. *In this case the class* M *is replaced by the class*

$$\begin{aligned} \mathrm{N} \ :=\ & \big\{(\sigma(x,|v|), k(x,v',v)) \in C(X\times V) \times C(X\times V\times V) \mid \\ & (\sigma, k) \text{ satisfies } (4.12),\ (4.7),\ \|\sigma_p\|_\infty \le M, \quad \text{and for any } \lambda \in (\lambda_1, \lambda_2), \\ & \sigma_{|X}(.,\lambda) \in H^{\frac{n}{2}+\tilde r}(X),\ \sup_{\lambda\in(\lambda_1,\lambda_2)} \|\sigma(.,\lambda)\|_{H^{\frac{n}{2}+\tilde r}(X)} \le M\big\}. \end{aligned} \tag{4.17}$$

Then the left-hand side of (4.14) *is replaced by* $\sup_{\lambda\in(\lambda_1,\lambda_2)} \|\sigma(.,\lambda)\|_{H^{\frac{n}{2}+\tilde r}(X)}$ *whereas the right-hand side of* (4.14) *and estimates* (4.15)–(4.16) *remain unchanged.*

5 Proof of the stability results

We now prove Lemma 3.1 and Theorems 3.1 and 3.2.

Proof of Lemma 3.1. Using the fact that X is a convex subset of $\mathbb{R}^n$ with C^1 boundary and using (3.1), we obtain that

$$\begin{aligned} & \text{the function } \mathbb{R}\times\mathbb{R}^n\times V \ni (t,x,v) \to \int_{-R}^{t} \sigma(x - s\hat v, v) ds \text{ is continous} \\ & \text{at any point } (\bar t, \bar x, \bar v) \text{ such that } \bar x + \eta \bar v \in X \text{ for some real } \eta. \end{aligned} \tag{5.1}$$

The same statement holds by replacing σ by $\tilde\sigma$. From (3.5), it follows that

$$I_1(\phi, \varepsilon) = \int_V \int_{\substack{xv=0 \\ |x|<R}} \Phi_1(x,v) f_\varepsilon(x - R\hat v, v)|v| dx dv, \tag{5.2}$$

where Φ_1 is the bounded function on F defined for $(x,v) \in F$ by

$$\Phi_1(x,v) = \phi(x + R\hat v, v)\left(e^{-|v|^{-1}\int_{-R}^{R}\sigma(x - s\hat v, v)ds} - e^{-|v|^{-1}\int_{-R}^{R}\tilde\sigma(x - s\hat v, v)ds}\right). \tag{5.3}$$

From (5.1) and the continuity of ϕ, it follows that Φ_1 is continuous at the point (x_0', v_0') in F. Therefore using (5.2) and the definition of the functions f_ε, we obtain $\lim_{\varepsilon\to 0^+} I_1(\phi,\varepsilon) = \Phi_1(x_0', v_0')$, which implies (3.11). Performing the change of variables $x - t\hat{v} = x' + t'\hat{v'}$ with $x'v' = 0$ ("$dtdx = dt'dx'$") in formula (3.6) and using (3.1), we obtain

$$I_2(\phi,\varepsilon) = \int_{-R}^{R}\int_V \left(\int_V \int_{\Pi_{v'}(R)} \Phi_{2,t',v}(x',v') f_\varepsilon(x' - R\hat{v'}, v')|v'|dx'dv'\right) dvdt', \tag{5.4}$$

where

$$\Phi_{2,t',v}(x',v') = 0 \text{ if } x' + t'\hat{v'} \notin X, \tag{5.5}$$

and

$$\begin{aligned}\Phi_{2,t',v}(x',v') = \frac{1}{|v'|}\Bigg(& k(x'+t'\hat{v'}, v', v)\,[\phi(x+R\hat{v}, v)E(x,t,v,v')]_{\substack{t=t(x',v',t',v)\\ x=x(x',v',t',v)}} \\ & - \tilde{k}(x'+t'\hat{v'}, v', v)\left[\phi(x+R\hat{v},v)\tilde{E}(x,t,v,v')\right]_{\substack{t=t(x',v',t',v)\\ x=x(x',v',t',v)}}\Bigg) \\ & \text{if } x'+t'\hat{v'} \in X,\end{aligned} \tag{5.6}$$

for $(t',v) \in (-R,R)\times V$, where

$$(t(x',v',t',v), x(x',v',t',v)) := \left(-(x'+t'\hat{v'})\hat{v}, x'+t'\hat{v'} - (x'+t'\hat{v'})\hat{v}\right), \tag{5.7}$$

for $x' \in \mathbb{R}^n$, $v, v' \in V$, $t' \in \mathbb{R}$. Let $t' \in (-R,R)$ such that $x_0' + t'v_0' \in X$, and let $v \in V$. Then from (5.1), (3.1)–(3.2) and (5.6)–(5.7) it follows that $\Phi_{2,t',v}$ is continuous at the point (x_0', v_0'). Hence

$$\int_V \int_{\substack{x'v'=0\\ |x'|<R}} \Phi_{2,t',v}(x',v') f_\varepsilon(x'-R\hat{v'}, v')|v'|dx'dv' \to \Phi_{2,t',v}(x_0', v_0'), \text{ as } \varepsilon \to 0^+. \tag{5.8}$$

Moreover using (5.5)–(5.6) and using the estimate $\sigma \geq 0$ (and (3.2)) and the equality $\|f_\varepsilon\|_{F_-} = 1$ we obtain

$$\left|\int_V\int_{\substack{x'v'=0\\ |x'|<R}} \Phi_{2,t',v}(x',v') f_\varepsilon(x'-R\hat{v'},v')|v'|dx'dv'\right| \leq \frac{\|k+\tilde{k}\|_\infty \|\phi\|_{L^\infty(F_+)}\chi_{\mathrm{supp}_V\phi}(v)}{\mathrm{v}_0}, \tag{5.9}$$

for $(t',v) \in (-R,R)\times V$, where $\mathrm{supp}_V\phi = \overline{\{v \in V \mid \exists x \in \mathbb{R}^n,\ x\hat{v} = R,\ \phi(x,v) \neq 0\}}$. From (5.8), (5.9), (5.4) it follows that $\lim_{\varepsilon\to 0^+} I_2(\phi,\varepsilon) = \int_{-R}^{R}\int_V \Phi_{2,t',v'}(x_0', v_0')dv'\ dt'$, which implies (3.12).

It remains to prove (3.16). We first estimate $\sup_{\varepsilon>0}|I_3^1(\phi,\varepsilon)|$. Using (2.28), the estimate $\|Jf_\varepsilon\|_{L^1(O,|v|dxdv)} \leq 2R\|f_\varepsilon\|_{F_-}$ and the equality $\|f_\varepsilon\|_{F_-} = 1$, we obtain

$$\|(I+K)^{-1}Jf_\varepsilon\|_{L^1(O,|v|dxdv)} \leq 2R\|(I+K)^{-1}\|_{\mathcal{L}(L^1(O,|v|dxdv))},\ \varepsilon > 0. \tag{5.10}$$

Let $\varepsilon' > 0$ and $1 < p < 1 + \frac{1}{n-1}$ and $p^{-1} + p'^{-1} = 1$. Using (2.31), (3.8), (2.32) and the estimate $\|\phi\|_{L^\infty(F_+)} \le 1$ we obtain

$$|I_3^1(\phi,\varepsilon)| = \left|\int_O ((I+K)^{-1}Jf_\varepsilon)(x',v')\int_V\int_{R\hat v+\Pi_v(R)}\phi(x,v)\beta(x,v,x',v')|v|dxdvdx'dv'\right|$$

$$\le \|(I+K)^{-1}Jf_\varepsilon\|_{L^1(O,|v|dxdv)}\left\|\int_V\int_{R\hat v+\Pi_v(R)}\phi(x,v)|v|\frac{1}{|v'|}\beta(x,v,x',v')dxdv\right\|_{L^\infty(O)}$$

$$\le \frac{1}{\mathrm{v}_0}C_1\left(C(\varepsilon',\delta,p)\left(\int_V\int_{R\hat v+\Pi_v(R)}\chi_{\mathrm{supp}\phi}(x,v)dxdv\right)^{\frac{1}{p'}}+\varepsilon'\right), \tag{5.11}$$

where $C(\varepsilon',\delta,p)$ is the constant from (2.32), and

$$C_1 = 2R\|(I+K)^{-1}\|_{\mathcal{L}(L^1(O,|v|dxdv))}. \tag{5.12}$$

Replacing K, J, σ and β by $\tilde K$, $\tilde J$, $\tilde\sigma$ and $\tilde\beta$ in (5.11)–(5.12), we obtain an estimate for $\sup_{\varepsilon>0} I_3^2(\phi,\varepsilon)$. Combining these estimates with (3.7), we obtain (3.16). □

