Small volume expansions for elliptic equations

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Abstract

This paper analyzes the influence of general, small volume, inclusions on the trace at the domain's boundary of the solution to elliptic equations of the form $\nabla \cdot D^\varepsilon \nabla u^\varepsilon = 0$ or $(-\Delta + q^\varepsilon)u^\varepsilon = 0$ with prescribed Neumann conditions. The theory is well-known when the constitutive parameters in the elliptic equation assume the values of different and smooth functions in the background and inside the inclusions. We generalize the results to the case of arbitrary, and thus possibly rapid, fluctuations of the parameters inside the inclusion and obtain expansions of the trace of the solution at the domain's boundary up to an order ε^{2d}, where d is dimension and ε is the diameter of the inclusion. We construct inclusions whose leading influence is of order at most ε^{d+1} rather than the expected ε^d. We also compare the expansions for the diffusion and Helmholtz equation and their relationship via the classical Liouville change of variables.

1 Introduction

Asymptotic expansions for the influence of small volume inclusions for elliptic and other equations is now well-established. We refer the reader to e.g. [2, 3, 4, 5, 6, 8] and their references for a few historic and recent works on the subject. A major advantage of such expansions is that they help us understand what details of the constitutive parameters in the equation may or may not be reconstructed from available boundary measurements. Indeed, in the elliptic equations of interest in this paper, namely the diffusion or conductivity equation and the Helmholtz equation, the reconstruction of the constitutive parameters X from knowledge of the full Dirichlet-to-Neumann map Λ, the most general type of information available at the domain's boundary, is an extremely ill-conditioned problem. Available stability estimates for both types of equations predict that the accuracy in the reconstruction is at best logarithmic in the accuracy of the measurements. More precisely, we have [1, 10]

$$\|X_1 - X_2\|_{L^\infty(\Omega)} \le C \Big| \log \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H^{\frac12}(\partial\Omega), H^{-\frac12}(\partial\Omega))} \Big|^{-\delta},$$

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for some positive constant C and $\delta \in (0, 1)$, where X_1 and X_2 are two sets of parameters and Λ_1 and Λ_2 their corresponding measurements.

For such severely ill-posed problems, only a limited number of degrees of freedom may be reconstructed from even quite accurate measurements. A natural way of limiting the number of degrees of freedom is to assume that the constitutive coefficients are known throughout the domain, except at some locations where unknown inclusions may be present. The asymptotic expansions in the size of the inclusion mentioned above thus provide a very efficient tool to understand what may or may not be reconstructed from data with a given level of noise.

For elliptic equations, the existing works on the subject, see e.g. [2, 4], typically assume the parameters jumps across the interface of the inclusion. One of the main objectives of this paper is to consider the case of more general inclusions whose coefficient may vary at the small scale ε and need not "jump" from the values of the background parameters. We also want to stress the similarities and differences between expansions for the diffusion equation $\nabla \cdot D^\varepsilon \nabla u^\varepsilon = 0$ and the Helmholtz equation $(-\Delta + q^\varepsilon)u^\varepsilon = 0$ with q^ε of order $\varepsilon^{-2+\eta}$ for $\eta \in [0, 2]$. In both cases of the diffusion equation and the Helmholtz equation when $\eta = 0$, we need to introduce local correctors and obtain a limiting influence at the domain's boundary that is non-linear in the parameters inside the inclusion.

The maximal leading term in the expansion is always of order $\mathcal{O}(\varepsilon^d)$, the volume of the inclusion. We construct expansions up to the order ε^{2d}. Going beyond this order of accuracy requires a more careful analysis of the decay properties of local correctors at infinity than is available here, or the use of single and double layer potentials as in [2] in the case of constant coefficients inside and outside of the inclusion. Note that the cross-talk between two inclusions of volume $\mathcal{O}(\varepsilon^d)$ is also a term of order ε^{2d}. It seems therefore natural to stop the expansion at the order $\mathcal{O}(\varepsilon^{2d})$ for the influence of any given well-separated inclusions.

Because our inclusions are modeled by somewhat arbitrary parameters that need not jump from the local value of the background parameter or are not constant, the limiting polarization tensors need not satisfy any property of positivity or definiteness. On the contrary, we show that the polarization tensors vanish to first order for some types of inclusions, whose influence at the domain's boundary is therefore at most of order ε^{d+1} rather than ε^d. Although we do not explore this aspect here, the proposed asymptotic expansions may be used to construct inclusions whose influence on the measurements is minimized in a prescribed manner.

The rest of the paper is structured as follows. Section 2 is devoted to the derivation of the asymptotic expansions for the diffusion equation. The main tool in the expansion is a decomposition of the corresponding Green's function given in proposition 2.1. The expansion obtained for smooth inclusions is presented in theorem 2.2 while the generalization to more singular inclusions with possible discontinuities of the coefficients across the inclusion's boundary is given in theorem 2.6. We compare our expansions with those obtained in [2] for constant coefficients inside and outside of the inclusions in proposition 2.8. Section 2.3 presents some properties of the polarization tensors that appear in the asymptotic expansions. In particular, proposition 2.12 shows that the leading polarization tensor vanishes for some non-vanishing diffusion coefficients inside the inclusion. Some proofs of the results are postponed until section 4.

Section 3 addresses local variations of the potential in a Helmholtz equation. The appropriate decomposition of the Green's function is shown in proposition 3.1 and the main result in theorem 3.3. The relationship between the expansions for diffusion and Helmholtz equations in regards of the Liouville change of variables is explored in section 3.2. We show that the expansions in both settings agree up to order ε^{d+2}. Most proofs are postponed to section 4.

2 Perturbations of the diffusion problem

In this section, we are interested in the analysis of small inclusions in the diffusion or conductivity problem. As we have mentioned in the introduction, the reconstruction of diffusion or conductivity coefficients from boundary measurements is a severely ill-posed problem. One possible way to overcome this difficulty is to assume that the background diffusion coefficient is known and that the unknown part of the coefficient is localized and has small volume.

Under such hypotheses, asymptotic expansions of the perturbed field in the volume of the inclusion have been derived in [6] when the inclusion is perfectly reflecting or insulating. These formulas have then been extended to more general inclusions in [5], and to higher orders in the volume and to domain with Lipschitz boundaries in [2]. In those references, the inclusion is modeled by a jump in the diffusion coefficient so that its first order effect on the boundary measurements is proportional to the inclusion's volume. The so-called polarization tensor contains the information about the inclusion that is available at this level of the asymptotic expansion.

Such a setting for the diffusion coefficient prevents us from using the well-known change of variable $q := \frac{\Delta\sqrt{D}}{\sqrt{D}}$ that allows us to relate the diffusion equation to the Helmholtz or Schrödinger equation. Since one of the objective of the paper is to show the equivalence of the asymptotic expansions within the diffusion and Helmholtz frameworks, we first consider a regular inclusion without jump and derive the corresponding asymptotic expansions in section 2.1. We next generalize these formulas to the case with jumps in the diffusion coefficient in section 2.2. We also recover the formulas in [2] in the special case of constant coefficients in the background and the inclusion. Finally, we present in section 2.3 some properties the polarization tensors involved in the asymptotic formula.

2.1 The case of smooth inclusions

We consider the following system of equations:

$$\left\{ \begin{array}{ll} \nabla \cdot D^\varepsilon \nabla u^\varepsilon = 0, & \text{in } \Omega, \\ D^\varepsilon \dfrac{\partial u^\varepsilon}{\partial \mathbf{n}} = g, & \text{on } \partial\Omega, \qquad \displaystyle\int_{\partial\Omega} u^\varepsilon d\sigma = 0, \end{array} \right. \tag{1}$$

where Ω is a bounded open domain of dimension $d \geq 2$ with Lipschitz boundary, σ is the surface measure on $\partial\Omega$, and $g \in L^2(\partial\Omega)$ such that the following compatibility condition holds $\int_{\partial\Omega} g d\sigma = 0$. It is assumed that D^ε is bounded from below by a positive constant independent of ε and that D^ε satisfies the decomposition $D^\varepsilon(\mathbf{x}) = D_0(\mathbf{x}) + D_1(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon})$, where $0 < C_0' \leq D_0 \in \mathcal{C}^\infty(\overline{\Omega})$, $D_1 \in L^\infty(\Omega)$ and D_1 vanishing in $\mathbb{R}^d \backslash \overline{B}$, B being a bounded set with Lipschitz boundary. The properties of D^ε are summarized below:

$$\left\{ \begin{array}{lll} D^\varepsilon(\mathbf{x}) \geq C_0 > 0, & & \Omega \ a.e., \\ D^\varepsilon(\mathbf{x}) = D_0(\mathbf{x}), & & \mathbf{x} \in \overline{\Omega} \backslash \overline{\mathbf{x}_0 + \varepsilon B}, \\ D^\varepsilon(\mathbf{x}) = D_0(\mathbf{x}) + D_1(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}), & & \mathbf{x} \in \mathbf{x}_0 + \varepsilon B, \\ D_0 \in \mathcal{C}^\infty(\overline{\Omega}), & D_1 \in L^\infty(\Omega). & \end{array} \right. \tag{2}$$

We assume in addition that the domain of the inclusion is located away from the boundary in the sense that there exists $d_0 > 0$ independent of ε such that

$$\operatorname{dist}(\partial\Omega, \mathbf{x}_0 + \varepsilon B) > d_0. \tag{3}$$

The Lax-Milgram lemma applied to (1)-(2) yields a unique variational solution $u^\varepsilon \in H^1(\Omega)$. Let us denote by U the solution with background diffusion coefficient D_0:

$$\left\{ \begin{array}{lll} \nabla \cdot D_0 \nabla U = 0, & \text{in } \Omega, & \\ D_0 \dfrac{\partial U}{\partial \mathbf{n}} = g, & \text{on } \partial\Omega, & \displaystyle\int_{\partial\Omega} U(\mathbf{x}) d\sigma(\mathbf{x}) = 0, \end{array} \right. \tag{4}$$

and introduce the related Green function $N \in \mathcal{D}'(\Omega \times \Omega)$ satisfying, for all fixed $\mathbf{y}$ in Ω,

$$\left\{ \begin{array}{lll} \nabla_{\mathbf{x}} \cdot D_0(\mathbf{x}) \nabla_{\mathbf{x}} N(\mathbf{x}, \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y}), & \text{in } \Omega, & \\ D_0(\mathbf{x}) \dfrac{\partial N(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} = -\dfrac{1}{|\partial\Omega|}, & \text{on } \partial\Omega, & \displaystyle\int_{\partial\Omega} N(\mathbf{x}, \mathbf{y}) d\sigma(\mathbf{x}) = 0. \end{array} \right. \tag{5}$$

For all $\mathbf{x} \in \overline{\Omega}$, the Lax-Milgram lemma yields again a unique variational solution $U \in H^1(\Omega)$ and standard elliptic regularity results [7] implies that $U \in \mathcal{C}^\infty(\Omega)$ since $D_0 \in \mathcal{C}^\infty(\overline{\Omega})$. We denote by Γ the fundamental solution of the Laplacian, namely

$$\Gamma(\mathbf{x}) = \left\{ \begin{array}{ll} -\dfrac{1}{2\pi} \log |\mathbf{x}|, & d = 2, \\ \dfrac{1}{(d-2)|S_{d-1}|} \dfrac{1}{|\mathbf{x}|^{d-2}}, & d \geq 3, \end{array} \right. \tag{6}$$

where $|S_{d-1}|$ is the measure of the $(d-1)$-dimensional unit sphere. Throughout the paper, we use the following multi-index notations: for $i = (i_1, \cdots, i_d) \in \mathbb{N}^d$, we define $|i| = i_1 + \cdots + i_d$, $\partial^i f = \partial_1^{i_1} f \cdots \partial_d^{i_1} f$ and $\mathbf{x}^i = x_1^{i_1} \cdots x_d^{i_d}$. We also define $i! = i_1! \cdots i_d!$.

One of the main tools in our asymptotic expansions is the following decomposition of the Green function N:

Proposition 2.1 *The Green function N can be decomposed, for $(\mathbf{x}, \mathbf{y}) \in \Omega \times \Omega$, as:*

$$N(\mathbf{x}, \mathbf{y}) = D_0^{-1}(\mathbf{x}) \Gamma(\mathbf{x} - \mathbf{y}) + R_1(\mathbf{x}, \mathbf{y}) + R_2(\mathbf{x}, \mathbf{y}) + R_3(\mathbf{y}), \tag{7}$$

where $R_3 \in \mathcal{C}^\infty(\Omega)$; for all $\mathbf{y}$ fixed in Ω, $R_1(\cdot, \mathbf{y}) \in W^{1,p}(\Omega)$, with $1 \leq p < \frac{d}{d-2}$ when $d \geq 3$ and $p < \infty$ when $d = 2$; and $R_2(\cdot, \mathbf{y}) \in H^1(\Omega)$. Moreover, R_1 is $\mathcal{C}^\infty$ when $\mathbf{x} \neq \mathbf{y}$, $R_2 \in \mathcal{C}^\infty(\Omega \times \Omega)$, and we have by construction that:

$$\nabla_{\mathbf{x}} N(\mathbf{x}, \mathbf{y}) = D_0^{-1}(\mathbf{x}) \nabla_{\mathbf{x}} \Gamma(\mathbf{x} - \mathbf{y}) + \nabla_{\mathbf{x}} R_2(\mathbf{x}, \mathbf{y}). \tag{8}$$

Also, N admits the following asymptotic expansion for $\mathbf{x} \in B$, $\mathbf{y}$ a.e. in $\partial\Omega$:

$$\nabla_{\mathbf{x}} N(\mathbf{x}_0 + \varepsilon \mathbf{x}, \mathbf{y}) = \sum_{|i|=1}^{d} \frac{\varepsilon^{|i|}}{i!} \nabla \mathbf{x}^i \partial_{\mathbf{x}}^i N(\mathbf{x}_0, \mathbf{y}) + \mathcal{O}(\varepsilon^{d+1}), \tag{9}$$

where $\mathcal{O}(\varepsilon^{d+1})$ denotes a term bounded in $L^2(\partial\Omega)$ by $C\varepsilon^{d+1}$ uniformly in $\mathbf{x}$.

Proof. Let R_1 be (uniquely) defined by

$$\nabla_{\mathbf{x}} R_1(\mathbf{x}, \mathbf{y}) = \frac{\nabla D_0(\mathbf{x})}{D_0^2(\mathbf{x})} \Gamma(\mathbf{x} - \mathbf{y}), \qquad \int_{\partial\Omega} R_1(\mathbf{x}, \mathbf{y}) d\sigma(\mathbf{x}) = 0,$$

and R_3 be defined as

$$|\partial\Omega| R_3(\mathbf{y}) = -\int_{\partial\Omega} D_0^{-1}(\mathbf{x}) \Gamma(\mathbf{x} - \mathbf{y}) d\sigma(\mathbf{x}).$$

Since $D_0 > 0$, $D_0 \in \mathcal{C}^\infty(\overline{\Omega})$ and $\Gamma \in L^p_{\text{loc}}(\mathbb{R}^d)$ for the values of p in the proposition, it follows that $R_1(\cdot, \mathbf{y}) \in W^{1,p}(\Omega)$. Moreover, R_1 is $\mathcal{C}^\infty$ as soon as $\mathbf{x} \neq \mathbf{y}$. In the same way, $R_3 \in \mathcal{C}^\infty(\Omega)$ since $\Gamma(\mathbf{x}) \in \mathcal{C}^\infty(\mathbb{R}^d \backslash \{0\})$. We then verify that (7) leads to (8) and that plugging (7) into (5) leads to the system, for $\mathbf{y} \in \Omega$:

$$\left\{ \begin{array}{lll} \nabla_{\mathbf{x}} \cdot D_0(\mathbf{x}) \nabla_{\mathbf{x}} R_2(\mathbf{x}, \mathbf{y}) = 0, & \text{in } \Omega, & \\ D_0 \dfrac{\partial R_2(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} = -\dfrac{1}{|\partial\Omega|} - \dfrac{\partial \Gamma(\mathbf{x} - \mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}}, & \text{on } \partial\Omega, & \displaystyle\int_{\partial\Omega} R_2(\mathbf{x}, \mathbf{y}) d\sigma(\mathbf{x}) = 0, \end{array} \right.$$

which admits a unique weak solution thanks to the Lax-Milgram lemma since we verify that $\int_{\partial\Omega} (\frac{1}{|\partial\Omega|} + \frac{\partial\Gamma}{\partial \mathbf{n}_{\mathbf{x}}}) d\sigma(\mathbf{x}) = 0$. Since $\partial_{\mathbf{y}}^\beta \frac{\partial \Gamma}{\partial \mathbf{n}_{\mathbf{x}}}(\cdot - \mathbf{y}) \in L^2(\partial\Omega)$ for any multi-index β and $\mathbf{y} \in \Omega$, we deduce that $\partial_{\mathbf{y}}^\beta R_2(\cdot, \mathbf{y}) \in H^1(\Omega)$, so that elliptic regularity yields $\partial_{\mathbf{y}}^\beta R_2(\cdot, \mathbf{y}) \in \mathcal{C}^\infty(\Omega)$, and finally $R_2 \in \mathcal{C}^\infty(\Omega \times \Omega)$.