Proof of Theorem 3.2. Let $\varepsilon_1 > 0$ and let ϕ_{ε_1} be any compactly supported continuous function on F_+ which satisfies $0 \le \phi_{\varepsilon_1} \le 1$ and

$$\phi_{\varepsilon_1}(x+R\hat v,v) = 1 \text{ for } (x,v)\in F,\ |x-x_0'|+|v-v_0'| < \frac{\varepsilon_1}{2}, \tag{5.13}$$

$$\mathrm{supp}\phi_{\varepsilon_1} \subseteq \{(x,v)\in F_+,\ |x-R\hat v-x_0'|+|v-v_0'|<\varepsilon_1\}. \tag{5.14}$$

From (3.11) and (5.13) it follows that

$$\lim_{\varepsilon_1\to0^+}\lim_{\varepsilon\to0^+} I_1(\varepsilon,\phi_{\varepsilon_1}) = e^{-|v_0'|^{-1}\int_{-R}^R\sigma(x_0'-s\hat v_0',v_0')ds} - e^{-|v_0'|^{-1}\int_{-R}^R\tilde\sigma(x_0'-s\hat v_0',v_0')ds}. \tag{5.15}$$

From (3.16) and (5.14) it follows that

$$\lim_{\varepsilon_1\to0^+}\limsup_{\varepsilon\to0^+}|I_3(\varepsilon,\phi_{\varepsilon_1})| = 0. \tag{5.16}$$

From (3.13), (3.14), it follows that

$$|I_2^1(\phi_{\varepsilon_1})+I_2^2(\phi_{\varepsilon_1})| \le \frac{2R}{\mathrm{v}_0}(\|k\|_\infty+\|\tilde k\|_\infty)\int_V\chi_{\mathrm{supp}_V\phi_{\varepsilon_1}}(v)dv, \tag{5.17}$$

where $\mathrm{supp}_V\phi_{\varepsilon_1} = \overline{\{v\in V \mid \exists x\in\mathbb{R}^n,\ x\hat v = R,\ \phi_{\varepsilon_1}(x,v)\neq 0\}}$. Note that using (5.14), we obtain

$$\int_V\chi_{\mathrm{supp}_V\phi_{\varepsilon_1}}(v)dxdv \le \int_{\substack{v\in V\\ |v-v_0'|<\varepsilon_1}} dv \to 0,\ \text{as } \varepsilon_1\to0^+. \tag{5.18}$$

From (5.18) and (5.17) it follows that

$$|I_2^1(\phi_{\varepsilon_1}) + I_2^2(\phi_{\varepsilon_1})| \to 0, \text{ as } \varepsilon_1 \to 0^+. \tag{5.19}$$

Note also that from (3.4) and (3.10), it follows that

$$|I_1(\phi_{\varepsilon_1},\varepsilon)| \le \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(F_-),L^1(F_+))} + |I_2^1(\phi_{\varepsilon_1},\varepsilon) + I_2^2(\phi_{\varepsilon_1},\varepsilon)| + |I_3(\phi_{\varepsilon_1},\varepsilon)|, \tag{5.20}$$

for $\varepsilon > 0$ and $\varepsilon_1 > 0$. Combining (5.20) (with "ϕ"$= \phi_{\varepsilon_1}$), (5.15), (5.16) and (5.19), we obtain (3.18). This provides us with a stability result for the absorption coefficient.

It remains to obtain a stability result for the scattering coefficient. We first construct an appropriate set of functions "ϕ" (see (5.28) below). The objective is to construct a sequence of such (smooth) functions whose support converges to the line in F_+ where single scattering is restricted; see Fig.1. Moreover, we want these functions to be good approximations of the sign of $k-\tilde{k}$ on that support. This is the main new ingredient that allows us to obtain stability for spatially dependent scattering coefficients. More precisely, let $U := \{(t',v) \in \mathbb{R}\times V \mid x_0' + t'\hat{v_0'} \in X \text{ and } (k-\tilde{k})(x_0'+t'\hat{v_0'}, v_0', v) > 0\}$. Using (3.1), it follows that U is an open subset of $\mathbb{R}\times V$. Let (K_m) a sequence of compact sets such that $\bigcup_{m\in\mathbb{N}} K_m = U$ and $K_m \subseteq K_{m+1}$ for $m \in \mathbb{N}$. For $m \in \mathbb{N}$ let $\chi_m \in C^\infty(\mathbb{R}\times V,\mathbb{R})$ such that $\chi_{K_m} \le \chi_m \le \chi_U$, and let

$$\rho_m = 2\chi_m - 1. \tag{5.21}$$

Thus using (3.1) we obtain

$$\lim_{m\to+\infty} (k-\tilde{k})(x_0'+t'\hat{v_0'}, v_0', v)\rho_m(t',v) = |k-\tilde{k}|(x_0'+t'\hat{v_0'}, v_0', v), \tag{5.22}$$

for $v \in V$ and $t' \in \mathbb{R}$ such that $x_0' + t'v_0' \in X \cup (\mathbb{R}^n\backslash\bar{X})$. For $(x,v) \in F_+$ such that v and v_0' are linearly independent, we define

$$\mathrm{d}(x,v) := \left| x - x_0' - ((x-x_0')\hat{v})\hat{v} - ((x-x_0')\frac{\hat{v}-(\hat{v_0'}\hat{v})\hat{v_0'}}{\sqrt{1-(\hat{v_0'}\hat{v})^2}})\frac{\hat{v}-(\hat{v_0'}\hat{v})\hat{v_0'}}{\sqrt{1-(\hat{v_0'}\hat{v})^2}} \right|. \tag{5.23}$$

For $(x,v) \in F_+$ such that v and v_0' are linearly independent, we verify that $\mathrm{d}(x,v) = \inf_{t,t'\in\mathbb{R}} |x + t\hat{v} - (x_0' + t'\hat{v_0'})|$ and the infimum is reached at

$$(t,t') = \Big(\frac{(x_0'-x)(\hat{v}-(\hat{v}\hat{v_0'})\hat{v_0'})}{1-(\hat{v}\hat{v_0'})^2}, \frac{(x-x_0')(\hat{v_0'}-(\hat{v}\hat{v_0'})\hat{v})}{1-(\hat{v}\hat{v_0'})^2}\Big). \tag{5.24}$$

Consider

$$\mathcal{V}_{\delta,l} := \{(x,v) \in F_+ \mid |x - R\hat{v}| < R-\delta,\ |\hat{v} - \frac{\hat{v}v_0'}{{v_0'}^2}v_0'| > \delta,\ |v| < \frac{1}{\delta},\ \mathrm{d}(x,v) < \frac{1}{l}\}, \tag{5.25}$$

$$\begin{aligned}\tilde{\mathcal{V}}_{\delta,l} &:= \{(x,v)\in F_+ \mid |x-R\hat{v}|\le R-\delta-\frac{1}{l},\ |\hat{v}-\frac{\hat{v}v_0'}{{v_0'}^2}v_0'|\ge \delta+\frac{1}{l},\\ &\quad |v|\le \delta^{-1}-l^{-1},\ \mathrm{d}(x,v)\le \frac{1}{2l}\},\end{aligned} \tag{5.26}$$

for $0<\delta<\min(R,\mathrm{v}_0^{-1})$ and $l\in\mathbb{N}$, $l>(R-\delta)^{-1}+\delta$. For $0<\delta<\min(R,\mathrm{v}_0^{-1})$ and $l\in\mathbb{N}$, $l>(R-\delta)^{-1}+\delta$, let $\chi_{\delta,l}\in C_0^\infty(F_+)$ be such that

$$\chi_{\tilde{\mathcal{V}}_{\delta,l}}\le \chi_{\delta,l}\le \chi_{\mathcal{V}_{\delta,l}}. \tag{5.27}$$

Finally, for $0<\delta<\min(R,\mathrm{v}_0^{-1})$ and $m,l\in\mathbb{N}$, $l>(R-\delta)^{-1}+\delta$, let $\phi_{\delta,m,l}\in C_0^\infty(F_+)$ be defined by

$$\phi_{\delta,m,l}(x,v):=\chi_{\delta,l}(x,v)\rho_m\left(t',v\right)\Big|_{t'=\frac{(x-x_0')(\hat{v_0'}-(\hat{v}\hat{v_0'})\hat{v})}{1-(\hat{v}\hat{v_0'})^2}}. \tag{5.28}$$

(See (5.22), (5.24) and (5.37)–(5.38) given below.) Note that from (5.28) and (5.27) it follows that

$$\mathrm{supp}\phi_{\delta,m,l}\subseteq \mathcal{V}_{\delta,l}. \tag{5.29}$$

Using (3.11), (5.29) and (5.25), it follows that

$$\lim_{\varepsilon\to 0^+} I_1(\phi_{\delta,m,l},\varepsilon)=0, \tag{5.30}$$

for $0<\delta<\min(R,\mathrm{v}_0^{-1})$ and $m,l\in\mathbb{N}$, $l>(R-\delta)^{-1}+\delta$ (we used that $(x_0'+R\hat{v}_0',v_0')\notin \mathcal{V}_{\delta,l}$).

Note that using (5.25) we obtain

$$\chi_{\mathcal{V}_{\delta,l}}(x,v) \le \chi_G(x,v)\le 1,\ l\in\mathbb{N},\ l>(R-\delta)^{-1}+\delta, \tag{5.31}$$

$$\lim_{l\to\infty}\chi_{\mathcal{V}_{\delta,l}}(x,v) = \chi_{\mathcal{V}_\delta}(x,v), \tag{5.32}$$

for $(x,v)\in F_+$ and $0<\delta<\min(R,\mathrm{v}_0^{-1})$, where G is the compact subset of F_+ given by $G=\{(x,v)\in F_+\mid |x-R\hat{v}|\le R-\delta,\ |\hat{v}-\frac{\hat{v}v_0'}{{v_0'}^2}v_0'|\ge\delta,\ |v|\le\delta^{-1}\}$ and

$$\mathcal{V}_\delta:=\big\{(x,v)\in F_+\mid |x-R\hat{v}|<R-\delta,\ |\hat{v}-\frac{\hat{v}v_0'}{{v_0'}^2}v_0'|>\delta,\ |v|<\delta^{-1},\ \mathrm{d}(x,v)=0\big\}. \tag{5.33}$$

Note also that, as $n\ge 3$, we obtain

$$\int_V\int_{\Pi_v(R)}\chi_{\mathcal{V}_\delta}(x+R\hat{v},v)dxdv=0,\ 0<\delta<\min(R,\mathrm{v}_0^{-1}). \tag{5.34}$$

From (3.16), (5.29), (5.31)–(5.34), it follows that

$$\limsup_{l\to\infty}\limsup_{\varepsilon\to 0^+}|I_3(\phi_{\delta,m,l},\varepsilon)|\le\varepsilon' \text{ for any } \varepsilon'>0. \tag{5.35}$$

Hence
$$\lim_{l\to\infty}\limsup_{\varepsilon\to 0^+}|I_3(\phi_{\delta,m,l},\varepsilon)|=0. \tag{5.36}$$

Let $0<\delta<\min(R,\mathrm{v}_0^{-1})$ and $m\in\mathbb{N}$. Using (5.28), (5.24) and (3.13), we obtain
$$I_2^1(\phi_{\delta,m,l})=\int_V\int_{-R}^R f_{\delta,m,l}(t',v)dt'dv, \tag{5.37}$$
for $l>(R-\delta)^{-1}+\delta$ where
$$\begin{aligned} f_{\delta,m,l}(t',v) \;:=\; & \frac{1}{|v_0'|}(k-\tilde{k})(x_0'+t'\hat{v_0'},v_0',v)\rho_m(t',v)\\ & \times\left[\chi_{\delta,l}(x+R\hat{v},v)E(x,t,v,v_0')\right]_{\substack{t=t(x_0',v_0',t',v)\\ x=x(x_0',v_0',t',v)}}, \end{aligned} \tag{5.38}$$
for $t'\in\mathbb{R}$ and $v\in V$ such that v and v_0' are linearly independent, where $(t(x_0',v_0',t',v),x(x_0',v_0',t',v))$ is defined by (3.15) for $v\in V$ and $t'\in\mathbb{R}$.

Using the estimates $\sigma\ge 0$, $|v|^{-1}\le \mathrm{v}_0^{-1}$, $0\le\chi_{\delta,l}\le 1$, we obtain
$$|f_{\delta,m,l}(t',v)|\le\frac{1}{\mathrm{v}_0}(k+\tilde{k})(x_0'+t'\hat{v_0'},v_0',v)), \tag{5.39}$$
for $l>(R-\delta)^{-1}+\delta$, $t'\in\mathbb{R}$ and $v\in V$ such that v and v_0' are linearly independent. Using (2.9) and (3.1), we obtain that the function arising on the right-hand side of (5.39) is integrable on $V\times(-R,R)$. In addition from (5.25)–(5.27), (5.32) and (5.33), it follows that
$$f_{\delta,m,l}(t',v)\to f_{\delta,m}(t',v)\text{ as } l\to+\infty, \tag{5.40}$$
for $t'\in\mathbb{R}$ and $v\in V$ such that v and v_0' are linearly independent, where
$$\begin{aligned} f_{\delta,m}(t',v) \;:=\; & \frac{1}{|v_0'|}(k-\tilde{k})(x_0'+t'\hat{v_0'},v_0',v)\rho_m(t',v)\\ & \times\left[\chi_{\mathcal{V}_\delta}(x+R\hat{v},v)E(x,t,v,v_0')\right]_{\substack{t=t(x_0',v_0',t',v)\\ x=x(x_0',v_0',t',v)}}, \end{aligned} \tag{5.41}$$
for $t'\in\mathbb{R}$ and $v\in V$ such that v and v_0' are linearly independent. Therefore we obtain by the Lebesgue dominated convergence theorem that
$$\lim_{l\to+\infty}I_2^1(\phi_{\delta,m,l})=\int_V\int_{-R}^R f_{\delta,m}(t',v)dt'dv. \tag{5.42}$$

Let $0<\delta<\min(R,\mathrm{v}_0^{-1})$. We also have $f_{\delta,m}(t',v)\le \mathrm{v}_0^{-1}(k+\tilde{k})(x_0'+t'\hat{v_0'},v_0',v))$, for $m\in\mathbb{N}$. From (5.42), (5.41) and (5.22), it follows that
$$\begin{aligned} \lim_{m\to+\infty}\lim_{l\to+\infty}I_2^1(\phi_{\delta,m,l}) = & \int_V\int_{-R}^R\frac{1}{|v_0'|}|k-\tilde{k}|(x_0'+t'\hat{v_0'},v_0',v)\\ & \left[\chi_{\mathcal{V}_\delta}(x+R\hat{v},v)E(x,t,v,v_0')\right]_{\substack{t=-(x_0'+t_0'\hat{v_0'})\hat{v}\\ x-t\hat{v}=x_0'+t'\hat{v_0'}}}dt'dv. \end{aligned} \tag{5.43}$$

From (5.43), we deduce

$$\lim_{\delta\to 0^+}\lim_{m\to+\infty}\lim_{l\to+\infty} I_2^1(\phi_{\delta,m,l}) = \int_V\int_{-R}^{R}\frac{1}{|v_0'|}|k-\tilde k|(x_0'+t'\hat{v_0'},v_0',v) \times E(x,v,t,v_0')_{\substack{t=-(x_0'+t'\hat{v_0'})\hat v\\ x=x_0'+t'\hat{v_0'}-((x_0'+t'\hat{v_0'})\hat v)\hat v}} dt'dv. \quad (5.44)$$