Moreover, $\partial_{\mathbf{y}}^\beta R_2(\cdot, \mathbf{y})$ is bounded in $H^1(\Omega)$ uniformly in $\mathbf{y}$ when $\mathbf{y} \in \Omega' \subset\subset \Omega$. To prove (9), we first remark from (7) that the trace $\partial_{\mathbf{y}}^\beta N(\mathbf{z}, \mathbf{y})\big|_{\partial\Omega}$ is defined in $L^2(\partial\Omega)$ uniformly in $\mathbf{y}$ when $\mathbf{y} \in \Omega'$ since $R_1 \in \mathcal{C}^\infty(\overline{\Omega} \backslash \overline{\Omega'} \times \Omega')$, $\partial_{\mathbf{y}}^\beta R_2(\cdot, \mathbf{y}) \in H^1(\Omega)$ uniformly in $\mathbf{y} \in \Omega'$, and $R_3 \in \mathcal{C}^\infty(\Omega')$. This allows us to apply Green's theorem and obtain, for any $(\mathbf{z}, \mathbf{y}) \in \Omega' \times \Omega$, that:

$$R_2(\mathbf{z}, \mathbf{y}) = -\int_{\partial\Omega} \left(\frac{1}{|\partial\Omega|} + \frac{\partial \Gamma(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} \right) N(\mathbf{x}, \mathbf{z}) d\sigma(\mathbf{x}).$$

As $\mathbf{y}$ goes to $\partial\Omega$, the boundary integral converges for Lipschitz domains Ω, see [2], to

$$-\frac{1}{|\partial\Omega|} \int_{\partial\Omega} N(\mathbf{x}, \mathbf{z}) d\sigma(\mathbf{x}) - \text{p.v} \int_{\partial\Omega} \frac{\partial \Gamma(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} N(\mathbf{x}, \mathbf{z}) d\sigma(\mathbf{x}) + \frac{1}{2} N(\mathbf{y}, \mathbf{z}), \quad (\mathbf{z}, \mathbf{y}) \in \Omega' \times \partial\Omega,$$

where p.v. stands for the Cauchy principal value and the above integral operator is bounded in $L^2(\partial\Omega)$. The first term belongs to $\mathcal{C}^\infty(\Omega')$ and the second and the third terms to $\mathcal{C}^\infty(\Omega')$ with values in $L^2(\partial\Omega)$. Using (8), this allows us to expand $\nabla_{\mathbf{x}} N(\mathbf{x}_0 + \varepsilon \mathbf{x}, \mathbf{y})$ and obtain (9). □

We first consider the case of a smooth inclusion by adding the hypothesis that D_1 is regular and compactly supported in B, that is $D_1 \in W^{1,\infty}(\Omega)$, with support $\text{supp} D_1 \subset B$. In such a context, the trace of D_1 vanishes on ∂B. We have the following result:

Theorem 2.2 *Assume that $D_1 \in W^{1,\infty}(\Omega)$ with support $\mathrm{supp} D_1 \subset B$. Then the solution u^ε to (1)-(2) verifies the following asymptotic expansion, a.e. on $\partial\Omega$:*

$$u^\varepsilon(\mathbf{y})|_{\partial\Omega} = U(\mathbf{y})|_{\partial\Omega} - \sum_{|i|=1}^{d}\sum_{|j|=1}^{d} \frac{\varepsilon^{d-2+|i|+|j|}}{i!j!} M_{ij}\, \partial^j U(\mathbf{x}_0)\ \partial^i_{\mathbf{x}} N(\mathbf{x}_0,\mathbf{y})\big|_{\partial\Omega} + \mathcal{O}(\varepsilon^{2d})$$
$$-\sum_{|i|=1}^{d}\sum_{|j|=1}^{d}\sum_{|k|=0}^{d}\sum_{l=0,\, l+|k|>0}^{d} \frac{\varepsilon^{d-2+|i|+|j|+|k|+l}}{i!j!k!l!} M^2_{ijkl}\partial^j U(\mathbf{x}_0)\left(\partial^k D_0^{-1}\right)(\mathbf{x}_0)\ \partial^i_{\mathbf{x}} N(\mathbf{x}_0,\mathbf{y})\big|_{\partial\Omega},$$

where M and M^2 are generalized polarization tensors given by

$$\begin{aligned} M_{ij} &= \int_B D_1(\mathbf{x})\, \nabla(\mathbf{x}^j + \phi^0_{j\,0}(\mathbf{x}))\cdot\nabla \mathbf{x}^i d\mathbf{x}, \qquad i,j\in\mathbb{N}^d, \\ M^2_{ijkl} &= \int_B D_1(\mathbf{x})\, \nabla\phi^l_{jk}(\mathbf{x})\cdot\nabla\mathbf{x}^i d\mathbf{x}, \qquad i,j,k\in\mathbb{N}^d, \quad l\in\mathbb{N}, \end{aligned} \tag{10}$$

and the functions ϕ^l_{jk} are the unique solutions in $H^1_{loc}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$ to:

$$\begin{cases} \nabla\cdot(D_0(\mathbf{x}_0)+D_1(\mathbf{x}))\,\nabla\phi^l_{jk} = -\delta^0_l\,\nabla\cdot\left(D_1(\mathbf{x})\,\mathbf{x}^k\nabla\mathbf{x}^j\right) \\ \qquad -D_0(\mathbf{x}_0)\displaystyle\sum_{|m|=1}^{l}\frac{l!\partial^m D_0^{-1}(\mathbf{x}_0)}{m!(l-|m|)!}\nabla\cdot\left(D_1(\mathbf{x})\,\mathbf{x}^m\nabla\phi^{l-|m|}_{jk}(\mathbf{x})\right), \\ \phi^l_{jk}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-d}) \quad as \quad |\mathbf{x}|\to\infty. \end{cases} \tag{11}$$

Here, δ^0_l is the Kronecker symbol and the notation $\mathcal{O}(\varepsilon^{2d})$ in the expansion represents a term bounded in $L^2(\partial\Omega)$ by a constant depending on $\|D_1\|_{L^\infty}$ and on $\|g\|_{L^2(\partial\Omega)}$.

Remark 2.3 The function ϕ^0_{jk} solves the following equation in $\mathbb{R}^d$:

$$\begin{aligned} \nabla\cdot(D_0(\mathbf{x}_0)+D_1(\mathbf{x}))\,\nabla\phi^0_{jk} &= -\nabla\cdot\left(D_1(\mathbf{x})\,\mathbf{x}^k\nabla\mathbf{x}^j\right), \\ \phi^0_{jk}(\mathbf{x}) &= \mathcal{O}(|\mathbf{x}|^{1-d}) \quad \text{as} \quad |\mathbf{x}|\to\infty, \end{aligned}$$

so that ϕ^l_{jk} is computed from ϕ^m_{jk}, $0\le m<l$, iteratively.

Remark 2.4 We may recast the expansion in theorem 2.2 as

$$u^\varepsilon(\mathbf{y})|_{\partial\Omega} = U(\mathbf{y})|_{\partial\Omega} - \sum_{|i|=1}^{d}\sum_{|j|=1}^{d}\frac{\varepsilon^{d-2+|i|+|j|}}{i!j!} M^\varepsilon_{ij}\,\partial^j U(\mathbf{x}_0)\ \partial^i_{\mathbf{x}} N(\mathbf{x}_0,\mathbf{y})\big|_{\partial\Omega} + \mathcal{O}(\varepsilon^{2d}),$$

where the ε-dependent tensor M^ε is given by:

$$M^\varepsilon_{ij} = \int_B D_1(\mathbf{x})\,\nabla(\mathbf{x}^j+\Psi^\varepsilon_j(\mathbf{x}))\cdot\nabla\mathbf{x}^i d\mathbf{x}, \qquad i,j\in\mathbb{N}^d,$$

and the functions Ψ^ε_j are the unique solutions in $H^1_{\mathrm{loc}}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$ to

$$\begin{aligned} \nabla\cdot\left(1+D_1(\mathbf{x})D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\right)\nabla\Psi^\varepsilon_j &= -\nabla\cdot\left(D_1(\mathbf{x})D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\nabla\mathbf{x}^j\right), \\ \Psi^\varepsilon_j(\mathbf{x}) &= \mathcal{O}(|\mathbf{x}|^{1-d}) \quad \text{as} \quad |\mathbf{x}|\to\infty. \end{aligned}$$

The asymptotic expansion of the theorem is then recovered by expanding Ψ_j^ε and $D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})$ in powers of ε.

There is another equivalent expansion to that of theorem 2.2 up to the order ε^{2d}. We sketch its derivation in the case where D_0 is constant. The right hand side of the equation for Ψ_j^ε is equal to $-D_0^{-1}\nabla D_1\cdot\nabla\mathbf{x}^j - D_0^{-1}D_1\Delta\mathbf{x}^j$. It turns out that an appropriate linear combination of $\Delta\mathbf{x}^j$ is of order ε^{d+1}, so that we can replace Ψ_j^ε in the definition of M_{ij}^ε by Φ_j solution to

$$\begin{aligned} \nabla\cdot\big(1+D_1(\mathbf{x})D_0^{-1}\big)\nabla\Phi_j &= -D_0^{-1}\nabla D_1(\mathbf{x})\cdot\nabla\mathbf{x}^j, \\ \Phi_j(\mathbf{x}) &= \mathcal{O}(|\mathbf{x}|^{1-d}) \quad \text{as} \quad |\mathbf{x}|\to\infty. \end{aligned}$$

The appropriate linear combination is deduced from $\Delta U(\mathbf{x}_0+\varepsilon\mathbf{x})=0$ and from Taylor expanding U so as to obtain:

$$0=\Delta U(\mathbf{x}_0+\varepsilon\mathbf{x}) = \Delta\sum_{|j|=0}^{d}\frac{\varepsilon^{|j|}}{j!}\partial^j U(\mathbf{x}_0)\mathbf{x}^j + \mathcal{O}(\varepsilon^{d+1}).$$

Remark 2.5 The leading order in the expansion is given by

$$\varepsilon^d\sum_{|i|=|j|=1} M_{ij}\partial^i N(\mathbf{x}_0,\mathbf{y})\partial^j U(\mathbf{x}_0).$$

The polarization tensor M^2 contributes only to higher orders. The polarization tensor M captures the correction when the background diffusion coefficient D_0 is constant in $\mathbf{x}_0+\varepsilon B$, whereas M^2 is the correction that needs to be added when D_0 is not constant in $\mathbf{x}_0+\varepsilon B$. When D_0 is constant in $\mathbf{x}_0+\varepsilon B$, then $M_{ijkl}^2 = M_{ijkl}^2\delta_l^0$ so that the expansion then reduces to the classical formula:

$$u^\varepsilon(\mathbf{y}) = U(\mathbf{y}) + \sum_{|i|=1}^{d}\sum_{|j|=1}^{d}\frac{\varepsilon^{d-2+|i|+|j|}}{i!j!}M_{ij}\partial_{\mathbf{x}}^i N(\mathbf{x}_0,\mathbf{y})\partial^j U(\mathbf{x}_0) + \mathcal{O}(\varepsilon^{2d}).$$

In this case, using the notation of remark 2.4, Ψ_j^ε no longer depends on ε and may be identified with ϕ_{j0}^0. Note that the latter formula also holds when D_0 is non-constant away from the support of the inclusion $\mathbf{x}_0+\varepsilon B$ as remark 2.4 makes clear since only the values of D_0^{-1} on the support of D_1 are involved.

The proof of the theorem is given in section 4. Its main ingredients are the integral formulation of (1) and the decomposition of the Green function given in proposition 2.1. Additional boundary effects, which are not considered here, appear at the order $\mathcal{O}(\varepsilon^{2d})$ when the geometry-dependent corrector $R_2(\mathbf{x},\mathbf{x}_0+\varepsilon\mathbf{y})$ of proposition 2.1 is expanded in powers of ε. When D_0 is constant, a proper factorization based on the technique of double layer potentials allow us to obtain arbitrarily accurate expansions; see [2].

2.2 The case of singular inclusions

In the preceding section, we assumed that the perturbed diffusion coefficient was regular. We may generalize the above theorem to include the case where D_1 is in $L^\infty(\mathbb{R}^d)$ with support in B and with a possibly non-vanishing trace (if it is defined) at the interior boundary ∂B. This generalization is achieved by regularizing the singular perturbation so that we can use the preceding result and then by computing the limiting polarization tensors. We have the following result:

Theorem 2.6 *Assume D_1 verifies* (2) *with no further assumption on its interior trace on ∂B. Then u^ε admits the same expansion as in theorem 2.2 with polarization tensors still given by* (10)*, where ϕ^l_{jk} is now the unique solution in $H^1_{loc}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$ to:*

$$\begin{cases} \Delta\phi^l_{jk}=0, \qquad \mathbf{x}\in\mathbb{R}^d/\overline{B} \qquad \textit{with} \qquad \phi^l_{jk}(\mathbf{x})=\mathcal{O}(|\mathbf{x}|^{1-d}) \quad \textit{as} \quad |\mathbf{x}|\to\infty, \\ \nabla\cdot(D_0(\mathbf{x}_0)+D_1(\mathbf{x}))\nabla\phi^l_{jk}=-\delta^0_l\,\nabla\cdot\big(D_1(\mathbf{x})\,\mathbf{x}^k\nabla\mathbf{x}^j\big) \\ \quad -D_0(\mathbf{x}_0)\displaystyle\sum_{|m|=1}^{l}\frac{l!\partial^m D_0^{-1}(\mathbf{x}_0)}{m!(l-|m|)!}\nabla\cdot\Big(D_1(\mathbf{x})\,\mathbf{x}^m\nabla\phi^{l-|m|}_{jk}(\mathbf{x})\Big), \qquad \mathbf{x}\in B, \\ D_0(\mathbf{x}_0)\displaystyle\frac{\partial\phi^l_{jk}}{\partial\mathbf{n}}\Big|_+-(D_0(\mathbf{x}_0)+D_1(\mathbf{x}))\frac{\partial\phi^l_{jk}}{\partial\mathbf{n}}\Big|_-=\delta^0_l\,D_1(\mathbf{x})\,\mathbf{x}^k\mathbf{n}\cdot\nabla\mathbf{x}^j \\ \quad +D_0(\mathbf{x}_0)\displaystyle\sum_{|m|=1}^{l}\frac{l!\partial^m D_0^{-1}(\mathbf{x}_0)}{m!(l-|m|)!}D_1(\mathbf{x})\,\mathbf{x}^m\frac{\partial\phi^{l-|m|}_{jk}}{\partial\mathbf{n}}\Big|_-, \qquad \mathbf{x}\in\partial B, \end{cases} \tag{12}$$

Here, $\mathbf{n}$ is the outer normal to the boundary of B, $\frac{\partial\phi^l_{jk}}{\partial\mathbf{n}}\Big|_+$ (resp. $\frac{\partial\phi^l_{jk}}{\partial\mathbf{n}}\Big|_-$) denotes the outer (resp. inner) trace of $\frac{\partial\phi^l_{jk}}{\partial\mathbf{n}}$ on ∂B as functions in $H^{-\frac{1}{2}}(\partial B)$.

The proof of the theorem is postponed to section 4.

Theorem 2.6 has been proved in [2] by using single and double layer potential techniques when the background diffusion coefficient D_0 is constant on the entire domain Ω and when D_1 is constant on B. Our result generalizes that of [2] to the case of non-constant D_0 and D_1 for which layers techniques are not available. The first order of the expansion can also be obtained from the general formula proved in [4] and in [5].

Remark 2.7 The expansion in remark 2.4 still holds for singular inclusions with Ψ^ε_j now the unique solution in $H^1_{\rm loc}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$ to

$$\begin{aligned} \Delta\Psi^\varepsilon_j &= 0 \qquad \mathbf{x}\in\mathbb{R}^d/\overline{B}, \\ \nabla\cdot\big(1+D_1(\mathbf{x})D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\big)\nabla\Psi^\varepsilon_j &= -\nabla\cdot\big(D_1(\mathbf{x})D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\nabla\mathbf{x}^j\big), \qquad \mathbf{x}\in B, \\ \Psi^\varepsilon_j(\mathbf{x}) &= \mathcal{O}(|\mathbf{x}|^{1-d}) \quad \text{as} \quad |\mathbf{x}|\to\infty, \end{aligned}$$

equipped with the jump condition on ∂B:

$$\frac{\partial\Psi^\varepsilon_j}{\partial\mathbf{n}}\Big|_+-(1+D_1(\mathbf{x})D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x}))\frac{\partial\Psi^\varepsilon_j}{\partial\mathbf{n}}\Big|_-=D_1(\mathbf{x})\,D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\mathbf{n}\cdot\nabla\mathbf{x}^j, \qquad \mathbf{x}\in\partial B.$$

As in the end of remark 2.4, we could also derive a modified asymptotic expansion in the case of singular inclusions.

The above asymptotic expansions are compatible with the slightly different expressions for the generalized polarization tensors obtained in [2]. We have the following proposition:

Proposition 2.8 *Assume that D_1 is a non vanishing constant on B and that D_0 is constant on the set $\mathbf{x}_0 + \varepsilon B$. Then u^ε verifies the following expansion, a.e. on $\partial\Omega$,*

$$u^\varepsilon(\mathbf{y})|_{\partial\Omega} \;=\; U(\mathbf{y})|_{\partial\Omega} - \sum_{|i|=1}^{d}\sum_{|j|=1}^{d} \frac{\varepsilon^{d-2+|i|+|j|}}{i!j!}\mathcal{M}_{ij}\,\partial^j U(\mathbf{x}_0)\;\partial^i_{\mathbf{x}} N(\mathbf{x}_0,\mathbf{y})\big|_{\partial\Omega} + \mathcal{O}(\varepsilon^{2d}),$$

where $\mathcal{M}$ is the generalized polarization tensor given in [2] by

$$\mathcal{M}_{ij} \;=\; D_1 \int_{\partial B} \mathbf{n}\cdot\nabla(\mathbf{x}^j + \phi_j(\mathbf{x}))\mathbf{x}^i d\sigma(\mathbf{x}), \qquad i,j\in\mathbb{N}^d.$$

The functions ϕ_j are the unique solutions in $H^1_{loc}(\mathbb{R}^d)\cap\mathcal{C}^\infty((\mathbb{R}^d/\overline{B})\cup B)$ to the problem:

$$\begin{cases} \Delta\phi_j = 0, \quad \mathbf{x}\in(\mathbb{R}^d/\overline{B})\cup B, \\ D_0\,\dfrac{\partial\phi_j}{\partial\mathbf{n}}\bigg|_+ - (D_0+D_1)\,\dfrac{\partial\phi_j}{\partial\mathbf{n}}\bigg|_- = D_1\,\mathbf{n}\cdot\nabla\mathbf{x}^j, \quad \mathbf{x}\in\partial B, \\ \phi_j(\mathbf{y}) - \Gamma(\mathbf{y})D_0^{-1}D_1\displaystyle\int_{\partial B}\mathbf{n}\cdot\nabla\mathbf{x}^j d\sigma(\mathbf{x}) = \mathcal{O}(|\mathbf{y}|^{1-d}), \quad \textit{when } |\mathbf{y}|\to\infty. \end{cases}$$

The proof of the proposition is also postponed to section 4.

2.3 Properties of the polarization tensor M

In this section, we give some symmetry properties and estimates satisfied by the tensors M in theorems 2.2 and 2.6:

Proposition 2.9 *Let $\alpha_i,\beta_i\in\mathbb{R}$, where i belongs to a set a of multi-index I. Then, the polarization tensor M verifies the following properties:*

$$(i) \qquad \sum_{i,j\in I}\alpha_i\beta_j M_{ij} = \sum_{i,j\in I}\alpha_i\beta_j M_{ji},$$
$$(ii) \quad \int_B \frac{D_0(\mathbf{x}_0)D_1(\mathbf{x})}{D_0(\mathbf{x}_0)+D_1(\mathbf{x})}\Big|\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)\Big|^2 d\mathbf{x} \le \sum_{i,j\in I}\alpha_i\alpha_j M_{ij} \le \int_B D_1(\mathbf{x})\Big|\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)\Big|^2 d\mathbf{x}.$$

Proof. Using the definition of M, we have,

$$\sum_{i,j\in I}\alpha_i\beta_j M_{ij} \;=\; \int_B D_1(\mathbf{x})\nabla\Big(\sum_{j\in I}\beta_j(\mathbf{x}^j + \phi^0_{j\,0}(\mathbf{x}))\Big)\cdot\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)d\mathbf{x},$$

and the system solved by $\phi^0_{j\,0}$ deduced from (12) imply that,

$$\int_{\mathbb{R}^d}\Big(D_0(\mathbf{x}_0)+D_1(\mathbf{x})\Big)\nabla\phi^0_{j0}\cdot\nabla\phi^0_{i0}\,d\mathbf{x} = -\int_B D_1(\mathbf{x})\nabla\mathbf{x}^j\cdot\nabla\phi^0_{i0}\,d\mathbf{x}. \qquad (13)$$

Consequently,

$$\sum_{i,j\in I}\alpha_i\beta_j M_{ij}=\int_B D_1(\mathbf{x})\nabla\Big(\sum_{j\in I}\beta_j\mathbf{x}^j\Big)\cdot\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)d\mathbf{x}$$
$$-\int_{\mathbb{R}^d}\Big(D_0(\mathbf{x}_0)+D_1(\mathbf{x})\Big)\nabla\Big(\sum_{j\in I}\beta_j\phi^0_{j0}\Big)\cdot\nabla\Big(\sum_{i\in I}\alpha_i\phi^0_{i0}\Big)\,d\mathbf{x}=\sum_{i,j\in I}\alpha_i\beta_j M_{ji}.$$

Concerning item (ii), we remark from the above equality that:

$$\sum_{i,j\in I}\alpha_i\alpha_j M_{ij}\le\int_B D_1(\mathbf{x})\Big|\nabla\Big(\sum_{j\in I}\alpha_j\mathbf{x}^j)\Big)\Big|^2 d\mathbf{x}.$$

For the other inequality, we split the sum as:

$$\sum_{i,j\in I}\alpha_i\alpha_j M_{ij}=\int_B D_1\Big|\nabla\Big(\sum_{j\in I}\alpha_j\mathbf{x}^j)\Big)\Big|^2 d\mathbf{x}+\int_B D_1\nabla\Big(\sum_{j\in I}\alpha_j\phi^0_{j0}\Big)\cdot\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)d\mathbf{x}.$$

Since $D_0(\mathbf{x}_0)+D_1(\mathbf{x})$ is strictly positive *a.e.* in Ω, the Cauchy-Schwarz inequality yields

$$\int_B D_1\nabla\Big(\sum_{j\in I}\alpha_j\phi^0_{j0}\Big)\cdot\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)d\mathbf{x}$$
$$\le\ \Big(\int_B(D_0(\mathbf{x}_0)+D_1)\Big|\nabla\Big(\sum_{j\in I}\alpha_j\phi^0_{j0}\Big)\Big|^2 d\mathbf{x}\Big)^{\frac12}\Big(\int_B\frac{D_1^2}{D_0(\mathbf{x}_0)+D_1}\Big|\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)\Big|^2 d\mathbf{x}\Big)^{\frac12}.$$

In the same way, equation (13) gives:

$$\Big(\int_B(D_0(\mathbf{x}_0)+D_1)\Big|\nabla\Big(\sum_{j\in I}\alpha_j\phi^0_{j0}\Big)\Big|^2 d\mathbf{x}\Big)^{\frac12}\le\Big(\int_B\frac{D_1^2}{D_0(\mathbf{x}_0)+D_1}\Big|\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)\Big|^2 d\mathbf{x}\Big)^{\frac12},$$

so that

$$\begin{aligned}\sum_{i,j\in I}\alpha_i\beta_j M_{ij}\ &\ge\ \int_B D_1\Big|\nabla\Big(\sum_{j\in I}\alpha_j\mathbf{x}^j)\Big)\Big|^2 d\mathbf{x}-\int_B\frac{D_1^2}{D_0(\mathbf{x}_0)+D_1}\Big|\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)\Big|^2 d\mathbf{x},\\ &=\ \int_B\frac{D_0(\mathbf{x}_0)D_1(\mathbf{x})}{D_0(\mathbf{x}_0)+D_1(\mathbf{x})}\Big|\nabla\Big(\sum_{i\in I}\alpha_i\mathbf{x}^i\Big)\Big|^2 d\mathbf{x}.\end{aligned}$$

This ends the proof. □

Item (ii) of the proposition is very similar to the estimates obtained at the first order in [4]. Such estimates can be applied to verify the definiteness or not of the polarization tensor. In particular, it gives:

$$|\alpha|^2\int_B\frac{D_0(\mathbf{x}_0)D_1(\mathbf{x})}{D_0(\mathbf{x}_0)+D_1(\mathbf{x})}d\mathbf{x}\le\sum_{|i|=1,|j|=1}\alpha_i\alpha_j M_{ij}\le|\alpha|^2\int_B D_1(\mathbf{x})d\mathbf{x},$$

so that for D_1 constant, M is positive definite when $D_1>0$ and negative definite when $D_1<0$, as it was shown in [2, 5]. The only possibility to cancel the above sum is then to set $D_1 = 0$, which means that there is no inclusion. Therefore, an inhomogeneity with constant diffusion coefficient always generates a perturbation of order ε^d on the measurements. The situation is different when D_1 is not constant. Indeed, when $\int_B D_1(\mathbf{x})d\mathbf{x} < 0$, then M is negative definite, and when $\int_B \frac{D_1(\mathbf{x})}{D_0(\mathbf{x}_0)+D_1(\mathbf{x})}d\mathbf{x} > 0$, then M is positive definite. But when $\int_B D_1(\mathbf{x})d\mathbf{x} > 0$ while at the same time $\int_B \frac{D_1(\mathbf{x})}{D_0(\mathbf{x}_0)+D_1(\mathbf{x})}d\mathbf{x} < 0$, then M might not be definite for a suitable choice of D_1 as we now show as an application of the intermediate value theorem. We show first that the functional $M_{ij} : L^\infty(\Omega) \to \mathbb{R}$, $D_1 \to M_{ij}[D_1]$ is continuous.

Lemma 2.10 *There exists a positive constant C, such that, for all finite multi-index i and j, we have:*

$$|M_{ij}[D_1^1] - M_{ij}[D_1^2]| \leq C\|D_1^1 - D_1^2\|_{L^\infty(B)}.$$

Proof. Take two perturbation D_1^1, D_1^2 in $L^\infty(\Omega)$ with support in B and denote by $M[D_1^1]$, $M[D_1^2]$ the corresponding polarization tensors. Then:

$$\begin{aligned} M_{ij}[D_1^1] - M_{ij}[D_1^2] &= \int_B (D_1^1 - D_1^2)\nabla\mathbf{x}^j \cdot \nabla\mathbf{x}^i d\mathbf{x} + \int_B (D_1^1 - D_1^2)\nabla\phi_{j0}^0[D_1^1] \cdot \nabla\mathbf{x}^i d\mathbf{x} \\ &\quad + \int_B D_1^2 \nabla\Big(\phi_{j0}^0[D_1^1] - \phi_{j0}^0[D_1^2]\Big) \cdot \nabla\mathbf{x}^i d\mathbf{x}. \end{aligned} \tag{14}$$

Introducing $w_j := \phi_{j0}^0[D_1^1] - \phi_{j0}^0[D_1^2]$ and using the equations verified by $\phi_{j0}^0[D_1^1]$ and $\phi_{j0}^0[D_1^2]$, we find the relation:

$$\int_{\mathbb{R}^d} (D_0(\mathbf{x}_0) + D_1^1)\big|\nabla w_j\big|^2 d\mathbf{x} = -\int_B (D_1^1 - D_1^2)\nabla w_j \cdot \big(\nabla\mathbf{x}^j + \nabla\phi_{j0}^0[D_1^2]\big)d\mathbf{x}.$$

Since $\nabla\phi_{j0}^0$ is bounded in $L^2(\mathbb{R}^d)$, this yields the estimate

$$\|\nabla w_j\|_{L^2(\mathbb{R}^d)} \leq C\|D_1^1 - D_1^2\|_{L^\infty(B)}.$$

Using (14), we obtain the desired result. □

Lemma 2.11 *There exists a perturbation $D_1 \in L^\infty(\Omega)$ with $\int_B D_1(\mathbf{x})d\mathbf{x} \neq 0$, such that, for a given $1 \leq l \leq d$, the component $M_{e_l,e_l}[D_1]$ of the polarization tensor M vanishes, where e_l is the l-th vector of the canonical basis of $\mathbb{R}^d$.*

Proof. Setting $\alpha_i = \delta_i^{e_l}$ in item (ii) of proposition 2.9 leads to

$$\int_B \frac{D_0(\mathbf{x}_0)D_1(\mathbf{x})}{D_0(\mathbf{x}_0) + D_1(\mathbf{x})}d\mathbf{x} \leq M_{e_l,e_l} \leq \int_B D_1(\mathbf{x})d\mathbf{x}.$$

Now take a D_1^1 such that $\int_B D_1^1(\mathbf{x})d\mathbf{x} < 0$. Therefore, $M_{e_l,e_l}[D_1^1] < 0$. We then continuously transform D_1^1 into D_1^2 such that $\int_B \frac{D_0(\mathbf{x}_0)D_1^2(\mathbf{x})}{D_0(\mathbf{x}_0)+D_1^2(\mathbf{x})}d\mathbf{x} > 0$ keeping $\int_B D_1^1 d\mathbf{x}$ non zero in the transformation. Such a transformation exists: let indeed D_1^1 be a bounded function in B with positive and negative parts D_+^1 and D_-^1. We set $\int_B D_-^1 d\mathbf{x} > \int_B D_+^1 d\mathbf{x}$ so that $\int_B D_1^1(\mathbf{x})d\mathbf{x} < 0$. Letting the negative part D_-^1 continuously go to zero then gives a possible transformation. For the resulting D_1^2, we have $M_{e_l,e_l}[D_1^2] > 0$. Since the functional $M_{e_l,e_l}[D_1]$ is continuous from $L^\infty(B)$ to $\mathbb{R}$, we deduce from the intermediate value theorem the existence of a D_1^* with $\int_B D_1^* d\mathbf{x} \neq 0$ such that $M_{e_l,e_l}[D_1^*] = 0$. This ends the proof of the proposition. □

As a corollary of the previous result, we have

Proposition 2.12 *There exists a perturbation $0 \not\equiv D_1 \in L^\infty(\Omega)$ with spherical symmetry such that $M_{ij} \equiv 0$.*

Proof. Consider an inclusion with spherical symmetry. We find that $M_{ij} = M_0 \delta_i^j$ when $|i| = |j| = 1$ so that the above lemma yields the existence of non-vanishing perturbation such that $M_0 = 0$ and consequently no term of order ε^d appears in the asymptotic expansion. □

The latter result is to be compared with the case where D_1 is constant for which there is always a contribution of order ε^d in the expansion provided the constant is not zero.

3 Perturbations in the Helmholtz equation

This section addresses the problem of small-volume inhomogeneities in the Helmholtz equation. As we did for the diffusion equation, we derive an asymptotic expansion of the perturbed solution in the volume of the inclusions.

3.1 Asymptotic expansion and polarization tensors

We consider the following Helmholtz (or Schrödinger) equation posed in a bounded Lipschitz domain Ω of $\mathbb{R}^d$, $d \geq 2$, and with $d \leq 5$ for technical reasons:

$$\left\{ \begin{array}{l} -\Delta v^\varepsilon(\mathbf{x}) + \Big(q_0(\mathbf{x}) + \dfrac{1}{\varepsilon^{2-\eta}} q_1\Big(\dfrac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\Big)\Big) v^\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \mathbf{n}} = g \in L^2(\partial\Omega) \quad \text{on } \partial\Omega, \end{array} \right. \tag{15}$$

where $\mathbf{x}_0$ is a given point in Ω, $q_0 \in L^\infty(\Omega)$ is the background index or potential, and $q_1 \in L^\infty(\Omega)$ is a local perturbation, with support localized in a bounded Lipschitz domain B. We consider the case with only one inclusion, knowing that the results below generalize to the setting with several well-separated inclusions so long as the maximal order in the expansion is sufficiently small so that the inclusions do not interact at that order. The perturbation has a magnitude of order $\varepsilon^{\eta-2}$, with $\eta \in [0, 2]$. The most interesting case is $\eta = 0$, which corresponds to the strongest type of perturbation. The latter case allows to relate the asymptotic formula given in the preceding section to the one that we propose below for a particular form of the potential q_1.