From (3.14), it follows that

$$|I_2^2(\phi_{\delta,m,l})| \le \frac{2R}{\mathrm{v}_0}\|\tilde\sigma_p(x_0'+t'\hat{v_0'},v_0')\|_{L^\infty(\mathbb{R}_{t'})}\sup_{\substack{(x,v)\in F\\ t\in\mathbb{R}}}\left|(E-\tilde E)(x,t,v,v_0')\right|, \quad (5.45)$$

for $0<\delta<\min(R,\mathrm{v}_0^{-1})$, $m\in\mathbb{N}$ and $l\in\mathbb{N}$, $l>(R-\delta)^{-1}+\delta$. Using (3.4) and (3.10), we obtain

$$|I_2(\phi_{\delta,m,l},\varepsilon)| \le \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(F_-),L^1(F_+))}+|I_1(\phi_{\delta,m,l},\varepsilon)|+|I_3(\phi_{\delta,m,l},\varepsilon)|, \quad (5.46)$$

for $0<\delta<\min(R,\mathrm{v}_0^{-1})$, $m\in\mathbb{N}$ and $l\in\mathbb{N}$, $l>(R-\delta)^{-1}+\delta$. From (5.46), (3.11) and (3.12), it follows that

$$\begin{aligned}|I_2^1(\phi_{\delta,m,l})| \;\le\; & \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(F_-),L^1(F_+))}+|I_2^2(\phi_{\delta,m,l})|+|\lim_{\varepsilon\to 0^+} I_1(\phi_{\delta,m,l},\varepsilon)|\\ &+\limsup_{\varepsilon\to 0^+}|I_3(\phi_{\delta,m,l},\varepsilon)|, \end{aligned} \quad (5.47)$$

for $0<\delta<\min(R,\mathrm{v}_0^{-1})$, $m\in\mathbb{N}$ and $l\in\mathbb{N}$, $l>(R-\delta)^{-1}+\delta$. Estimate (3.19) follows from (5.47), (5.30), (5.36), (3.12), (5.44) and (5.45). □

Proof of Theorem 3.4. The method we use to prove (3.22) is the same as in [13]. Let (σ,k), $(\tilde\sigma,\tilde k)\in\mathcal{M}$. Let $f=\sigma-\tilde\sigma$ and consider Pf the X-ray transform of $f=\sigma-\tilde\sigma$ defined by $Pf(x,\theta):=\int_{-\infty}^{+\infty}f(t\theta+x)dt$ for a.e. $(x,\theta)\in T\mathbb{S}^{n-1}$.

From (3.1) and $f_{|X}\in H^{\frac{n}{2}+\tilde r}(X)$, it follows that

$$\|f\|_{H^{-\frac{1}{2}}(X)} \le D_1(n,X)\|Pf\|_*, \quad (5.48)$$

where

$$\|Pf\|_* := \left(\int_{\mathbb{S}^{n-1}}\int_{\Pi_\theta}|Pf(x,\theta)|^2dxd\theta\right)^{\frac{1}{2}}$$

and $D_1(n,X)$ is a real constant which does not depend on f and $\Pi_\theta:=\{x\in\mathbb{R}^n\mid x\theta=0\}$ for $\theta\in\mathbb{S}^{n-1}$. Using (3.1) (and (σ,k), $(\tilde\sigma,\tilde k)\in\mathcal{M}$), it follows that $Pf(x,\theta)=0$ for $(x,\theta)\in T\mathbb{S}^{n-1}$ and $|x|\ge R$. Therefore using also (5.48) we obtain

$$\|f\|_{H^{-\frac{1}{2}}(X)} \le D_2(n,X)\|Pf\|_{L^\infty(T\mathbb{S}^{n-1})}, \quad (5.49)$$

where $D_2(n, X)$ is a real constant which does not depend on σ.

We also use the following interpolation inequality:

$$\|f\|_{H^s(X)} \le \|f\|_{H^{\frac{n}{2}+\tilde r}}^{\frac{2s+1}{n+1+2\tilde r}} \|f\|_{H^{-\frac{1}{2}}}^{\frac{n+2\tilde r}{n+1+2\tilde r}}, \tag{5.50}$$

for $-\frac{1}{2} \le s \le \frac{n}{2} + \tilde r$. As $\sigma \in \mathcal{M}$, it follows that

$$\|\sigma\|_\infty \le D_3(n, \tilde r)\|\sigma\|_{H^{\frac{n}{2}+\tilde r}} \le D_3(n, \tilde r)M. \tag{5.51}$$

Therefore,

$$\int_{-R}^{R} \sigma(x_0' - s\hat{v_0'})ds \le 2RD_3(n, \tilde r)M, \tag{5.52}$$

for a.e. $(x_0', \hat{v_0'}) \in T\mathbb{S}^{n-1}$. From (5.52) it follows that

$$\begin{aligned} &\left| e^{-|v_0'|^{-1}\int_{-R}^{R}\sigma(x_0'-s\hat{v_0'},v_0')ds} - e^{-|v_0'|^{-1}\int_{-R}^{R}\tilde\sigma(x_0'-s\hat{v_0'},v_0')ds} \right| \\ &\ge \frac{e^{-2\mathrm{v}_0^{-1}RD_3(n,\tilde r)M}}{\mathrm{V}_0} |P(\sigma - \tilde\sigma)(x_0', \hat{v_0'})|, \end{aligned} \tag{5.53}$$

for a.e. $(x_0', v_0') \in \mathbb{R}^n \times V$, $x_0'v_0' = 0$ (we used the equality $e^{t_1} - e^{t_2} = e^c(t_2 - t_1)$ for $t_1 < t_2 \in \mathbb{R}$ and for some $c \in [t_1, t_2]$ which depends on t_1 and t_2). (In fact, the estimate (5.53) is valid for any $(x_0', v_0') \in \mathbb{R}^n \times V$, $x_0'v_0' = 0$, such that $\{x_0' + tv_0' \mid t \in \mathbb{R}\} \cap X \ne \emptyset$ or $\{x_0' + tv_0' \mid t \in \mathbb{R}\} \cap \bar X = \emptyset$.) Combining (5.53), (5.49), and (3.18), we obtain

$$\frac{e^{-2\mathrm{v}_0^{-1}RD_3(n,\tilde r)M}}{D_2(n, X)\mathrm{V}_0} \|\sigma - \tilde\sigma\|_{H^{-\frac{1}{2}}(X)} \le \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(F_-), L^1(F_+))}. \tag{5.54}$$

Combining (5.54) and (5.50), we obtain (3.22).

We now prove (3.23). Using $|v_0'|^{-1} \ge \mathrm{V}_0^{-1}$ for $v \in V$, (3.2), and (5.51), we obtain that

$$\begin{aligned} &\frac{1}{|v_0|'} \int_V \int_{-R}^{R} \left| k(x_0' + t'\hat{v_0'}, v_0', v) - \tilde k(x_0' + t'\hat{v_0'}, v_0', v) \right| E(x, v, t, v_0')_{\substack{t=t(x_0',v_0',t',v)\\ x=x(x_0',v_0',t',v)}} dt'dv \\ &\ge \frac{e^{-4\mathrm{v}_0^{-1}RD_3(n,\tilde r)M}}{\mathrm{V}_0} \int_V \int_{-R}^{R} |k - \tilde k|(x_0' + t'\hat{v_0'}, v_0', v)|dt'dv, \end{aligned} \tag{5.55}$$

for any $(x_0', v_0') \in \mathbb{R}^n \times V$, $x_0'v_0' = 0$, such that $x_0' + sv_0' \in X$ for some $s \in \mathbb{R}$, and where $(t(x_0', v_0', t', v), x(x_0', v_0', t', v))$ is defined by (3.15) for $v \in V$ and $t' \in (-R, R)$.

As $(\tilde\sigma,\tilde k)\in\mathcal{M}$ we have $\|\tilde\sigma_p\|_\infty\le M$. Using the latter estimate, (3.2), and $|v|^{-1}\le \mathrm{v}_0^{-1}$ for all $v\in V$, we obtain

$$
\begin{aligned}
&\|\tilde\sigma_p(x_0'+t'\hat{v_0'},v_0')\|_{L^\infty(\mathbb{R}_{t'})}\sup_{\substack{(x,v)\in F\\ t\in\mathbb{R}}}|E-\tilde E|(x,v,t,v_0')\le Me^{4\mathrm{v}_0^{-1}RD_3(n,\tilde r)M}\\
&\times\sup_{\substack{(x,v)\in F\\ t\in\mathbb{R}}}\left[\frac{1}{|v|}\int_{-R}^{t}|\sigma-\tilde\sigma|(x-s\hat v,v)ds+\frac{1}{|v_0'|}\int_0^{R+(x-t\hat v)\hat{v_0'}}|\sigma-\tilde\sigma|(x-t\hat v-s\hat{v_0'},v_0')ds\right]\\
&\le 4R\mathrm{v}_0^{-1}Me^{4\mathrm{v}_0^{-1}RD_3(n,\tilde r)M}\|\sigma-\tilde\sigma\|_\infty, \qquad (5.56)
\end{aligned}
$$

for any $(x_0',v_0')\in\mathbb{R}^n\times V$, $x_0'v_0'=0$, such that $x_0'+t'v_0'\in X$ for some $t'\in\mathbb{R}$. (We also used $|e^u-e^{\tilde u}|\le e^{\max(|u|,|\tilde u|)}|u-\tilde u|$ where $u=-|v|^{-1}\int_{-R}^t\sigma(x-s\hat v,v)ds-|v_0'|^{-1}\int_0^{R+(x-t\hat v)\hat{v_0'}}\tilde\sigma(x-t\hat v-s\hat{v_0'},v_0')ds$ and $\tilde u$ denotes the real number obtained by replacing σ by $\tilde\sigma$ on the right-hand side of the latter equality which defines u; using (5.51) (for σ and for $\tilde\sigma$) we obtain $\max(|u|,|\tilde u|)\le 4R\mathrm{v}_0^{-1}D_3(n,\tilde r)M$.) Note that $\|\sigma-\tilde\sigma\|_\infty\le D_3(n,r)\|\sigma-\tilde\sigma\|_{H^{\frac{n}{2}+r}}$ for $0<r<\tilde r$ (see (5.51)). Therefore, combining (5.55), (5.56), (3.19) and (3.22), we obtain (3.23).

Let us finally prove (3.24). Let $0<r<\tilde r$ and let $\theta=\frac{2(\tilde r-r)}{n+1+2\tilde r}$. From (3.23) it follows that

$$
\begin{aligned}
&\int_{\substack{x_0'v_0'=0\\ |x_0'|<R}}\int_{-R}^{R}\int_V\left|k(x_0'+t'\hat{v_0'},v_0',v)-\tilde k(x_0'+t'\hat{v_0'},v_0',v)\right|dvdt'dx_0'\\
\le\ & D_4\|\mathcal{A}-\tilde{\mathcal{A}}\|^\theta_{\mathcal{L}(L^1(F_-),L^1(F_+))}\left(1+\|\mathcal{A}-\tilde{\mathcal{A}}\|^{1-\theta}_{\mathcal{L}(L^1(F_-),L^1(F_+))}\right), \qquad (5.57)
\end{aligned}
$$

where $D_4=C_2\int_{\substack{x_0'v_0'=0\\ |x_0'|<R}}dx_0'$ and C_2 is the constant that appears on the right-hand side of (3.23). From (3.1), (5.57) and the change of variables "$x=x_0'+t'v_0'$", it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^n\times V}\left|k(x,v_0',v)-\tilde k(x,v_0',v)\right|dvdx\\
\le\ & D_4\|\mathcal{A}-\tilde{\mathcal{A}}\|^\theta_{\mathcal{L}(L^1(F_-),L^1(F_+))}\left(1+\|\mathcal{A}-\tilde{\mathcal{A}}\|^{1-\theta}_{\mathcal{L}(L^1(F_-),L^1(F_+))}\right). \qquad (5.58)
\end{aligned}
$$

Integrating on $v_0'\in V$ both sides of (5.58), we obtain (3.24).

□

6 Decomposition of the albedo operator

We now prove Lemmas 2.7 and 2.8.

Proof of Lemma 2.7. Let $\psi_- \in L^1(O, |v|dxdv)$. Using the definition of K, we obtain

$$\begin{aligned}(\mathcal{R}\psi_-)(x,v) &= \left(K^2\psi_-\right)_{|F_+}(x,v) \\ &= \int_{V\times V}\frac{1}{|v|}\frac{1}{|v_1|}\int_0^{2R}\int_0^{R+(x-t\hat v)\hat v_1} k(x-t\hat v, v_1, v) \\ &\quad \times k(x-t\hat v - t_1\hat v_1, v', v_1)E_0(x,v,x-t\hat v, v_1, x-t\hat v-t_1\hat v_1) \\ &\quad \times \psi_-(x-t\hat v-t_1\hat v_1, v')dt_1dtdv'dv_1,\end{aligned} \tag{6.1}$$

for a.e. $(x,v)\in F_+$, where

$$E_0(x,v,x-t\hat v,v_1,x-t\hat v-t_1\hat v_1) = e^{-|v|^{-1}\int_0^t\sigma(x-s\hat v,v)ds-|v_1|^{-1}\int_0^{t_1}\sigma(x-t\hat v-s\hat v_1,v_1)ds}, \tag{6.2}$$

for $x\in\mathbb{R}^n$, t, $t_1\in\mathbb{R}$ and v, $v_1\in V$. We recall that $\mathcal{R}$ is a bounded operator from $L^1(O,|v|dxdv)$ to $L^1(F_+)$, i.e.

$$\|\mathcal{R}\psi\|_{F_+} \le C\||v|\psi\|_O, \text{ for any } \psi\in L^1(O,|v|dxdv). \tag{6.3}$$

Hence we obtain, in particular, that the integral in t, t_1, v' and v_1, on the right-hand side of (6.1) is absolutely convergent for a.e. $(x,v)\in F_+$.

Let us assume first that $V=\mathbb{S}^{n-1}$. Performing the changes of variables "$x' = x-tv-t_1v_1$" ("$dx' = t_1^{n-1}dt_1dv_1$"), we obtain

$$(\mathcal{R}\psi_-)(x,v) = \int_O \beta(x,v,x',v')\psi_-(x',v')dx'dv', \tag{6.4}$$

where

$$\beta(x,v,x',v') := \int_0^{2R}\left[\frac{k(x-tv,v_1,v)k(x',v',v_1)}{|x-tv-x'|^{n-1}}E_0(x,v,x-tv,v_1,x-tv-t_1v_1)\right]_{\substack{t_1=|x-tv-x'|\\ v_1=\frac{x-tv-x'}{t_1}}}dt, \tag{6.5}$$

for a.e. $(x,v)\in F_+$, $(x',v')\in O$, where E_0 is defined by (6.2).

Now assume that V is an open subset of $\mathbb{R}^n$, which satisfies $\mathrm{v}_0 = \inf_{v\in V}|v| > 0$. From (6.1), it follows that

$$\begin{aligned}(\mathcal{R}\psi_-)(x,v) &= \left(K^2\psi_-\right)_{|F_+}(x,v) \\ &= \int_{V\times\mathbb{S}^{n-1}}|v|^{-1}\int_{\mathrm{v}_0}^{+\infty} r^{n-2}\chi_V(r\omega)\int_0^{2R}\int_0^{R+(x-t\hat v)\omega} k(x-t\hat v, r\omega, v) \\ &\quad \times k(x-t\hat v-t_1\omega, v', r\omega)E_0(x,v,x-t\hat v, r\omega, x-t\hat v-t_1\omega) \\ &\quad \times \psi_-(x-t\hat v-t_1\omega, v')dt_1dtdrdv'd\omega,\end{aligned} \tag{6.6}$$

for $(x,v)\in F_+$, where for $r>0$ and $\omega\in\mathbb{S}^{n-1}$. Performing the changes of variables "$x'=x-t\hat v-t_1\omega$" ("$dx'=t_1^{n-1}dt_1d\omega$"), we obtain

$$(\mathcal{R}\psi_-)(x,v)=\int_O\beta(x,v,x',v')\psi_-(x',v')dx'dv', \tag{6.7}$$

where

$$\begin{aligned}\beta(x,v,x',v') &= \frac{1}{|v|}\int_{\mathrm{v}_0}^{+\infty}r^{n-2}\int_0^{2R}\Big[\chi_V(r\omega)\frac{k(x-t\hat v,r\omega,v)k(x',v',r\omega)}{|x-x'-t\hat v|^{n-1}}\\ &\quad\times E_0(x,v,x-t\hat v,r\omega,x-t\hat v-t_1\omega)\Big]_{\substack{t_1=|x-x'-t\hat v|\\ t_1\omega=x-x'-t\hat v}}dtdr,\end{aligned} \tag{6.8}$$

for a.e. $(x,v)\in F_+$, $(x',v')\in O$, where E_0 is defined by (6.2).