When q_0 is negative, the above system models waves propagating in a medium perturbed by a small inclusion of diameter ε with a refractive index of order $\varepsilon^{\eta-2}$. We refer to [9] and [8] for the case of high-frequency waves in dimension two perturbed by small inclusions with index of order one. The case q_0 and q_1 constant with q_0 negative and $\eta = 2$ has been treated in [2] with Dirichlet conditions instead of Neumann conditions at the domain's boundary. When q_0 is positive, (15) models e.g. diffusive light propagating in a medium with background absorption q_0 and zones of different absorption coefficients in a small volume. The case $\eta = 2$ has been investigated in dimension three in [3] for a constant background q_0 and a constant perturbation q_1.

We denote by V the solution of the unperturbed equation

$$\left\{ \begin{array}{l} -\Delta V + q_0 V = 0, \quad \mathbf{x} \in \Omega, \\ \dfrac{\partial V}{\partial \mathbf{n}} = g \quad \text{on } \partial\Omega. \end{array} \right. \tag{16}$$

When $q_0 \equiv 0$, we assume the normalizing and compatibility conditions:

$$\int_{\partial\Omega} V d\sigma = 0 \qquad \text{and} \qquad \int_{\partial\Omega} g d\sigma = 0, \tag{17}$$

where σ denotes the surface measure on $\partial\Omega$. According to (15), this also implies:

$$\int_\Omega q_1\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\Big)\, v^\varepsilon(\mathbf{x}) d\mathbf{x} = 0, \qquad \text{when } q_0 = 0. \tag{18}$$

In order to obtain the existence and uniqueness of a variational solution to (16), we make the following classical assumption:

(H-1) Let $u \in H^1(\Omega)$. Then

$$\int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} + \int_\Omega q_0 \, u \, v d\mathbf{x} = 0, \qquad \text{for all } v \in H^1(\Omega),$$

implies that $u = 0$.

Under **(H-1)**, an application of lemma 4.4 of the appendix yields a unique weak solution $V \in H^1(\Omega)$ to (16). When $q_0 := 0$, the same holds thanks to conditions (17). Since we need high-order Taylor expansions of V in the sequel, we make the additional assumption that the restriction of q_0 to a neighborhood $\mathbf{x}_0 + \varepsilon B'$ of the set $\mathbf{x}_0 + \varepsilon B$, with $B \subset\subset B'$, belongs to $\mathcal{C}^\infty(\mathbf{x}_0 + \varepsilon B')$. Using standard elliptic regularity [7] and (3), we obtain that $V \in \mathcal{C}^\infty(\mathbf{x}_0 + \varepsilon B')$. When first order expansions are considered, then a $L^\infty(\Omega)$ regularity for V is sufficient. Existence and uniqueness for (15) uniformly in ε for ε small enough will be given in the sequel. When $\eta \in]0, 2]$, no additional condition is required on q_1. When $\eta = 0$, we add the following assumption:
(H-2) -1 is not an eigenvalue of the bounded operator T defined as:

$$T : L^2(B) \to L^2(B), \quad \varphi \to T\varphi(\mathbf{y}) = \int_B q_1(\mathbf{x})\varphi(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}.$$

Here, Γ is the fundamental solution of the Laplacian given in (6). **(H-2)** is verified for instance when $q_1 > 0$ *a.e.* in B or when the following Rollnick type [11] norm of q_1 is less than one,

$$\int_B \int_B \Big(\sqrt{|q_1(\mathbf{x})|}\sqrt{|q_1(\mathbf{y})|}|\Gamma(\mathbf{x},\mathbf{y})|\Big)^p d\mathbf{x} d\mathbf{y} < 1,$$

for some $p \geq 1$, or when q_1 is a Bohm-like potential of the form

$$q_1(\mathbf{x}) = \frac{\Delta\sqrt{1 + D_1(\mathbf{x})}}{\sqrt{1 + D_1(\mathbf{x})}},$$

for some $\mathcal{C}^2(\mathbb{R}^d)$ function D_1 with support in B such that $1 + D_1 > 0$ in $\mathbb{R}^d$.

The case $d = 2$ and $\eta = 0$ is particular in the sense that

$$\frac{1}{\varepsilon^2}\int_\Omega q_1\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\Big)\, d\mathbf{x} = \int_B q_1(\mathbf{x})d\mathbf{x} = \mathcal{O}(1),$$

so that we cannot expect the perturbation caused by the inclusion to be small in the general case. We thus need to add an additional hypothesis to be able to treat q_1 as a

perturbation. It is the case under the following symmetry assumption:
(H-3) When $d = 2$ and $\eta = 0$, we assume that the solution v^ε to (15) verifies that

$$\int_\Omega q_1\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\Big)\, v^\varepsilon(\mathbf{x})d\mathbf{x} = 0.$$

Note that **(H-3)** is verified when e.g. $q_0 \equiv 0$ thanks to (17). We introduce the Green function $N(\mathbf{x},\mathbf{y}) \in \mathcal{D}'(\Omega\times\Omega)$ of (16), which for each fixed $\mathbf{y}$ in Ω, solves:

$$\begin{cases} -\Delta_{\mathbf{x}} N(\mathbf{x},\mathbf{y}) + q_0(\mathbf{x})N(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \quad \mathbf{x}\in\Omega, \\ \dfrac{\partial N(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{19}$$

When $q_0 \equiv 0$, N has to be defined as in (5). N is symmetric in its arguments. Hypothesis **(H-1)** is verified e.g. when $q_0 \geq 0$, Ω *a.e.* (with the normalizing condition when $q_0 \equiv 0$), when q_0 is constant and not an eigenvalue of the Laplacian equipped with homogeneous Neumann conditions, or when the following Rollnick-type norm of q_0 is less than one,

$$\int_\Omega\int_\Omega \Big(\sqrt{|q_0(\mathbf{x})|}\sqrt{|q_0(\mathbf{y})|}|N(\mathbf{x},\mathbf{y})|\Big)^p d\mathbf{x}d\mathbf{y} < 1,$$

for some $p \geq 1$. We have the following proposition, which allows us to decompose N as the sum of the whole space Green function Γ and a regular function:

Proposition 3.1 *We have $N(\mathbf{x},\mathbf{y}) := \Gamma(\mathbf{x}-\mathbf{y}) + R(\mathbf{x},\mathbf{y})$, where $R(\cdot,\mathbf{y}) \in H^1(\Omega)\cap W^{2,p}(\Omega')$ with $p < \frac{d}{d-2}$ when $3 \leq d \leq 5$ and $p < \infty$ when $d = 2$ for any $\Omega' \subset\subset \Omega$ uniformly in $\mathbf{y}\in\Omega'$. When $q_0 \equiv 0$, then R belongs to $\mathcal{C}^\infty(\Omega\times\Omega)$. Moreover, N admits the following asymptotic expansion for $\mathbf{x}\in B$, $\mathbf{y}$ a.e. in $\partial\Omega$:*

$$\nabla_{\mathbf{x}} N(\mathbf{x}_0+\varepsilon\mathbf{x},\mathbf{y}) = \sum_{|i|=1}^{d} \frac{\varepsilon^{|i|}}{i!}\nabla \mathbf{x}^i \partial_{\mathbf{x}}^i N(\mathbf{x}_0,\mathbf{y}) + \mathcal{O}(\varepsilon^{d+1}), \tag{20}$$

where $\mathcal{O}(\varepsilon^{d+1})$ denotes a term bounded in $L^2(\partial\Omega)$ by $C\varepsilon^{d+1}$, uniformly in $\mathbf{x}$.

Proof. We consider only the case $q_0 \neq 0$ since the case $q_0 \equiv 0$ follows from proposition 2.1. Plugging $N(\mathbf{x},\mathbf{y}) := \Gamma(\mathbf{x}-\mathbf{y}) + R(\mathbf{x},\mathbf{y})$ into (19) leads for any $\mathbf{y}$ fixed in Ω to the equation:

$$\begin{cases} -\Delta_{\mathbf{x}} R(\mathbf{x},\mathbf{y}) + q_0(\mathbf{x})R(\mathbf{x},\mathbf{y}) = -q_0(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y}), \quad \mathbf{x}\in\Omega, \\ \dfrac{\partial R(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} = -\dfrac{\partial \Gamma(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}}, \quad \text{on } \partial\Omega. \end{cases} \tag{21}$$

Pick an $\mathbf{y}\in\Omega' \subset\subset \Omega$ and for any $v \in H^1(\Omega)$, consider the linear form:

$$l(v) := -\int_\Omega q_0(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y})v(\mathbf{x})d\mathbf{x} - \int_{\partial\Omega} \frac{\partial\Gamma(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} v(\mathbf{x})d\sigma(\mathbf{x}).$$

Then l is continuous in $H^1(\Omega)$. Indeed, on the one hand, $\Gamma(\mathbf{x}-\mathbf{y})$ is uniformly bounded for $(\mathbf{x},\mathbf{y}) \in \partial\Omega \times \Omega'$ which allows us to treat the second integral. On the other hand, $\Gamma \in L^p_{\text{loc}}(\mathbb{R}^d)$ with $p < \frac{d}{d-2}$ when $d \geq 3$ and $p < \infty$ when $d = 2$ so that the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$, for $q \leq \frac{2d}{d-2}$ when $d \geq 3$ and $q < \infty$ when $d = 2$ implies

$$|l(v)| \leq C(\|\Gamma\|_{L^{q'}(B_R)} + 1)\|v\|_{H^1(\Omega)},$$

for $q' \geq \frac{2d}{d+2}$ when $d \geq 3$ and $q' > 1$ when $d = 2$, where B_R is a ball of radius R large enough. Since $\frac{d}{d-2} > \frac{2d}{d+2}$ for $d < 6$, we get the desired result. Note that for $d \geq 7$, the above linear form is not continuous as we may construct functions $v \in H^1(\Omega)$ of the form $|\mathbf{x}|^{-\alpha}$ such that $\Gamma(\mathbf{x})v(\mathbf{x})$ is not integrable in the vicinity of 0. Lemma 4.4 then yields a unique $R(\cdot,\mathbf{y}) \in H^1(\Omega)$ uniformly bounded in $\mathbf{y}$ when $\mathbf{y} \in \Omega'$ by choosing $a_0(u,v) = \int_\Omega(\nabla u \cdot \nabla v + uv)d\mathbf{x}$ and $a_1(u,v) = \int_\Omega(q_0(\mathbf{x}) - 1)uvd\mathbf{x}$. Standard elliptic regularity [7] gives, for $1 < p < \frac{d}{d-2}$ when $d \geq 3$ and $p < \infty$ when $d = 2$, that:

$$\|R(\cdot,\mathbf{y})\|_{W^{2,p}(\Omega')} \leq C\left(\|R(\cdot,\mathbf{y})\|_{H^1(\Omega)} + \|\Gamma\|_{L^p(B_R)}\right),$$

so that $R(\cdot,\mathbf{y}) \in W^{2,p}(\Omega')$ uniformly in $\mathbf{y} \in \Omega'$.

To prove (20), we decompose R as $R(\mathbf{x},\mathbf{y}) := R_1(\mathbf{x},\mathbf{y}) + R_2(\mathbf{x},\mathbf{y})$ with

$$\begin{cases} -\Delta_{\mathbf{x}} R_1(\mathbf{x},\mathbf{y}) + q_0(\mathbf{x})R_1(\mathbf{x},\mathbf{y}) = -q_0(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y}), \quad \mathbf{x} \in \Omega, \\ \dfrac{\partial R_1(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{22}$$

$$\begin{cases} -\Delta_{\mathbf{x}} R_2(\mathbf{x},\mathbf{y}) + q_0(\mathbf{x})R_2(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \Omega, \\ \dfrac{\partial R_2(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} = -\dfrac{\partial \Gamma(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}}, \quad \text{on } \partial\Omega. \end{cases} \tag{23}$$

Consider first (22) for $\mathbf{y} \in \partial\Omega$. According to lemma 4.4, $R_1(\cdot,\mathbf{y})$ belongs to $H^1(\Omega)$ and is uniformly bounded with respect to $\mathbf{y}$. Let B' be a neighborhood of B such that $B \subset\subset B'$. Since $\Gamma(\cdot - \mathbf{y}) \in \mathcal{C}^\infty(\mathbf{x}_0 + \varepsilon\overline{B'})$ uniformly in $\mathbf{y} \in \partial\Omega$, and $q_0 \in \mathcal{C}^\infty(\mathbf{x}_0 + \varepsilon B')$, we obtain from elliptic regularity that $R_1(\cdot,\mathbf{y}) \in \mathcal{C}^\infty(\mathbf{x}_0 + \varepsilon\overline{B})$ uniformly in $\mathbf{y} \in \partial\Omega$. Now, R_2 is treated almost exactly as the term R_2 in proposition 2.1, so we highlight the differences. According to the previous results on R, the trace $N(\mathbf{x},\mathbf{z})|_{\partial\Omega}$ exists in $L^2(\partial\Omega)$ uniformly for $\mathbf{z} \in \Omega' \subset\subset \Omega$. Thus we have the following integral equation:

$$R_2(\mathbf{z},\mathbf{y}) = -\int_{\partial\Omega} \frac{\partial\Gamma(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} N(\mathbf{x},\mathbf{z})d\sigma(\mathbf{x}), \qquad (\mathbf{z},\mathbf{y}) \in \Omega' \times \Omega.$$

As $\mathbf{y}$ goes to $\partial\Omega$, the integral converges to

$$-\text{p.v}\int_{\partial\Omega} \frac{\partial\Gamma(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_{\mathbf{x}}} N(\mathbf{x},\mathbf{z})d\sigma(\mathbf{x}) + \frac{1}{2}N(\mathbf{y},\mathbf{z}),$$

where p.v. stands for the Cauchy principal value and the above quantity makes sense in $L^2(\partial\Omega)$ uniformly in $\mathbf{z} \in \Omega'$ so that $R_2(\mathbf{z},\cdot) \in L^2(\partial\Omega)$ for all $\mathbf{z} \in \Omega'$. Moreover, we verify that $R_2(\mathbf{z},\mathbf{y})$ satisfies in the distributional sense, for $\mathbf{z} \in \Omega'$, $\mathbf{y} \in \partial\Omega$,

$$-\Delta_{\mathbf{z}} R_2(\mathbf{z},\mathbf{y}) + q_0(\mathbf{z})R_2(\mathbf{z},\mathbf{y}) = 0,$$

so that we conclude from elliptic regularity that $R_2(\cdot,\mathbf{y}) \in \mathcal{C}^\infty(\mathbf{x}_0+\varepsilon\overline{B})$ with values in $L^2(\partial\Omega)$. Classical Taylor expansions then yield (20). □
We come back to (15) and state the following result.