From (6.3), (6.4)–(6.5), and (6.7)–(6.8), it follows that for a.e. $(x',v')\in O$, $\beta(x,v,x',v')\in L^1(F_+)$. Moreover from (6.3), it follows that the function $O\ni(x',v')\to\beta(x,v,x',v')\psi(x',v')\in L^1(F_+)$ belongs to $L^1(O,|v|dxdv)$ for any $\psi\in L^1(O,|v'|dx'dv')$. Therefore

$$|v'|^{-1}\beta\in L^\infty(O,L^1(F_+)). \tag{6.9}$$

Now we prove (2.32). Assume $k\in L^\infty(\mathbb{R}^n\times V\times V)$ and let $1<p<1+\frac{1}{n-1}$, $p'^{-1}+p^{-1}=1$, be fixed for the rest of the proof of Lemma 2.7. We use (6.10). Using Hölder estimate, the change of variables "$y=x-t\hat v$" ($dy=dxdt$) and the spherical coordinates, we obtain

$$\begin{aligned}&\int_{V_\delta}\int_{R\hat v+\Pi_v(R)}\left(\int_0^{2R}\frac{1}{|x-x'-t\hat v|^{n-1}}dt\right)^p dxdv\\ &\le(2R)^{\frac{p}{p'}}\int_{V_\delta}\int_{R\hat v+\Pi_v(R)}\int_0^{2R}\frac{dtdxdv}{|x-x'-t\hat v|^{p(n-1)}}\\ &=(2R)^{\frac{p}{p'}}\int_{V_\delta}\int_{\substack{y\in\mathbb{R}^n\\(y,v)\in O}}\frac{dydv}{|y-x'|^{p(n-1)}}\le(2R)^{\frac{p}{p'}}\mathrm{Vol}(V_\delta)\mathrm{Vol}(\mathbb{S}^{n-1})\frac{(4R)^{n-(n-1)p}}{n-(n-1)p},\end{aligned} \tag{6.10}$$

for $x'\in\mathbb{R}^n$, $|x'|<2R$, where

$$V_\delta:=\{v\in V\mid |v|<\delta^{-1}\}. \tag{6.11}$$

Assume first that $V=\mathbb{S}^{n-1}$ and let ϕ be a continuous function on F_+. Then using

(6.5), $\sigma \geq 0$, Hölder estimate and (6.10) (with $\delta = \frac{1}{2}$), we obtain

$$\left|\int_V \int_{R\hat{v}+\Pi_v(R)} \phi(x,v)\beta(x,v,x',v')dxdv\right| \leq \|k\|_\infty^2 \int_V \int_{R\hat{v}+\Pi_v(R)} |\phi(x,v)| \int_0^{2R} \frac{dtdxdv}{|x-tv-x'|^{n-1}}$$

$$\leq \|k\|_\infty^2 \left(\int_V \int_{R\hat{v}+\Pi_v(R)} |\phi(x,v)|^{p'} dxdv\right)^{\frac{1}{p'}}$$

$$\times \left(\int_V \int_{R\hat{v}+\Pi_v(R)} \left(\int_0^{2R} \frac{1}{|x-tv-x'|^{n-1}} dt\right)^p dxdv\right)^{\frac{1}{p}}$$

$$\leq C \left(\int_V \int_{R\hat{v}+\Pi_v(R)} |\phi(x,v)|^{p'} dxdv\right)^{\frac{1}{p'}}, \tag{6.12}$$

where $C = (2R)^{\frac{1}{p'}} \|k\|_\infty^2 \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{2}{p}} \left(\frac{(4R)^{n-(n-1)p}}{n-(n-1)p}\right)^{\frac{1}{p}}$, which proves (2.32) for $V = \mathbb{S}^{n-1}$.

Now assume that V is an open subset of $\mathbb{R}^n$ which satisfies $\inf_{v\in V} |v| > 0$. Let $\varepsilon' > 0$ an $\delta > 0$ be positive real numbers. Let ϕ be a compactly supported and continuous function on F_+ such that $\mathrm{supp}\phi \subseteq \{(x,v) \in F_+ \mid |v| < \delta^{-1}\}$. We use the following lemma, whose proof is postponed to the end of this section.

Lemma 6.1. *The nonnegative measurable function β_1 defined for a.e. $(x,v,x',v') \in F_+ \times O$ by*

$$\beta_1(x,v,x',v') = \frac{1}{|v|} \int_{\mathrm{v}_0}^{+\infty} r^{n-1} \int_0^{2R} \left[\chi_V(r\omega) \frac{k(x-t\hat{v}, r\omega, v)k(x',v',r\omega)}{|x-x'-t\hat{v}|^{n-1}} \times E_0(x,v,x-t\hat{v}, r\omega, x-t\hat{v}-t_1\omega)\right]_{\substack{t_1=|x-x'-t\hat{v}| \\ t_1\omega = x-x'-t\hat{v}}} dtdr, \tag{6.13}$$

belongs to $L^\infty(O, L^1(F_+))$, where E_0 is defined by (6.2).

Let $M_{\varepsilon'} > \mathrm{v}_0$ be defined by

$$M_{\varepsilon'} = \mathrm{v}_0 + {\varepsilon'}^{-1} \left\|\int_V \int_{R\hat{v}+\Pi_v(R)} \beta_1(x,v,x',v')|v|dxdv\right\|_{L^\infty(O)}. \tag{6.14}$$

From (6.8), it follows that

$$\int_V \int_{R\hat{v}+\Pi_v(R)} \phi(x,v)|v|\beta(x,v,x',v')dxdv = I_1(x',v') + I_2(x',v'), \tag{6.15}$$

for a.e. $(x',v')\in O$ and where

$$I_1(x',v') = \int_{V_\delta}\int_{R\hat{v}+\Pi_v(R)} \phi(x,v) \int_{\mathrm{v}_0}^{M_{\varepsilon'}} r^{n-2}\int_0^{2R}\Big[\chi_V(r\omega)\frac{k(x-t\hat{v},r\omega,v)k(x',v',r\omega)}{|x-x'-t\hat{v}|^{n-1}} \times E_0(x,v,x-t\hat{v},r\omega,x-t\hat{v}-t_1\omega)\big]_{t_1\omega=x-x'-t\hat{v}}\, dtdrdxdv, \tag{6.16}$$

$$I_2(x',v') = \int_{V_\delta}\int_{R\hat{v}+\Pi_v(R)} \phi(x,v) \int_{M_{\varepsilon'}}^{+\infty} r^{n-2}\int_0^{2R}\Big[\chi_V(r\omega)\frac{k(x-t\hat{v},r\omega,v)k(x',v',r\omega)}{|x-x'-t\hat{v}|^{n-1}} \times E_0(x,v,x-t\hat{v},r\omega,x-t\hat{v}-t_1\omega)\big]_{t_1\omega=x-x'-t\hat{v}}\, dtdrdxdv. \tag{6.17}$$

Using (6.17) and the estimates $r^{n-2}=r^{n-1}r^{-1}\le M_{\varepsilon'}^{-1}r^{n-1}$ for $v\in V$ and $r\ge M_{\varepsilon'}$, and using (6.14), we obtain

$$\begin{aligned}|I_2(x',v')| &\le \|\phi\|_{L^\infty(F_+)}M_{\varepsilon'}^{-1}\left\|\int_{v\in V}\int_{R\hat{v}+\Pi_v(R)}\beta_1(x,v,x'',v'')|v|dxdv\right\|_{L^\infty(O)}\\ &\le \varepsilon'\|\phi\|_{L^\infty(F_+)},\ \text{for a.e. } (x',v')\in F_+. \end{aligned}\tag{6.18}$$

From (6.16) and Hölder estimate, it follows that

$$\begin{aligned}|I_1(x',v')| &\le \|k\|_\infty^2\int_{V_\delta}\int_{R\hat{v}+\Pi_v(R)}|\phi(x,v)|\int_{\mathrm{v}_0}^{M_{\varepsilon'}} r^{n-2}dr\int_0^{2R}\frac{1}{|x-x'-t\hat{v}|^{n-1}}dtdxdv.\\ &\le \|k\|_\infty^2\frac{M_{\varepsilon'}^{n-1}}{n-1}\left(\int_{V_\delta}\int_{R\hat{v}+\Pi_v(R)}\left(\int_0^{2R}\frac{1}{|x-x'-t\hat{v}|^{n-1}}dt\right)^p dxdv\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_{V_\delta}\int_{R\hat{v}+\Pi_v(R)}|\phi(x,v)|^{p'}dxdv\right)^{\frac{1}{p'}},\end{aligned}\tag{6.19}$$

for a.e. $(x',v')\in O$. Combining (6.15), (6.18)–(6.19) and (6.10), we obtain (2.32) with

$$C(\varepsilon',\delta,p)=\|k\|_\infty^2\frac{M_{\varepsilon'}^{n-1}}{n-1}(2R)^{\frac{1}{p'}}\left(\mathrm{Vol}(V_\delta)\mathrm{Vol}(\mathbb{S}^{n-1})\right)^{\frac{1}{p}}\left(\frac{(4R)^{n-(n-1)p}}{n-(n-1)p}\right)^{\frac{1}{p}}.$$

□

Proof of Lemma 2.8. Let $\phi_-\in C_0^1(F_-)$ (which denotes the spaces of C^1 compactly supported functions on F_-). Let $\phi:=(I+K)^{-1}J\phi_-$. Then note that

$$\phi := J\phi_- - KJ\phi_- + K^2(I+K)^{-1}J\phi_-. \tag{6.20}$$

Thus

$$\mathcal{A}\phi_- = \phi_{|F_+} := (J\phi_-)_{|F_+} - (KJ\phi_-)_{|F_+} + \mathcal{R}(I+K)^{-1}J\phi_-. \tag{6.21}$$

From (2.18) and (2.35), it follows that

$$(J\phi_-)_{|F_+}(x,v) = \int_V \int_{\substack{x'v'=0\\|x'|<R}} \alpha_1(x,v,x',v')\phi_-(x'-R\hat{v}',v')dx'dv',\ (x,v)\in F_+. \tag{6.22}$$

From the definitions of K and J, we obtain

$$-(KJ\phi_-)_{|F_+}(x,v) = \int_V \int_{\substack{x'v'=0\\|x'|<R}} \alpha_2(x,v,x',v')\phi_-(x'-R\hat{v}',v')dx'dv',\ (x,v)\in F_+. \tag{6.23}$$

Lemma 2.8 follows from (6.21)–(6.23) and Lemma 2.7. □

Proof of Lemma 6.1. Using (6.13) and the estimate $\sigma \geq 0$ and using the change of variables "$y = x - t\hat{v}$" ($dy = dtdx$) and (2.9), and spherical coordinates, we obtain

$$\int\limits_{v\in V}\int\limits_{R\hat{v}+\Pi_v(R)} |v|\beta_1(x,v,x',v')dxdv$$

$$\leq \int\limits_V \int\limits_{R\hat{v}+\Pi_v(R)} \int\limits_{\mathrm{v}_0}^{+\infty} r^{n-1}\int\limits_0^{2R}\left[\chi_V(r\omega)\frac{k(x-t\hat{v},r\omega,v)k(x',v',r\omega)}{|x-x'-t\hat{v}|^{n-1}}\right]_{\omega=\frac{x-t\hat{v}-x'}{|x-t\hat{v}-x'|}} dtdrdxdv,$$

$$= \int\limits_V \int\limits_{\substack{y\in\mathbb{R}^n\\|y|<R}} \frac{1}{|x'-y|^{n-1}}\int\limits_{\mathrm{v}_0}^{+\infty} r^{n-1}\left[\chi_V(r\omega)k(y,r\omega,v)k(x',v',r\omega)\right]_{\omega=\frac{y-x'}{|y-x'|}} drdydv$$

$$= \int\limits_{\mathbb{S}^{n-1}}\int_0^R \chi_{|y|<R}(x'+r'\omega)\int\limits_{\mathrm{v}_0}^{+\infty} r^{n-1}\chi_V(r\omega)\sigma_p(x'+r'\omega,r\omega)k(x',v',r\omega)drdr'd\omega$$

$$\leq \|\sigma_p\|_\infty R\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\mathrm{v}_0}^{+\infty} r^{n-1}\chi_V(r\omega)k(x',v',r\omega)drd\omega = R\|\sigma_p\|_\infty\sigma_p(x',v') \leq R\|\sigma_p\|_\infty^2,$$

for a.e. $(x',v')\in O$. The lemma is proved. □

7 Proof of existence of the albedo operator

In this section, we prove Lemma 2.4 and Propositions 2.5 and 2.6.

Proof of Lemma 2.4. Using the definition of ${\mathbf{T_1}}^{-1}$, the estimate $\sigma \geq 0$ and (2.11), we have

$$\begin{aligned}\||v|{\mathbf{T_1}}^{-1}f\|_O &\leq \int_V\int_{\Pi_v(R)}\int_{-R}^{R}\int_0^{R+w}|f(y+(w-t)\hat{v},v)|dtdwdydv\\ &= \int_V\int_{\Pi_v(R)}\int_{-R}^{R}\int_{-R}^{w}|f(y+t\hat{v},v)|dtdwdydv \leq 2R\|f\|_O,\end{aligned}$$

for $f\in L^1(O)$.

Using the definition of A_2 and (2.11), we have

$$\begin{aligned}\|A_2|v|^{-1}f\|_O &\leq \int_{V\times V}\int_{\Pi_v(R)}\int_{-R}^{R}k(y+t\hat{v},v',v)|v'|^{-1}|f(y+t\hat{v},v')|dtdydvdv'\\ &= \int_V\int_{\Pi_{v'}(R)}\int_{-R}^{R}\sigma_p(y'+t'\hat{v}',v')|v'|^{-1}|f(y'+t'\hat{v}',v')|dt'dy'dv'\\ &\leq \||v'|^{-1}\sigma_p(x',v')\|_{L^\infty(O)}\|f\|_O,\end{aligned}$$

for $f\in L^1(O)$. We also used (2.9) and the change of variables

$$\int_{yv=0}\int_{-\infty}^{+\infty}f(y+t\hat{v})dtdy = \int_{y'v'=0}\int_{-\infty}^{+\infty}f(y'+t'\hat{v}')dt'dy', \tag{7.1}$$

for $f\in L^1(\mathbb{R}^n)$ and $v,v'\in V$.

Using the definition of ${\mathbf{T_1}}^{-1}$ and A_2 and Lemma 2.1, (2.23) and (7.1), we obtain

$$\begin{aligned}&\|A_2{\mathbf{T_1}}^{-1}f\|_O\\ &\leq \int_{V\times V}\frac{1}{|v'|}\int_{\Pi_v(R)}\int_{-R}^{R}k(y+w\hat{v},v',v)\int_0^{R+(y+w\hat{v})\hat{v'}}e^{-|v'|^{-1}\int_0^t\sigma_p(y+w\hat{v}-s\hat{v'},v')ds}\\ &\qquad\times|f(y+w\hat{v}-t\hat{v'},v')|dtdwdydvdv'\\ &= \int_{V\times V}\frac{1}{|v'|}\int_{\Pi_{v'}(R)}\int_{-R}^{R}k(y'+w'\hat{v'},v',v)\int_0^{R+w'}e^{-|v'|^{-1}\int_0^t\sigma_p(y'+(w'-s)\hat{v'},v')ds}\\ &\qquad\times|f(y'+(w'-t)\hat{v'},v')|dtdw'dy'dvdv'\\ &= \int_V\int_{\Pi_{v'}(R)}\int_{-R}^{R}\left(\int_t^R(-\frac{d}{dw'}e^{-|v'|^{-1}\int_t^{w'}\sigma_p(y'+s\hat{v'},v')ds}dw'\right)|f(y'+t\hat{v'},v')|dtdy'dv'\end{aligned}$$

$$= \int_O \left(1 - e^{-|v'|^{-1}\int_0^{R-x\hat{v'}} \sigma_p(x+sv',v')ds}\right)|f(x',v')|dx'dv' \le (1 - e^{-2R\mathrm{v}_0^{-1}\|\sigma_p\|_\infty})\|f\|_O,$$

for $f \in L^1(O)$.