Proposition 3.2 *Assume that **(H-2)** is satisfied when $\eta = 0$ and **(H-3)** is satisfied when $d = 2$ and $\eta = 0$. Then, under assumption **(H-1)**, there exists $\varepsilon_0 > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, the system (15) admits a unique variational solution $v^\varepsilon \in H^1(\Omega)$. Moreover, the restriction of v^ε to the set $\mathbf{x}_0 + \varepsilon B$ verifies the following decomposition*

$$v^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{y}) = V(\mathbf{x}_0+\varepsilon\mathbf{y}) + \varepsilon^\eta \Psi^\varepsilon(\mathbf{y}) + \varepsilon^{d-2+\eta}\, r^\varepsilon(\mathbf{y}) + \mathcal{O}(\varepsilon^{d+2}), \qquad \mathbf{y} \text{ a.e. in } B, \quad (24)$$

where $\Psi^\varepsilon(\mathbf{y}) := \sum_{|j|=0}^{d+1} \frac{\varepsilon^{|j|}}{j!}\partial^j V(\mathbf{x}_0)\phi_j^\eta(\mathbf{y})$ *and* ϕ_j^η *is the unique solution in* $H^1(B)$ *to*

$$\phi_j^\eta + \varepsilon^\eta T\phi_j^\eta = -T\mathbf{x}^j, \qquad \mathbf{y} \in B, \quad (25)$$

and r^ε *the unique solution in* $H^1(B)$, *for* $\mathbf{y} \in B$, *to*

$$r^\varepsilon(\mathbf{y}) + \varepsilon^\eta T r^\varepsilon(\mathbf{y}) = \int_B q_1(\mathbf{x})\, v^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{x}) \left(R(\mathbf{x}_0+\varepsilon\mathbf{x},\mathbf{x}_0+\varepsilon\mathbf{y}) - \delta_d^2\,(2\pi)^{-1}\log\varepsilon\right) d\mathbf{x}.$$

*The operator T is defined in **(H-2)** and the function R in proposition 3.1 whereas δ_d^2 is the Kronecker symbol. The notation $\mathcal{O}(\varepsilon^{d+2})$ represents a term bounded in $H^1(B)$ by $C\varepsilon^{d+2}$. The remainder r^ε is bounded in $L^2(B)$ independently of ε when $d = 3$, by $C\varepsilon^{-\alpha}$, for any $\alpha > 0$ when $d = 4$, by $C\varepsilon^{-1}$ when $d = 5$, and by $C|\log\varepsilon|$ when $d = 2$. When $d = 2$ and $\eta = 0$, then r^ε is of order $\mathcal{O}(\varepsilon)$ thanks to **(H-3)**. When $q_0 \equiv 0$, then r^ε is bounded in $H^1(B)$ independently of ε for any d.*

We then have the following theorem:

Theorem 3.3 *Under the hypotheses of proposition 3.2, the solution to v^ε to (15) satisfies the following asymptotic expansion, almost everywhere on $\partial\Omega$:*

$$\begin{aligned} v^\varepsilon(\mathbf{y})|_{\partial\Omega} &= V(\mathbf{y})|_{\partial\Omega} - \sum_{|j|=0}^{d+1}\sum_{|i|=0}^{d+1} \frac{\varepsilon^{d-2+\eta+|i|+|j|}}{i!j!}\left(Q_{ij}+\varepsilon^\eta Q_{ij}^\eta\right)\partial^j V(\mathbf{x}_0)\,\partial^i N(\mathbf{x}_0,\mathbf{y})\big|_{\partial\Omega} \\ &\quad + \varepsilon^{2(d-2+\eta)} f^\varepsilon(\mathbf{y}) + \mathcal{O}(\varepsilon^{2d}), \end{aligned}$$

where $\mathcal{O}(\varepsilon^{2d})$ is a term bounded in $L^2(\partial\Omega)$ by $C\varepsilon^{2d}$ and for $(i,j) \in \mathbb{N}^d \times \mathbb{N}^d$,

$$\begin{aligned} Q_{ij} &= \int_B q_1(\mathbf{x})\mathbf{x}^j\mathbf{x}^i d\mathbf{x}, \qquad Q_{ij}^\eta = \int_B q_1(\mathbf{x})\phi_j^\eta(\mathbf{x})\mathbf{x}^i d\mathbf{x}, \\ f^\varepsilon(\mathbf{y}) &= \int_B q_1(\mathbf{x}) r^\varepsilon(\mathbf{x}) N(\mathbf{x}_0+\varepsilon\mathbf{x},\mathbf{y}) d\mathbf{x}. \end{aligned}$$

The remainder $\|f^\varepsilon\|_{L^2(\partial\Omega)}$ is of order: $\mathcal{O}(|\log\varepsilon|)$ when $d = 2$; $\mathcal{O}(1)$ when $d = 3$; $\mathcal{O}(\varepsilon^{-\alpha})$ for any $\alpha > 0$ when $d = 4$; and $\mathcal{O}(\varepsilon^{-1})$ when $d = 5$.

The proofs of the proposition and the theorem are given in section 4.2. When $\eta > 0$, ϕ_j^η still depends on ε. We may then expand the operator $(\mathcal{I} + \varepsilon^\eta T)^{-1}$ in terms of Neumann series up to the right order. We include the term f^ε in the formula because we need its explicit expression below to make the link between the asymptotic expansion for the diffusion equation and that for the Helmholtz equation.

In the particular case where q_0 constant and positive, $\eta = 2$, q_1 is constant, and the inclusion is centered at $\mathbf{x}_0$ so that $\int_B \mathbf{x} d\mathbf{x} = 0$, we find for $d = 3$ that

$$\begin{aligned} v^\varepsilon(\mathbf{y}) &= V(\mathbf{y}) - \varepsilon^3 q_1 \left(\int_B \left(1 + \varepsilon^2 \phi_0^2\right) d\mathbf{x} \right) V(\mathbf{x}_0) N(\mathbf{x}_0, \mathbf{y}) \\ &\quad - q_1 \sum_{|j|=0}^{2} \sum_{|i|+|j|=2} \frac{\varepsilon^5}{i!j!} \left(\int_B \mathbf{x}^i \mathbf{x}^j d\mathbf{x} \right) \partial^i N(\mathbf{x}_0, \mathbf{y}) \partial^j V(\mathbf{x}_0) + \mathcal{O}(\varepsilon^6). \end{aligned}$$

According to (25), ϕ_0^2 verifies $\phi_0^2 = -T1 + \mathcal{O}(\varepsilon^2)$ so that we recover the asymptotic expansion given in [3].

The tensor Q is clearly symmetric. When q_1 is constant and not identically zero, there is always a contribution of order $\varepsilon^{d-2+\eta}$ in the expansion, while for spatially varying q_1, the first order contribution can vanish for instance by choosing q_1 such that $\int_B q_1 d\mathbf{x} = 0$.

3.2 Relation between the diffusion and Helmholtz equations

We now compare the asymptotic expansions for the solution u^ε to the diffusion equation (1) given in theorem 2.2 and for the solution v^ε to the Helmholtz equation (15) given in theorem 3.3. It is well-known that a solution to the diffusion equation

$$\nabla \cdot D \nabla u = 0,$$

with $D \in \mathcal{C}^2(\mathbb{R}^d)$ for instance and strictly positive, also satisfies a Helmholtz or Schrödinger equation of the form

$$\Delta\big(\sqrt{D}u\big) + \Big(\frac{\Delta\sqrt{D}}{\sqrt{D}}\Big)\big(\sqrt{D}u\big) = 0.$$

Our purpose here is to verify that the polarization tensors obtained in the diffusion and Helmholtz frameworks are indeed the same for the specific form of the potential q_1 that allows us to transform one equation into the other. As in section 2, we define $D^\varepsilon(\mathbf{x}) = D_0(\mathbf{x}) + D_1(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon})$ and to simplify the presentation, assume that D_0 is constant in Ω. We assume that $D_1 \in \mathcal{C}^2(\Omega)$ with support included in B and that $D_0 + D_1$ is strictly positive in Ω, so that we can define

$$q_1(\mathbf{x}) := \frac{\Delta\sqrt{D_0 + D_1(\mathbf{x})}}{\sqrt{D_0 + D_1(\mathbf{x})}}. \tag{26}$$

We then consider the function v^ε which satisfies (15) with $q_0 = 0$, $\eta = 0$ and q_1 defined as above. With such a choice, the quantity

$$\frac{v^\varepsilon(\mathbf{x})}{\sqrt{D_0 + D_1(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon})}}$$

solves (1). Since $\eta = 0$, we may expect from the expansion given in theorem 3.3 that the inclusion induces a correction of order ε^{d-2} whereas the same inclusion induces a correction of order ε^d in the diffusion equation. Some simplifications due to the particular form of the potential q_1 must render the correction of order ε^d in the Helmholtz framework as well. We state the main result of this section:

Proposition 3.4 *When q_1 has the form (26), then we have*

$$\sum_{j=0}^{d+1} \frac{\varepsilon^{|j|}}{j!} \partial^j V(\mathbf{x}_0) \left(Q_{0j} + Q^0_{0j}\right) = \mathcal{O}(\varepsilon^{d+2}), \tag{27}$$

$$\sum_{i=0}^{d+1} \frac{\varepsilon^{|i|}}{i!} \partial^i N(\mathbf{x}_0, \mathbf{y}) \left(Q_{i0} + Q^0_{i0}\right) = \mathcal{O}(\varepsilon^{d+2}). \tag{28}$$

Here, the index 0 of the polarization tensors represents the vector of $\mathbb{N}^d$ with components all equal to zero. We have the following relation between the polarization tensor M in the context of theorem 2.2 and the polarization tensor $\tilde{M} := \sqrt{D_0}(Q+Q^0)$ in the context of the Helmholtz equation:

$$M_{ij} = \tilde{M}_{ij}, \qquad |i| = |j| = 1, \tag{29}$$

$$\sum_{|j|=1}^{d+1} \sum_{|i|=1}^{d+1} \frac{\varepsilon^{|i|+|j|}}{i!j!} \partial^i N(\mathbf{x}_0, \mathbf{y}) \partial^j V(\mathbf{x}_0) \big(M_{ij} - \tilde{M}_{ij}\big) = \mathcal{O}(\varepsilon^{d+2}). \tag{30}$$

The proof of the proposition is given in section 4.2. Equations (27) and (28) imply that the two first orders in the expansion of theorem 3.3 vanish so that the correction is of order ε^d. Equations (29) and (30) show the equivalence of the tensors M_{ij} and $\tilde{M}_{ij}$ for $|i|, |j| \leq d+1$ up to an error of order ε^{d+2}, which is sufficient to show that the asymptotic expansions on u^ε and v^ε agree up to the order ε^{2d}. The proofs can in fact be modified to show the equivalence at higher orders as well, *e.g.*, for any $r \in \mathbb{N}$,

$$\sum_{|j|=1}^{r+1} \sum_{|i|=1}^{r+1} \frac{\varepsilon^{|i|+|j|}}{i!j!} \partial^i N(\mathbf{x}_0, \mathbf{y}) \partial^j V(\mathbf{x}_0) \big(M_{ij} - \tilde{M}_{ij}\big) = \mathcal{O}(\varepsilon^{r+2}).$$

Furthermore, denoting by (m_{ij}) the modified polarization tensor obtained from Φ_j at the end of remark 2.4, we can show in this context the strict equality between the Helmholtz and diffusion tensors, that is $\tilde{M}_{ij} = m_{ij}$, for all i, j.

4 Proofs of the main results

4.1 Asymptotic expansions for the diffusion equation

We now prove theorems 2.2 and 2.6 and proposition 2.8.

Proof of Theorem 2.2. The starting point of the proof is the formulation of (1) as the following integral equation:

$$\begin{aligned} u^\varepsilon(\mathbf{y}) &= U(\mathbf{y}) - \int_{\mathbf{x}_0+\varepsilon B} D_1\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right) \nabla u^\varepsilon(\mathbf{x}) \cdot \nabla_{\mathbf{x}} N(\mathbf{x}, \mathbf{y}) d\mathbf{x}, \\ &= U(\mathbf{y}) - \varepsilon^d \int_B D_1(\mathbf{x}) \nabla u^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x}) \cdot \nabla_{\mathbf{x}} N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y}) d\mathbf{x}. \end{aligned} \tag{31}$$

The above equation is justified rigorously as in the derivation of (75) in lemma 4.2 of the appendix. We highlight the main differences. According to proposition 2.1, we have $\nabla_{\mathbf{x}} N(\mathbf{x},\mathbf{y}) = D_0^{-1}(\mathbf{x})\nabla\Gamma(\mathbf{x}-\mathbf{y}) + \nabla_{\mathbf{x}} R_2(\mathbf{x},\mathbf{y})$, with $\nabla_{\mathbf{x}} R_2(\cdot,\mathbf{y}) \in L^2(\Omega)$ for every $\mathbf{y}$ in Ω so that the above equation makes sense in $L^2(\Omega)$ and therefore almost everywhere in Ω thanks to the Young inequality since $\nabla u^\varepsilon \in L^2(\Omega)$ and $\nabla\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. The integral equation (31) is obtained from the variational formulations of (1) and (4):

$$\int_\Omega D^\varepsilon \nabla u^\varepsilon \cdot \nabla v \, d\mathbf{x} \;=\; \int_{\partial\Omega} gv \, d\sigma(\mathbf{x}) \;=\; \int_\Omega D_0 \nabla U \cdot \nabla v \, d\mathbf{x}, \tag{32}$$

for all $v \in H^1(\Omega)$. Then, let $\varphi \in L^2(\Omega)$ and set $v(\mathbf{x}) := \int_\Omega N(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})d\mathbf{y}$. Thus v is the unique solution in $H^1(\Omega)$ to $-\nabla\cdot D_0\nabla v = \varphi$ equipped with homogeneous Neumann conditions and the normalization $\int_{\partial\Omega} v d\sigma(\mathbf{x})$=0. As in the proof of (75) or in the proof of proposition 2.8, we verify that Fubini's theorem applies and that

$$\int_\Omega \left(\int_\Omega D_0(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla_{\mathbf{x}} N(\mathbf{x},\mathbf{y}) d\mathbf{x} - u(\mathbf{y}) \right) \varphi(\mathbf{y}) d\mathbf{y} = 0, \qquad \forall u \in H^1(\Omega).$$

Applying the latter equality to both u^ε and U, gives (31) together with (32).

To continue the proof of theorem, we write $u^\varepsilon = U + w^\varepsilon$ as the sum of the unperturbed solution U and a corrector w^ε, solution of

$$\begin{aligned} \nabla\cdot\left(D_0(\mathbf{x}) + D_1\left(\tfrac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)\right)\nabla w^\varepsilon = -\nabla\cdot D_1\left(\tfrac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)\nabla U, \quad \text{in } \Omega,\\ \frac{\partial w^\varepsilon}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega, \quad \int_{\partial\Omega} w^\varepsilon(\mathbf{x}) d\sigma(\mathbf{x}) = 0. \end{aligned} \tag{33}$$

Since both u^ε and U belong to $H^1(\Omega)$, then $w^\varepsilon \in H^1(\Omega)$ and we deduce from (33) that:

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{d}{2}} \|D_1\|_{L^\infty(B)} \|\nabla U\|_{L^\infty(B_0)},$$

for some $\mathbf{x}_0 + \varepsilon_0 B \subset B_0 \subset\subset \Omega$ with $\varepsilon_0 > 0$ so that, from standard elliptic regularity,

$$\|\nabla w^\varepsilon(\mathbf{x}_0 + \varepsilon\cdot)\|_{L^2(B)} \le C\|D_1\|_{L^\infty(B)} \|\nabla U\|_{L^\infty(B_0)} \le C\|D_1\|_{L^\infty(B)} \|g\|_{L^2(\partial\Omega)}, \tag{34}$$

for some constant $C > 0$. We need an approximation of the corrector w^ε up to the order ε^d and so that we decompose it as $w^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x}) = \Psi^\varepsilon(\mathbf{x}) + r^\varepsilon(\mathbf{x})$, where r^ε is a remainder of order ε^d in a sense made precise below. Finding an asymptotic expression for u^ε then amounts to calculating $\Psi^\varepsilon(\mathbf{x})$ and showing that r^ε is indeed of order ε^d. To this aim, we use (31) to obtain an integral equation for w^ε verified *a.e.* in Ω:

$$w^\varepsilon(\mathbf{y}) \;=\; -\varepsilon^d \int_B D_1(\mathbf{x}) \nabla\big[w^\varepsilon + U\big](\mathbf{x}_0 + \varepsilon\mathbf{x}) \cdot \nabla_{\mathbf{x}} N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y}) d\mathbf{x}. \tag{35}$$

We then decompose $N(\mathbf{x},\mathbf{y})$ following (8). Plugging (8) into (35), setting $\mathbf{y} := \mathbf{x}_0 + \varepsilon\mathbf{y}$ for $\mathbf{y} \in B$, and using the homogeneity $\nabla\Gamma(\varepsilon\mathbf{x}) = \varepsilon^{1-d}\nabla\Gamma(\mathbf{x})$, we find

$$\begin{aligned} w^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{y}) \;=\;& -\varepsilon \int_B D_1(\mathbf{x}) D_0^{-1}(\mathbf{x}_0 + \varepsilon\mathbf{x}) \nabla\big[w^\varepsilon + U\big](\mathbf{x}_0 + \varepsilon\mathbf{x}) \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x} \\ & -\varepsilon^d \int_B D_1(\mathbf{x}) \nabla\big[w^\varepsilon + U\big](\mathbf{x}_0 + \varepsilon\mathbf{x}) \cdot \nabla_{\mathbf{x}} R_2(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{x}_0 + \varepsilon\mathbf{y}) d\mathbf{x}. \end{aligned}$$

We shall prove that the contribution involving R_2 above is of order $\mathcal{O}(\varepsilon^d)$ and that up to an error of the same order, we may replace $D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})$ and $U(\mathbf{x}_0+\varepsilon\mathbf{x})$ by $D_{0,d}^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})$ and $U_d(\mathbf{x}_0+\varepsilon\mathbf{x})$, respectively, where for $H = D_0^{-1}$ and $H = U$, we have defined the Taylor expansion to order d:

$$H_d(\mathbf{x}_0+\varepsilon\mathbf{x}) = \sum_{|m|=0}^{d} \frac{\varepsilon^{|m|}}{m!} \left(\partial^m H\right)(\mathbf{x}_0)\, \mathbf{x}^m. \tag{36}$$