Item iii follows from items i and ii (under (2.24), we also use that $\|A_2{\mathbf{T_1}}^{-1}\| \le \|A_2|v|^{-1}\|\||v|{\mathbf{T_1}}^{-1}\|$). □

Proof of Proposition 2.5. We first prove item i.

Assume (2.27). For all $f \in D(\mathbf{T})$,

$$\mathbf{T}f = (\mathbf{T_1} + A_2)f = (I + A_2{\mathbf{T_1}}^{-1})\mathbf{T_1}f. \tag{7.2}$$

From (2.27) it follows that $\mathbf{T}$ admits a bounded inverse in $L^1(O)$ given by $\mathbf{T}^{-1} := {\mathbf{T_1}}^{-1}(I + A_2{\mathbf{T_1}}^{-1})^{-1}$. Using the latter equality, we obtain

$$\begin{aligned}(I+K)(I-\mathbf{T}^{-1}A_2) =& I + {\mathbf{T_1}}^{-1}A_2 - {\mathbf{T_1}}^{-1}(I + A_2{\mathbf{T_1}}^{-1})^{-1}A_2 \\ & -{\mathbf{T_1}}^{-1}(I + A_2{\mathbf{T_1}}^{-1} - I)(I + A_2{\mathbf{T_1}}^{-1})^{-1}A_2 = I. \end{aligned} \tag{7.3}$$

The proof that $(I - \mathbf{T}^{-1}A_2)(I + K) = I$ is similar. We now prove that (2.26) implies (2.27). For $f \in D(\mathbf{T})$,

$$\mathbf{T}f = (\mathbf{T_1} + A_2)f = \mathbf{T_1}(I + {\mathbf{T_1}}^{-1}A_2)f = \mathbf{T_1}(I + K)f. \tag{7.4}$$

Let us prove $(I + K)(D(\mathbf{T})) = D(\mathbf{T})$. From the latter equality and (7.4) it follows that $\mathbf{T}$ admits a bounded inverse in $L^1(O)$ given by

$$\mathbf{T}^{-1} = (I + K)^{-1}{\mathbf{T_1}}^{-1}. \tag{7.5}$$

As $K = {\mathbf{T_1}}^{-1}A_2$, we have $(I + K)(D(\mathbf{T})) \subseteq D(\mathbf{T})$. Let $g \in D(\mathbf{T})$, and let $f = (I + K)^{-1}{\mathbf{T_1}}^{-1}g \in L^1(O)$. Then $f = -K{\mathbf{T_1}}^{-1}g + g = -{\mathbf{T_1}}^{-1}A_2{\mathbf{T_1}}^{-1}g + g \in D(\mathbf{T})$ (we recall that $g \in D(\mathbf{T})$).

Equality (7.2) still holds. Using (7.2), (7.5) and the fact that $\mathbf{T_1} : D(\mathbf{T}) \to L^1(O)$ is one-to-one and onto $L^1(O)$, we obtain (2.27). Item i is thus proved. Item ii follows from item iii of Lemma 2.4 and item i. We shall prove item iii. Note that (see (7.3))

$$(I + K)(I - \mathbf{T}^{-1}A_2) = I = (I - \mathbf{T}^{-1}A_2)(I + K) \text{ in } \mathcal{L}(L^1(O)). \tag{7.6}$$

Note also that $L^1(O, |v|dxdv) \subseteq L^1(O)$ and recall that K is a bounded operator in $L^1(O, |v|dxdv)$. Therefore, we only have to prove that $\mathbf{T}^{-1}A_2$ defines a bounded operator in $L^1(O, |v|dxdv)$. Note that

$$\mathbf{T}^{-1} = {\mathbf{T_1}}^{-1}(I + A_2{\mathbf{T_1}}^{-1})^{-1}. \tag{7.7}$$

From item i, (7.7) and item i of Lemma 2.4, it follows that $\mathbf{T}^{-1}A_2$ defines a bounded operator in $L^1(O, |v|dxdv)$. Thus item iii is proved. □

Proof of Proposition 2.6. Let $f_- \in L^1(F_-)$. From (2.19), it follows that $Jf_- \in \mathcal{W}$. Hence $Jf_- \in L^1(O, |v|dxdv)$ and from (2.28) it follows that (2.25) is uniquely solvable in $L^1(O, |v|dxdv)$ and its solution is given by $(I+K)^{-1}Jf_-$ which satisfies

$$\|(I+K)^{-1}Jf_-\|_{L^1(O,|v|dxdv)} \leq C_0\|f_-\|_{F_-}, \tag{7.8}$$

where $C_0 = 2R(1 + \mathrm{v}_0^{-1}\|\sigma\|_\infty)\|(I+K)^{-1}\|_{\mathcal{L}(L^1(O,|v|dxdv))}$.

Let $f := (I+K)^{-1}Jf_-$. Hence by definition

$$f = Jf_- - Kf \text{ in } L^1(O, |v|dxdv). \tag{7.9}$$

Using (7.9), we check that the following equality is valid in the sense of distributions:

$$T_0 f = -A_1 f - A_2 f. \tag{7.10}$$

Using (7.10), we obtain $T_0 f \in L^1(O)$ and

$$\|T_0 f\| \leq (\|\sigma|v|^{-1}\|_\infty + \||v|^{-1}\sigma_p\|_\infty)\|f\|_{L^1(O,|v|dxdv)}. \tag{7.11}$$

Therefore $f \in \mathcal{W}$ (item i is thus proved), and using (2.14) and (7.11) we obtain

$$\|f_{|F_+}\|_{F_+} \leq \max((2R)^{-1}, 1)(\||v|^{-1}\sigma\|_\infty + \||v|^{-1}\sigma_p\|_\infty + 1)\|f\|_{L^1(O,|v|^{-1}dxdv)}. \tag{7.12}$$

Item ii follow from (7.8) and (7.12).

□

Acknowledgments

This work was funded in part by the National Science Foundation under Grants DMS-0239097 and DMS-0554097.

References

[1] G. Bal, *Inverse problems for homogeneous transport equations. Part II: Multidimensional case*, Inverse Problems, 16 (2000), pp. 1013–1028.

[2] M. Choulli and P. Stefanov, *Inverse scattering and inverse boundary value problems for the linear Boltzmann equation*, Comm. Partial Diff. Equ., 21 (1996), pp. 763–785.

[3] ——, *Reconstruction of the coefficients of the stationary transport equation from boundary measurements*, Inverse Problems, 12 (1996), pp. L19–L23.

[4] ——, *An inverse boundary value problem for the stationary transport equation*, Osaka J. Math., 36 (1999), pp. 87–104.

[5] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Vol.6*, Springer Verlag, Berlin, 1993.

[6] M. Mokhtar-Kharroubi, *Mathematical Topics in Neutron Transport Theory*, World Scientific, Singapore, 1997.

[7] F. Natterer and F. Wübbeling, *Mathematical Methods in Image Reconstruction*, SIAM monographs on Mathematical Modeling and Computation, Philadelphia, 2001.

[8] M. Reed and B. Simon, *Methods of modern mathematical physics. III.*, Academic Press, Inc., New York, 1979.

[9] V. G. Romanov, *Stability estimates in the three-dimensional inverse problem for the transport equation*, J. Inverse Ill-Posed Probl., 5 (1997), pp. 463–475.

[10] P. Stefanov, *Inside Out: Inverse problems and applications*, vol. 47 of MSRI publications, Ed. G. Uhlmann, Cambridge University Press, Cambridge, UK, 2003, ch. Inverse Problems in Transport Theory.

[11] P. Stefanov and G. Uhlmann, *Optical tomography in two dimensions*, Methods Appl. Anal., 10 (2003), pp. 1–9.

[12] A. Tamasan, *An inverse boundary value problem in two-dimensional transport*, Inverse Problems, 18 (2002), pp. 209–219.

[13] J.-N. Wang, *Stability estimates of an inverse problem for the stationary transport equation*, Ann. Inst. Henri Poincaré, 70 (1999), pp. 473–495.