Note that $\varepsilon\nabla w^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{y}) = \nabla\Psi^\varepsilon(\mathbf{y}) + \nabla r^\varepsilon(\mathbf{y})$. We thus want $\Psi^\varepsilon(\mathbf{y})$ to solve:

$$\Psi^\varepsilon(\mathbf{y}) + T_{0,d}\Psi^\varepsilon(\mathbf{y}) \;=\; -\varepsilon \int_B D_1(\mathbf{x})\, D_{0,d}^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\nabla U_d(\mathbf{x}_0+\varepsilon\mathbf{x}) \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}, \tag{37}$$

where we have introduced the notation

$$T_{0,d}\Psi(\mathbf{y}) = \int_B D_1(\mathbf{x})\, D_{0,d}^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\nabla\Psi(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}. \tag{38}$$

The above equation is the integral formulation of

$$\Delta\Psi^\varepsilon + \nabla\cdot\big(D_1(\mathbf{x})D_{0,d}^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\big)\nabla\Psi^\varepsilon = -\varepsilon\nabla\cdot\big(D_1(\mathbf{x})D_{0,d}^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})(\nabla U_d)(\mathbf{x}_0+\varepsilon\mathbf{x})\big).$$

We now thus expand $D_{0,d}^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})$ in the definition of $T_{0,d}$ to obtain:

$$\begin{aligned} T_{0,d}\Psi(\mathbf{y}) &= T_0\Psi(\mathbf{y}) + \sum_{|m|=1}^{d} \frac{\varepsilon^{|m|}}{m!}\left(\partial^m D_0^{-1}\right)(\mathbf{x}_0) \int_B D_1(\mathbf{x})\, \mathbf{x}^m \nabla\Psi(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}, \\ T_0\Psi(\mathbf{y}) &:= \int_B D_1(\mathbf{x})\, D_0^{-1}(\mathbf{x}_0)\nabla\Psi(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}. \end{aligned}$$

Expanding U_d and $D_{0,d}^{-1}$ in (37), and setting

$$\Psi^\varepsilon(\mathbf{y}) \;=\; D_0(\mathbf{x}_0) \sum_{|j|=1}^{d}\sum_{|k|=0}^{d} \frac{\varepsilon^{|j|+|k|}}{j!k!}\left(\partial^j U\right)(\mathbf{x}_0)\left(\partial^k D_0^{-1}\right)(\mathbf{x}_0)\, \Psi^\varepsilon_{jk}(\mathbf{y}),$$

leads to the following equation for Ψ^ε_{jk}:

$$\begin{aligned} (I+T_0)\Psi^\varepsilon_{jk}(\mathbf{y}) &= -\sum_{|m|=1}^{d} \frac{\varepsilon^{|m|}}{m!}\left(\partial^m D_0^{-1}\right)(\mathbf{x}_0)\int_B D_1(\mathbf{x})\, \mathbf{x}^m \nabla\Psi^\varepsilon_{jk}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}, \\ &\quad -D_0(\mathbf{x}_0)^{-1}\int_B D_1(\mathbf{x})\, \mathbf{x}^k \nabla\mathbf{x}^j \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}. \end{aligned}$$

Equating like powers of ε, we verify that $\Psi^\varepsilon_{jk}(\mathbf{y}) = \sum_{l=0}^{d} \frac{\varepsilon^l}{l!}\phi^l_{jk}(\mathbf{y})$, where ϕ^l_{jk} solves the following integral equation *a.e.* in every bounded set of $\mathbb{R}^d$:

$$\begin{aligned} (I+T_0)\phi^l_{jk}(\mathbf{y}) &= -\sum_{|m|=1}^{l} \frac{l!(\partial^m D_0^{-1})(\mathbf{x}_0)}{m!(l-|m|)!}\int_B D_1(\mathbf{x})\, \mathbf{x}^m \nabla\phi^{l-|m|}_{jk}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x} \\ &\quad -\delta^0_l D_0^{-1}(\mathbf{x}_0)\int_B D_1(\mathbf{x})\, \mathbf{x}^k \nabla\mathbf{x}^j \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}. \end{aligned}$$

Existence and uniqueness of solutions in $H^1_{\text{loc}}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$ to the above equations follows from lemma 4.2 of the appendix: we first prove the result for ϕ^0_{jk}, then for ϕ^1_{jk} which depends only on ϕ^0_{jk}, and finally for all ϕ^m_{jk} iteratively. Moreover, according to the lemma, ϕ^l_{jk} solves the system of differential equations given in (11). The function Ψ^ε thus belongs to the space $H^1_{\text{loc}}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$ by construction. When D_0 is constant and equal to $D_0(\mathbf{x}_0)$ in the set $\mathbf{x}_0+\varepsilon B$, we do not need to expand D_0^{-1}. We thus have $D^{-1}_{0,d}=D_0^{-1}(\mathbf{x}_0)$ and ϕ^0_{j0} can be identified with Ψ^ε_{j0}.

We then verify that the remainder $r^\varepsilon(\mathbf{y})=w^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{y})-\Psi^\varepsilon(\mathbf{y})$ belongs to $H^1(B)$ by construction and moreover solves the integral equation:

$$(I+T_{0,d})r^\varepsilon(\mathbf{y})=S^\varepsilon(\varepsilon\mathbf{y})-\varepsilon^{d+2}\int_B D_1(\mathbf{x})\big[S_1(\mathbf{x})\nabla w^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{x})+\mathbf{S}_2(\mathbf{x})\big]\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}$$

where S_1 is the remainder of the $d+1$ order Taylor expansion of $D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})$ (so that $D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})=D^{-1}_{0,d}(\mathbf{x}_0+\varepsilon\mathbf{x})+\varepsilon^{d+1}S_1(\mathbf{x})$), $\mathbf{S}_2$ the remainder of $D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\nabla U(\mathbf{x}_0+\varepsilon\mathbf{x})$ and where we have defined

$$S^\varepsilon(\varepsilon\mathbf{y})=-\varepsilon^d\int_B D_1(\mathbf{x})\nabla u^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{x})\cdot\nabla_{\mathbf{x}}R_2(\mathbf{x}_0+\varepsilon\mathbf{x},\mathbf{x}_0+\varepsilon\mathbf{y})d\mathbf{x}.$$

We may now decompose r^ε as $r^\varepsilon(\mathbf{y}):=r^\varepsilon_1(\mathbf{y})+r^\varepsilon_2(\mathbf{y})+S^\varepsilon(\varepsilon\mathbf{y})$ with:

$$\begin{aligned}(I+T_{0,d})r^\varepsilon_1(\mathbf{y}) &= -\varepsilon\int_B D_1(\mathbf{x})D^{-1}_{0,d}(\mathbf{x}_0+\varepsilon\mathbf{x})\nabla S^\varepsilon(\varepsilon\mathbf{y})\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x},\\ (I+T_{0,d})r^\varepsilon_2(\mathbf{y}) &= -\varepsilon^{d+2}\int_B D_1(\mathbf{x})\big[S_1(\mathbf{x})\nabla w^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{x})+\mathbf{S}_2(\mathbf{x})\big]\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}.\end{aligned}$$

We know from the hypotheses in (2) that for all $\mathbf{y}\in\overline{B}$, $D_0(\mathbf{x}_0+\varepsilon\mathbf{y})+D_1(\mathbf{x})\geq C_0>0$, so that setting $0<\varepsilon\leq\varepsilon_0$ for ε_0 small enough, we have $1+D_1(\mathbf{x})D^{-1}_{0,d}(\mathbf{x}_0+\varepsilon\mathbf{y})\geq C_1>0$, for another constant C_1 independent of ε. An application of lemma 4.2 then yields that r^ε_1 and r^ε_2 are uniquely defined in $H^1_{\text{loc}}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$. Moreover, following lemma 4.2, we have the estimates:

$$\begin{aligned}\|\nabla r^\varepsilon_1\|_{L^2(\mathbb{R}^d)} &\leq C\varepsilon\|D_1\|_{L^\infty(B)}\|\nabla S^\varepsilon(\varepsilon\cdot)\|_{L^\infty(B)},\\ \|\nabla r^\varepsilon_2\|_{L^2(\mathbb{R}^d)} &\leq C\varepsilon^{d+2}\|D_1\|_{L^\infty(B)}\left(\|\nabla w^\varepsilon(\mathbf{x}_0+\varepsilon\cdot)\|_{L^2(B)}+\|D_0^{-1}\nabla U\|_{\mathcal{C}^{d+1}(B_0)}\right),\\ &\leq C\varepsilon^{d+2}\|D_1\|^2_{L^\infty(B)}\|g\|_{L^2(\partial\Omega)},\end{aligned}$$

according to (34) and by elliptic regularity, where B_0 is as above (34). It thus remains to estimate S^ε. From proposition 2.1, we know that $R_2\in\mathcal{C}^\infty(\Omega\times\Omega)$, which yields:

$$\begin{aligned}\|\nabla S^\varepsilon(\varepsilon\cdot)\|_{L^\infty(B)} &\leq C\varepsilon^d\|D_1\|_{L^\infty(B)}\left(\|\nabla w^\varepsilon(\mathbf{x}_0+\varepsilon\cdot)\|_{L^2(B)}+\|\nabla U\|_{L^\infty(B_0)}\right)\times\\ &\quad\times\|\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}R_2\|_{L^\infty(B_0\times B_0)}\quad\leq\quad\varepsilon^d\|D_1\|^2_{L^\infty(B)}\|g\|_{L^2(\partial\Omega)}.\end{aligned}$$

Gathering the different estimates for r^ε_1, r^ε_2 and S^ε, we obtain that

$$\|\nabla r^\varepsilon\|_{L^2(B)}\leq C\varepsilon^{d+1}\|D_1\|^2_{L^\infty(B)}\|g\|_{L^2(\partial\Omega)}.$$

To conclude the proof, we go back to (31) and take the trace on $\partial\Omega$. Plugging $\nabla w^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{x}) = \varepsilon^{-1}(\nabla\Psi^\varepsilon(\mathbf{x})+\nabla r^\varepsilon(\mathbf{x}))$ into (31), it just remains to expand $\nabla U(\mathbf{x}_0+\varepsilon\mathbf{x}) \in \mathcal{C}^\infty(\overline{B})$ and $\nabla_{\mathbf{x}} N(\mathbf{x}_0+\varepsilon\mathbf{x},\mathbf{y})$ thanks to (9) since we find, *a.e.* in $\partial\Omega$, that:

$$
\begin{aligned}
u^\varepsilon(\mathbf{y})|_{\partial\Omega} \;=\;& U(\mathbf{y})|_{\partial\Omega} - \varepsilon^d \int_B D_1(\mathbf{x})\Big(\nabla U(\mathbf{x}_0+\varepsilon\mathbf{x}) + \frac{1}{\varepsilon}\nabla\Psi^\varepsilon(\mathbf{x})\Big)\cdot\nabla_{\mathbf{x}}\, N(\mathbf{x}_0+\varepsilon\mathbf{x},\mathbf{y})|_{\partial\Omega}\, d\mathbf{x} \\
& + \mathcal{O}(\varepsilon^{2d}).
\end{aligned}
$$

The asymptotic expansion of remark 2.4 is obtained by decomposing w^ε slightly differently. We write $w^\varepsilon(\mathbf{x}_0+\varepsilon\mathbf{x}) = \Psi^\varepsilon(\mathbf{x}) + r^\varepsilon(\mathbf{x})$, where Ψ^ε is now given by

$$
\begin{aligned}
\Psi^\varepsilon(\mathbf{y}) + T^\varepsilon\Psi^\varepsilon(\mathbf{y}) \;&=\; -\varepsilon\int_B D_1(\mathbf{x})\, D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\nabla U_d(\mathbf{x}_0+\varepsilon\mathbf{x})\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}, \\
T^\varepsilon\Psi(\mathbf{y}) \;&:=\; \int_B D_1(\mathbf{x})\, D_0^{-1}(\mathbf{x}_0+\varepsilon\mathbf{x})\nabla\Psi(\mathbf{x})\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}.
\end{aligned}
$$

We then verify that the remainder r^ε is of order ε^d and that expanding U_d and setting $\Psi^\varepsilon(\mathbf{x}) = \sum_{|j|=1}^d \frac{\varepsilon^{|j|}}{j!}\,(\partial^j U)(\mathbf{x}_0)\,\Psi_j^\varepsilon$, leads to the desired result. □

Proof of Theorem 2.6. Let D_1 be a non-regular perturbation and let χ^η be the cut-off function with support in B defined as

$$
\left\{
\begin{array}{ll}
\chi^\eta(\mathbf{x}) = 1, & \text{for } \mathbf{x}\in B \text{ such that } \mathrm{dist}(\mathbf{x},\partial B) > \eta, \\
\chi^\eta(\mathbf{x}) = 0, & \text{otherwise.}
\end{array}
\right. \tag{39}
$$

The parameter η will be adjusted according to ε. Let now $\rho^\eta(\mathbf{x}) := \eta^{-d}\rho(\eta^{-1}\mathbf{x})$ be a standard mollifier and let $D_1^\eta := \rho^\eta * (\chi^\eta D_1)$. We verify that $D_1^\eta \in \mathcal{C}^\infty(\mathbb{R}^d)$ and that its support is included in B with a vanishing and continuous trace at the boundary. We can then apply theorem 2.2 to obtain an asymptotic expansion for the solution u_η^ε associated to D_1^η. Since the error term of order ε^{2d} depends only on $\|D_1^\eta\|_{L^\infty(\mathbb{R}^d)}$ - which is bounded by $\|D_1\|_{L^\infty(\mathbb{R}^d)}$ - it suffices to look at the limit of the different polarization tensors to find the limiting asymptotic expansion.

Since $D_0(\mathbf{x}) + D_1\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)$ is bounded from below by C_0, this property is still verified by the regularized diffusion coefficient so that, according to (74) of lemma 4.2 of the appendix, the function $\phi_{jk}^{l,\eta}$ associated to D_1^η satisfy by induction the estimates, for $l = 0,\cdots,d$:

$$
\|\nabla\phi_{jk}^{l,\eta}\|_{L^2(\mathbb{R}^d)} \le C\|D_1^\eta\|_{L^\infty(\mathbb{R}^d)}^{l+1}, \qquad \|\phi_{jk}^{l,\eta}\|_{L^2(A)} \le C\|D_1^\eta\|_{L^\infty(\mathbb{R}^d)}^{l+2},
$$

for any bounded set A. This yields that $\nabla\phi_{jk}^{l,\eta}$ is bounded in $L^2(\mathbb{R}^d)$ independently of η and so is $\phi_{jk}^{l,\eta}$ in $H^1(A)$. Defining the set $E := \{(j,k)\in\mathbb{N}^{2d}, l\in\mathbb{N}, 0\le |j|,|k|,l\le d\}$ with cardinal $|E|$, we may thus see $\{\phi_{jk}^{l,\eta}\}_E$ as bounded in $(H^1(A))^{|E|}$ and extract a subsequence as $\eta\to 0$ converging strongly in $(L^2(A))^{|E|}$ and with gradient converging weakly in $(L^2(\mathbb{R}^d))^{|E|}$ to a limit $\{\phi_{jk}^l\}_E$. We obtain that $\phi_{jk}^l \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla\phi_{jk}^l \in L^2(\mathbb{R}^d)$. To find the equation solved by $\nabla\phi_{jk}^l$, we consider the weak formulation verified by $\phi_{jk}^{l,\eta}$, which is, for all functions $\varphi\in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\nabla\varphi\in L^2(\mathbb{R}^d)$, $R^{-d}\|\varphi\|_{L^1(S_R)} \to$

0 as $R \to \infty$, where S_R denotes the sphere of radius R,

$$\int_{\mathbb{R}^d} (D_0(\mathbf{x}_0) + D_1^\eta(\mathbf{x})) \nabla \phi_{jk}^{l,\eta} \cdot \nabla \varphi \, d\mathbf{x} = -\, \delta_l^0 \int_B D_1^\eta (\mathbf{x}) \, \mathbf{x}^k \nabla \mathbf{x}^j \cdot \nabla \varphi \, d\mathbf{x}$$
$$-D_0(\mathbf{x}_0) \sum_{|m|=1}^{l} \frac{l!}{m!(l-|m|)!} \left(\partial^m D_0^{-1}\right)(\mathbf{x}_0) \int_B D_1^\eta (\mathbf{x}) \, \mathbf{x}^m \nabla \phi_{jk}^{l-|m|,\eta}(\mathbf{x}) \cdot \nabla \varphi \, d\mathbf{x}.$$

The above formulation is justified in lemma 4.2 below; see (77). Since D_1^η converges strongly in any $L^p(\mathbb{R}^d)$ for $1 \le p < \infty$, we can pass to the limit in the non-linear terms above and obtain the following limiting equation:

$$\begin{aligned}\int_{\mathbb{R}^d} (D_0(\mathbf{x}_0) + D_1(\mathbf{x})) \nabla \phi_{jk}^{l} \cdot \nabla \varphi \, d\mathbf{x} = -\, \delta_l^0 \int_B D_1 (\mathbf{x}) \, \mathbf{x}^k \nabla \mathbf{x}^j \cdot \nabla \varphi \, d\mathbf{x} \\ -D_0(\mathbf{x}_0) \sum_{|m|=1}^{l} \frac{l!}{m!(l-|m|)!} \left(\partial^m D_0^{-1}\right)(\mathbf{x}_0) \int_B D_1 (\mathbf{x}) \, \mathbf{x}^m \nabla \phi_{jk}^{l-|m|}(\mathbf{x}) \cdot \nabla \varphi \, d\mathbf{x}.\end{aligned} \tag{40}$$

To obtain the behavior of ϕ_{jk}^l at infinity, we use the integral formulation given in (75) of lemma 4.2 for the subsequence $\phi_{jk}^{l,\eta}$ and obtain, *a.e.* in every bounded set $\Omega' \subset \mathbb{R}^d$:

$$\begin{aligned}(I+T_0)\phi_{jk}^{l,\eta}(\mathbf{y}) &= -\sum_{|m|=1}^{l} \frac{l! \partial^m D_0^{-1}(\mathbf{x}_0)}{m!(l-|m|)!} \int_B D_1^\eta (\mathbf{x}) \, \mathbf{x}^m \nabla \phi_{jk}^{l-|m|,\eta}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x} \\ &\quad -\delta_l^0 D_0^{-1}(\mathbf{x}_0) \int_B D_1^\eta (\mathbf{x}) \, \mathbf{x}^k \nabla \mathbf{x}^j \cdot \nabla_{\mathbf{x}} \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x}.\end{aligned}$$

The above equation makes sense in $L^2(\Omega')$ and therefore almost everywhere in Ω' since $\nabla \phi_{jk}^{l,\eta} \in L^2(\Omega)$, $l = 0, \cdots, d$, and $\nabla \Gamma \in L^1_{\text{loc}}(\mathbb{R}^d)$ so that the right hand side is finite thanks to the Young inequality. Consider now a compact set $K \subset \mathbb{R}^d$ such that $\text{dist}(K, B) > C > 0$. The above equation is then verified uniformly in K and moreover $\phi_{jk}^{l,\eta} \in \mathcal{C}^0(K)$. Since $\nabla \phi_{jk}^{l,\eta}$ converges weakly to $\nabla \phi_{jk}^l$ for $0 \le l \le d$ and D_1^η converges strongly, it follows from the above equation that $\phi_{jk}^{l,\eta}$ is a Cauchy sequence in $\mathcal{C}^0(K)$ so that it converges uniformly to the solution, for all $\mathbf{x} \in K$, to

$$\begin{aligned}(I+T_0)\phi_{jk}^{l}(\mathbf{y}) &= -\sum_{|m|=1}^{l} \frac{l! \partial^m D_0^{-1}(\mathbf{x}_0)}{m!(l-|m|)!} \int_B D_1 (\mathbf{x}) \, \mathbf{x}^m \nabla \phi_{jk}^{l-|m|}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x} \\ &\quad -\delta_l^0 D_0^{-1}(\mathbf{x}_0) \int_B D_1 (\mathbf{x}) \, \mathbf{x}^k \nabla \mathbf{x}^j \cdot \nabla_{\mathbf{x}} \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x}.\end{aligned} \tag{41}$$

The fact that $\nabla_{\mathbf{x}} \Gamma(\mathbf{x}-\mathbf{y}) = \mathcal{O}(|\mathbf{y}|^{1-d})$ for $\mathbf{x} \in B$ and $\mathbf{y} \in K$ yields that $\phi_{jk}^l(\mathbf{y}) = \mathcal{O}(|\mathbf{y}|^{1-d})$ for such values of $\mathbf{y}$. It is then not difficult to see that (40) is the weak formulation of the problem given in the theorem. Notice that equation (41) is also valid *a.e.* in A since $\phi_{jk}^l \in L^2(A)$ for any bounded set A. Uniqueness follows from (40) and the behavior at infinity: the right hand side of (40) vanishes when we consider the difference of two possible solutions. Since those solutions are sufficiently regular, taking that difference as a test function implies the difference is a constant which must be equal to zero according to the vanishing limit at infinity.

Now that we have the expression of the limiting ϕ^l_{jk}, it suffices to pass to the limit in the polarization tensors using the weak convergence of $\nabla\phi^{0;\eta}_{j\,0}$ and the strong convergence of D_1^η and to choose η small enough such that all the errors terms coming from the different passages to the limit are smaller than $C\varepsilon^{2d}$. $\square$

Proof of Proposition 2.8. When D_0 is constant on the set $\mathbf{x}_0 + \varepsilon B$, only the sum involving the polarization tensor M remains in theorem 2.2 as we have mentioned in remark 2.5. We thus start from the expression of M given in theorem 2.6 and define $f_j := \phi^0_{j0} - \phi_j$. A proof of the existence and uniqueness for ϕ_j can be found in [2]. Owing the definitions of ϕ^0_{j0} and ϕ_j, we find that $f_j \in H^1_{\text{loc}}(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d/\overline{B})$ by construction and is the unique weak solution to

$$\nabla\cdot(D_0 + \mathbb{1}_B(\mathbf{x})D_1)\nabla f_j \;\;= -\mathbb{1}_B(\mathbf{x})D_1\Delta\mathbf{x}^j, \quad \mathbf{x}\in\mathbb{R}^d,$$

equipped with the condition at infinity:

$$f_j(\mathbf{y}) + \Gamma(\mathbf{y})D_0^{-1}D_1\int_{\partial B}\mathbf{n}\cdot\nabla\mathbf{x}^j d\sigma(\mathbf{x}) = \mathcal{O}(|\mathbf{y}|^{1-d}).$$

Here, $\mathbb{1}_B$ is the characteristic function of the set B. When $|j| = 1$, we obtain $f_j = 0$. When $|j| \geq 2$, we need to sum over j to show that f_j is small in an appropriate sense. To this aim, we derive an integral equation for f_j from that of ϕ^0_{j0} and ϕ_j. As we mentioned in the proof of theorem 2.6, (41) is verified *a.e.* by ϕ^0_{j0} so that we have

$$D_0D_1^{-1}\phi^0_{j0}(\mathbf{y}) = -\int_B\nabla\phi^0_{j0}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x} - \int_B\nabla\mathbf{x}^j\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}. \qquad (42)$$

Since ϕ_j is harmonic in $B\cup\mathbb{R}^d\backslash\overline{B}$, we deduce from elliptic regularity in Lipschitz domains (see e.g. [2]) that $\phi_j \in H^{\frac{3}{2}}(B)$ so that its inner normal derivative at the boundary ∂B belongs to $L^2(\partial B)$. This allows us to express ϕ_j in terms of single layer potential, using the jump of its normal derivative at the boundary given in proposition 2.8, as

$$D_0D_1^{-1}\phi_j(\mathbf{y}) = -\int_{\partial B}\left(\frac{\partial\phi_j}{\partial\mathbf{n}}\bigg|_-(\mathbf{x}) + \mathbf{n}\cdot\nabla\mathbf{x}^j\right)\Gamma(\mathbf{x}-\mathbf{y})d\sigma(\mathbf{x}). \qquad (43)$$

The latter equation is verified in $L^1(A)$ for any bounded set $A\subset\mathbb{R}^d$, and thus *a.e.* since $\|\Gamma(\mathbf{x}-\cdot)\|_{L^1(A)}$ is uniformly bounded in $\mathbf{x}\in\partial B$. Moreover, since ϕ_j is harmonic in B, we have for any $\varphi\in H^1(B)$:

$$\int_B\nabla\phi_j\cdot\nabla\varphi\,d\mathbf{x} = \int_{\partial B}\frac{\partial\phi_j}{\partial\mathbf{n}}\bigg|_-\varphi\,d\sigma(\mathbf{x}).$$

Let $\psi\in\mathcal{C}^0_c(A)$ and set $\varphi(\mathbf{x}) = \int_A\psi(\mathbf{y})\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y}$. Using the Young inequality and the fact that Γ and $\nabla\Gamma$ belong to $L^1_{\text{loc}}(\mathbb{R}^d)$, we verify that $\varphi\in H^1(B)$ so that it can be used as a test function. Moreover, to be able to use the Fubini theorem, we apply as in the proof of lemma 4.2 the Sobolev inequality 4.3 to conclude that $\nabla\phi_j(\mathbf{x})\cdot\nabla\Gamma(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})$ belongs to $L^1(B\times A)$. In the same way, $\frac{\partial\phi_j}{\partial\mathbf{n}}\Big|_-(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})$ belongs to $L^1(\partial B\times A)$ since

$$\int_{\partial B}\int_A\left|\frac{\partial\phi_j}{\partial\mathbf{n}}\bigg|_-(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\right|d\sigma(\mathbf{x})d\mathbf{y} \leq C\left\|\frac{\partial\phi_j}{\partial\mathbf{n}}\bigg|_-\right\|_{L^2(\partial B)}\|\Gamma\|_{B_a}\|\psi\|_{L^\infty(A)},$$

for a ball of radius a large enough. We may thus write:

$$\int_A \left(\int_B \nabla\phi_j(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x} - \int_{\partial B} \frac{\partial \phi_j}{\partial \mathbf{n}}\Big|_- \Gamma(\mathbf{x}-\mathbf{y})\, d\sigma(\mathbf{x}) \right) \psi(\mathbf{y}) d\mathbf{y} = 0.$$

Plugging (43) into the latter equation yields:

$$\int_A \left(D_0 D_1^{-1} \phi_j(\mathbf{y}) + \int_B \nabla\phi_j(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x} + \int_{\partial B} \mathbf{n} \cdot \nabla \mathbf{x}^j\, \Gamma(\mathbf{x}-\mathbf{y})\, d\sigma(\mathbf{x}) \right) \psi(\mathbf{y}) d\mathbf{y} = 0.$$

Integrating (42) against ψ, subtracting the equation above and performing an integration by parts, we find:

$$\int_A \left(D_0 D_1^{-1} f_j(\mathbf{y}) + \int_B \nabla f_j(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x} - \int_B \Delta \mathbf{x}^j\, \Gamma(\mathbf{x}-\mathbf{y})\, d\mathbf{x} \right) \psi(\mathbf{y}) d\mathbf{y} = 0.$$

The quantity under parentheses belongs to $L^2(A)$. Thus, we deduce by density that the above relation holds also for any $\psi \in L^2(A)$ so that f_j solves the following integral equation, *a.e.* in every bounded set $\Omega' \subset \mathbb{R}^d$,

$$D_0 D_1^{-1} f_j(\mathbf{y}) \;=\; \int_B \nabla f_j(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{x} + \int_B \Delta \mathbf{x}^j \Gamma(\mathbf{x}-\mathbf{y}) d\sigma(\mathbf{x}). \qquad (44)$$

We now show that an appropriate linear combination of the f_j's is of order ε^{d+1}. First, since D_0 is constant in $\mathbf{x}_0 + \varepsilon B$, $\Delta U(\mathbf{x}_0 + \varepsilon\mathbf{x}) = 0$ for $\mathbf{x} \in B$ according to (4), so that, using the notation in (36), we get that $\Delta U_d(\mathbf{x}) = \Delta(U_d(\mathbf{x}) - U(\mathbf{x}_0 + \varepsilon\mathbf{x})) = \mathcal{O}(\varepsilon^{d+1})$ uniformly in B. As a consequence, we have

$$R^\varepsilon(\mathbf{x}) := \Delta U_d(\mathbf{x}) = \sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) \Delta \mathbf{x}^j = \mathcal{O}(\varepsilon^{d+1}).$$

Thus, defining

$$F^\varepsilon(\mathbf{x}) := \sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) f_j(\mathbf{x}),$$

it follows:

$$\begin{aligned} \sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) M_{ij} &= D_1 \sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) \int_B \nabla(\mathbf{x}^j + \phi_j(\mathbf{x}) + f_j(\mathbf{x})) \cdot \nabla \mathbf{x}^i d\mathbf{x}, \\ &= D_1 \sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) \int_B \nabla(\mathbf{x}^j + \phi_j(\mathbf{x})) \cdot \nabla \mathbf{x}^i d\mathbf{x} + D_1 \int_B \nabla F^\varepsilon(\mathbf{x}) \cdot \nabla \mathbf{x}^i d\mathbf{x}. \end{aligned}$$

According to the definition of f_j, $F^\varepsilon \in H^1_{\text{loc}}(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d/\overline{B})$ solves:

$$\nabla \cdot (D_0(\mathbf{x}_0) + \mathbb{1}_B(\mathbf{x}) D_1) \nabla F^\varepsilon \;= -\mathbb{1}_B(\mathbf{x}) D_1 R^\varepsilon, \quad \mathbf{x} \in \mathbb{R}^d, \qquad (45)$$

equipped with the condition at infinity:

$$F^\varepsilon(\mathbf{y}) + \Gamma(\mathbf{y}) D_1 \int_B R^\varepsilon d\mathbf{x} = \mathcal{O}(|\mathbf{y}|^{1-d}). \qquad (46)$$

Following (44), F^ε thus solves the integral equation, *a.e.* in every bounded set $\Omega' \subset \mathbb{R}^d$:

$$F^\varepsilon(\mathbf{y}) \;\;=\;\; -D_1 D_0^{-1}(\mathbf{x}_0)\int_B \big[\nabla F^\varepsilon(\mathbf{x})\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y}) - R^\varepsilon(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y})\big]d\mathbf{x}, \qquad (47)$$

so that Young's inequality gives

$$\|F^\varepsilon\|_{L^2(B)} \le C\|\nabla F^\varepsilon\|_{L^2(B)} + C\|R^\varepsilon\|_{L^2(B)} \le C\|\nabla F^\varepsilon\|_{L^2(B)} + \mathcal{O}(\varepsilon^{d+1}). \qquad (48)$$

Let B_R be the ball of radius R with $B \subset\subset B_R$ and denote by S_R its boundary. Integrating (45) on B_R against F^ε leads to

$$\int_{B_R}(D_0(\mathbf{x}_0) + \mathbb{1}_B D_1)|\nabla F^\varepsilon|^2 d\mathbf{x} = -D_1\int_B R^\varepsilon F^\varepsilon d\mathbf{x} + \int_{S_R}\frac{\partial F^\varepsilon}{\partial \mathbf{n}}F^\varepsilon d\sigma(\mathbf{x}), \qquad (49)$$

where σ is the surface measure on S_R. We may recast condition (46) using the integral equation (47) for F^ε and its derivative as

$$\partial^\alpha F^\varepsilon(\mathbf{y}) + \partial^\alpha\Gamma(\mathbf{y})D_1\int_B R^\varepsilon d\mathbf{x} = \mathcal{O}\left((\|\nabla F^\varepsilon\|_{L^2(B)} + \|R^\varepsilon\|_{L^2(B)})|\mathbf{y}|^{1-d-|\alpha|}\right), \qquad (50)$$

for a multi-index α with $|\alpha| \le 1$. Consider first $d \ge 3$. Then $\nabla F^\varepsilon \in L^2(\mathbb{R}^d)$ and the boundary integral in (49) goes to zero as R tends to infinity so that

$$\int_{\mathbb{R}^d}(D_0(\mathbf{x}_0) + \mathbb{1}_B D_1)|\nabla F^\varepsilon|^2 d\mathbf{x} = -D_1\int_B R^\varepsilon F^\varepsilon d\mathbf{x}. \qquad (51)$$

This yields, together with (48):

$$\|\nabla F^\varepsilon\|^2_{L^2(\mathbb{R}^d)} \le \mathcal{O}(\varepsilon^{2(d+1)}) + \|R^\varepsilon\|_{L^2(B)}\|\nabla F^\varepsilon\|_{L^2(\mathbb{R}^d)},$$

so that $\|\nabla F^\varepsilon\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{d+1})$. Consider now the case $d = 2$. We cannot use the same approach since F^ε does not vanish at infinity. Using (50) for $\mathbf{y} \in S_R$, we have

$$\begin{aligned}
|F^\varepsilon(\mathbf{y})| &\le C\Big(\Big(\log R + \frac{1}{R}\Big)\|R^\varepsilon\|_{L^2(B)} + \frac{1}{R}\|\nabla F^\varepsilon\|_{L^2(B)}\Big),\\
\Big|\frac{\partial F^\varepsilon}{\partial \mathbf{n}}(\mathbf{y})\Big| &\le C\Big(\frac{1}{R}\Big(1 + \frac{1}{R}\Big)\|R^\varepsilon\|_{L^2(B)} + \frac{1}{R^2}\|\nabla F^\varepsilon\|_{L^2(B)}\Big),
\end{aligned}$$

so that, since $\|R^\varepsilon\|_{L^2(B)} = \mathcal{O}(\varepsilon^{d+1})$,

$$\Big|\int_{S_R}\frac{\partial F^\varepsilon}{\partial \mathbf{n}}F^\varepsilon d\sigma(\mathbf{x})\Big| \;\le\; C\varepsilon^{2(d+1)} + C_R\,\varepsilon^{d+1}\|\nabla F^\varepsilon\|_{L^2(B)} + \frac{C}{R^3}\|\nabla F^\varepsilon\|^2_{L^2(B)}.$$

Since, according to hypothesis 2, $D_0(\mathbf{x}_0) + \mathbb{1}_B D_1 \ge C_0 > 0$ *a.e.* in $\mathbb{R}^d$, it follows from (48), (49) and the above inequality that:

$$C_0\|\nabla F^\varepsilon\|^2_{B_R} \le C\varepsilon^{2(d+1)} + \Big(\frac{C}{R^3} + \eta\Big)\|\nabla F^\varepsilon\|^2_{L^2(B)},$$

for any $\eta > 0$. It suffices finally to set η small enough and R large enough so that $\frac{C}{R^3} + \eta < C_0$ to obtain

$$\|\nabla F^\varepsilon\|_{L^2(B)} \le \|\nabla F^\varepsilon\|_{L^2(B_R)} = \mathcal{O}(\varepsilon^{d+1}).$$

We end the proof with the following integration by parts:

$$
\begin{aligned}
\sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) M_{ij} &= D_1 \sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) \int_{\partial B} \mathbf{n} \cdot \nabla(\mathbf{x}^j + \phi_j(\mathbf{x}))\mathbf{x}^i d\sigma(\mathbf{x}) \\
&\quad - D_1 \int_B R^\varepsilon(\mathbf{x})\, \mathbf{x}^i d\mathbf{x} + D_1 \int_B \nabla F^\varepsilon(\mathbf{x}) \cdot \nabla \mathbf{x}^i d\mathbf{x}, \\
&= D_1 \sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) \int_{\partial B} \mathbf{n} \cdot \nabla(\mathbf{x}^j + \phi_j(\mathbf{x}))\mathbf{x}^i d\sigma(\mathbf{x}) + \mathcal{O}(\varepsilon^{d+1}), \\
&= \sum_{|j|=1}^{d} \frac{\varepsilon^{|j|}}{j!} \partial^j U(\mathbf{x}_0) \mathcal{M}_{ij} + \mathcal{O}(\varepsilon^{d+1}),
\end{aligned}
$$

which shows that the error terms generated by M and $\mathcal{M}$ agree up to an order $\mathcal{O}(\varepsilon^{d+1})$. □

4.2 Asymptotic expansions for the Helmholtz equation

We now prove proposition 3.2, theorem 3.3, and proposition 3.4.

Proof of proposition 3.2. We write $v^\varepsilon := V + w^\varepsilon$ so that the corrector w^ε satisfies:

$$
\begin{aligned}
&-\Delta w^\varepsilon(\mathbf{x}) + \left(q_0(\mathbf{x}) + \frac{1}{\varepsilon^{2-\eta}} q_1 \left(\frac{\mathbf{x} - \mathbf{x}_0}{\varepsilon} \right) \right) w^\varepsilon(\mathbf{x}) = -\frac{1}{\varepsilon^{2-\eta}} q_1 \left(\frac{\mathbf{x} - \mathbf{x}_0}{\varepsilon} \right) V(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\
&\frac{\partial w^\varepsilon}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega.
\end{aligned}
\tag{52}
$$

We need to show the existence of w^ε. We first show the existence and uniqueness of a solution to the integral formulation of (52), which formally reads, $\mathbf{y}$ *a.e.* in Ω:

$$
\begin{aligned}
w^\varepsilon + T^\varepsilon w^\varepsilon &= -T^\varepsilon V, \\
T^\varepsilon \varphi(\mathbf{y}) &= \int_{\mathbf{x}_0 + \varepsilon B} q_1 \left(\frac{\mathbf{x} - \mathbf{x}_0}{\varepsilon} \right) \varphi(\mathbf{x}) N(\mathbf{x}, \mathbf{y}) d\mathbf{x}.
\end{aligned}
$$

We consider first the case $d \geq 3$. Using the decomposition of N given in proposition 3.1 and denoting by w^* the restriction of $w^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{y})$ to B (we do not write the dependence on ε to simplify), we recast the above system as

$$
\left\{
\begin{aligned}
&w^* + \varepsilon^\eta T w^* + \varepsilon^{d-2+\eta} R^\varepsilon w^* = -T^\varepsilon V(\mathbf{x}_0 + \varepsilon\mathbf{y}), \qquad \mathbf{y} \in B, \\
&T w^*(\mathbf{y}) = \int_B q_1(\mathbf{x})\, w^*(\mathbf{x}) \Gamma(\mathbf{x} - \mathbf{y}) d\mathbf{x}, \\
&R^\varepsilon w^*(\mathbf{y}) = \int_B q_1(\mathbf{x})\, w^*(\mathbf{x}) R(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{x}_0 + \varepsilon\mathbf{y}) d\mathbf{x}.
\end{aligned}
\right.
\tag{53}
$$

We have used the homogeneity $\Gamma(\varepsilon\mathbf{x}) = \varepsilon^{2-d}\Gamma(\mathbf{x})$ when $d \geq 3$. Since T and R^ε are compact operators in $L^2(B)$, they have discrete spectra. Indeed, since $\Gamma, \nabla\Gamma \in L^1_{\text{loc}}(\mathbb{R}^d)$, we have, using the Young inequality for any $\varphi \in L^2(B)$,

$$
\|T\varphi\|_{L^2(B)} \leq \|q_1\varphi\|_{L^2(B)} \|\Gamma\|_{L^1(B_a)} \leq \|q_1\|_{L^\infty(B)} \|\varphi\|_{L^2(B)} \|\Gamma\|_{L^1(B_a)},
$$

where B_a is a ball of radius a large enough. Thus, proceeding analogously for ∇T,

$$\|T\|_{\mathcal{L}(L^2(B))} \leq \|q_1\|_{L^\infty(B)}\|\Gamma\|_{L^1(B_a)}, \qquad \|\nabla T\|_{\mathcal{L}(L^2(B))} \leq \|q_1\|_{L^\infty(B)}\|\nabla\Gamma\|_{L^1(B_a)},$$

and compactness stems from the Rellich theorem. The same holds for R^ε since it is Hilbert-Schmidt as $R(\mathbf{x}_0+\varepsilon\cdot,\mathbf{x}_0+\varepsilon\cdot)$ belongs to $L^2(B\times B)$ (though not necessarily uniformly in ε; see below) according to proposition 3.1. In the same way, we obtain that

$$\|T^\varepsilon V(\mathbf{x}_0+\varepsilon\cdot)\|_{H^1(B)} \leq C\|V(\mathbf{x}_0+\varepsilon\cdot)\|_{L^2(B)},$$

where C is independent of ε. It remains to show that the operator $I+\varepsilon^\eta T+\varepsilon^{d-2+\eta}R^\varepsilon$ is injective and to use the Fredholm alternative to obtain the existence of a unique $w^* \in H^1(B)$ verifying (53). Injectivity is obvious when $\eta \in]0,2]$ since he operator norm of $\varepsilon^\eta T+\varepsilon^{1+\eta}R^\varepsilon$ in $\mathcal{L}(L^2(B))$ is of order $\mathcal{O}(\varepsilon^\eta)<1$ for $\varepsilon<\varepsilon_0$ small enough.

When $\eta=0$, we need to use assumption **(H-2)**. Since -1 is not an eigenvalue of T, it suffices to fix ε_0 small enough such that the distance between -1 and the nearest eigenvalue of T is larger than $\varepsilon_0^{d-2}\|R^\varepsilon\|_{\mathcal{L}(L^2(B))}$. To do so, we remark, following proposition 3.1, that uniformly in $\mathbf{y}\in B$, $R(\cdot,\mathbf{y}) \in W^{2,p}(B)$ with $p<\frac{d}{d-2}$ when $3\leq d\leq 5$ and $p<\infty$ when $d=2$. The Sobolev embedding then yields that $R(\cdot,\mathbf{y})\in\mathcal{C}^0(\overline{B})$ when $d\leq 3$ and $R(\cdot,\mathbf{y})\in L^q(B)$ with $q<\infty$ when $d=4$ and $q=5$ when $d=5$. Hence,

$$\|R^\varepsilon\|_{\mathcal{L}(L^2(B))} \leq C\|R(\mathbf{x}_0+\varepsilon\cdot,\mathbf{x}_0+\varepsilon\cdot)\|_{L^2},$$

which is $\mathcal{O}(1)$ for $d\leq 3$, $\mathcal{O}(\varepsilon^{-\alpha})$ for any $\alpha>0$ when $d=4$, and $\mathcal{O}(\varepsilon^{-1})$ for $d=5$. For the particular case $q_0\equiv 0$, proposition 3.1 gives $R\in\mathcal{C}^\infty(\overline{B}\times\overline{B})$ so that $\|R^\varepsilon\|_{\mathcal{L}(L^2(B))}$ is bounded independently of ε for any d. In any event, $\varepsilon_0^{d-2}\|R^\varepsilon\|_{\mathcal{L}(L^2(B))}=o(\varepsilon_0)$ so that the Fredholm alternative yields again a unique $w^*\in L^2(B)$ solution to (53) for ε_0 small enough. In addition, w^* satisfies the estimate:

$$\|w^*\|_{L^2(B)} \leq C\varepsilon^\eta\|V(\mathbf{x}_0+\varepsilon\cdot)\|_{L^2(B)}. \tag{54}$$

Then w^ε is given, for $\mathbf{y}\in\Omega$, by:

$$\begin{cases} w^\varepsilon(\mathbf{y}) = w^*\left(\dfrac{\mathbf{y}-\mathbf{x}_0}{\varepsilon}\right), & \mathbf{y}\in\mathbf{x}_0+\varepsilon B, \\ w^\varepsilon(\mathbf{y}) = \left(-\varepsilon^\eta T w^* - \varepsilon^{d-2+\eta}R^\varepsilon w^*\right)\left(\dfrac{\mathbf{y}-\mathbf{x}_0}{\varepsilon}\right) - T^\varepsilon V(\mathbf{y}), & \text{otherwise,} \end{cases}$$

so that $w^\varepsilon\in H^1(\Omega)$. We verify that w^ε is then a solution to the variational formulation of (52). To prove uniqueness, we show that, for a given $u\in H^1(\Omega)$, the assertion

$$\int_\Omega \nabla u\cdot\nabla v\,d\mathbf{x} + \int_\Omega \left(q_0+\frac{1}{\varepsilon^{2-\eta}}q_1\left(\frac{\cdot-\mathbf{x}_0}{\varepsilon}\right)\right)u\,v d\mathbf{x}=0, \qquad \forall v\in H^1(\Omega), \tag{55}$$

implies $u=0$. Indeed, for $\varphi\in L^2(\Omega)$, consider the weak solution $v\in H^1(\Omega)$ of

$$-\Delta v+q_0 v=\varphi, \quad \mathbf{x}\in\Omega,$$

augmented with homogeneous Neumann conditions on $\partial\Omega$. Thus, v is given by $v(\mathbf{y})=\int_\Omega N(\mathbf{x},\mathbf{y})\varphi(\mathbf{x})d\mathbf{x}$. Plugging v into (55) leads to

$$\int_\Omega (u+T^\varepsilon u)\varphi d\mathbf{x}=0, \qquad \forall\varphi\in L^2(\Omega),$$

so that $u + T^\varepsilon u = 0$, which implies that $u = 0$. This ends the proof of existence of a unique solution of the variational formulation of (52) when $d \geq 3$.

We treat now the case $d = 2$. When $\eta > 0$, existence and uniqueness can be established in the same manner as above. When $\eta = 0$, we use assumption **(H-3)**. We first notice that for $d = 2$, we have

$$T^\varepsilon w^*(\mathbf{x}_0 + \varepsilon\mathbf{y}) = \int_B q_1(\mathbf{x})\, w^*(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y} - \frac{\log\varepsilon}{2\pi}\int_B q_1(\mathbf{x})\, w^*(\mathbf{x})d\mathbf{x} + R^\varepsilon w^*(\mathbf{y}).$$

In the same way, proposition 3.1 gives, uniformly in $\mathbf{y} \in B$, $R(\cdot, \mathbf{y}) \in W^{2,p}(B) \subset \mathcal{C}^1(\overline{B})$ with $p < \infty$, so that we can recast $R^\varepsilon w^*$ as

$$\begin{aligned} R^\varepsilon w^*(\mathbf{y}) &= R(\mathbf{x}_0, \mathbf{x}_0 + \varepsilon\mathbf{y})\int_B q_1(\mathbf{x})\, w^*(\mathbf{x})d\mathbf{x} + \widetilde{R}^\varepsilon w^*(\mathbf{y}) \\ \widetilde{R}^\varepsilon w^*(\mathbf{y}) &= \int_B q_1(\mathbf{x})\, w^*(\mathbf{x})\left(R(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{x}_0 + \varepsilon\mathbf{y}) - R(\mathbf{x}_0, \mathbf{x}_0 + \varepsilon\mathbf{y})\right)d\mathbf{x}. \end{aligned}$$

The system (53) can then be reformulated as:

$$w^* + Tw^* + \widetilde{R}^\varepsilon w^* = -TV - \widetilde{R}^\varepsilon V - \left(\frac{\log\varepsilon}{2\pi} + R(\mathbf{x}_0, \mathbf{x}_0 + \varepsilon\cdot)\right)C^\varepsilon,$$

where the constant C^ε is equal to

$$C^\varepsilon = \int_B q_1(\mathbf{x})\left(V(\mathbf{x}_0 + \varepsilon\mathbf{x}) + w^*(\mathbf{x})\right)d\mathbf{x} = \int_B q_1(\mathbf{x})\, v^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x})d\mathbf{x}.$$

Under assumption **(H-3)**, we have $C^\varepsilon = 0$ so that we just need to show that

$$\|\widetilde{R}^\varepsilon\|_{\mathcal{L}(L^2(B))} = o(1)$$

to apply the Fredholm alternative. Since $R(\cdot, \mathbf{y}) \in \mathcal{C}^1(\overline{B})$, uniformly in $\mathbf{y}$, we have, for all $(\mathbf{x}, \mathbf{y}) \in \overline{B} \times \overline{B}$, that $|R(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{x}_0 + \varepsilon\mathbf{y}) - R(\mathbf{x}_0, \mathbf{x}_0 + \varepsilon\mathbf{y})| \leq C\varepsilon$, which gives $\|\widetilde{R}^\varepsilon\|_{\mathcal{L}(L^2(B))} = \mathcal{O}(\varepsilon)$ and ends the proof of existence when $d = 2$.

We now prove decomposition (24), which is the corner stone of the proof of theorem 3.3. Since $w^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x}) = w^*(\mathbf{x})$ when $\mathbf{x} \in B$, it suffices to obtain an expression for w^*. We consider first the case $d \geq 3$. Defining $V^\varepsilon(\mathbf{x}) := V(\mathbf{x}_0 + \varepsilon\mathbf{x})$, we recast (53) as:

$$w^* + \varepsilon^\eta Tw^* = -\varepsilon^\eta TV^\varepsilon - \varepsilon^{d-2+\eta}R^\varepsilon\left(V^\varepsilon + w^*\right).$$

We expand V^ε in the first term of the right hand side and set $w^* = \varepsilon^\eta\Psi^\varepsilon + \varepsilon^{d-2+\eta}\, r^\varepsilon + r^\varepsilon_V$, so as to obtain:

$$\begin{aligned} \Psi^\varepsilon + \varepsilon^\eta T\Psi^\varepsilon &= -T\sum_{|j|=0}^{d+1}\frac{\varepsilon^{|j|}}{j!}\mathbf{x}^j\partial^j V(\mathbf{x}_0), \\ r^\varepsilon + \varepsilon^\eta Tr^\varepsilon &= -R^\varepsilon\left(V^\varepsilon + w^*\right), \qquad\qquad r^\varepsilon_V + \varepsilon^\eta Tr^\varepsilon_V = -TR^\varepsilon_V, \end{aligned}$$

where R_V^ε is the remainder of the Taylor expansion of $V^\varepsilon \in \mathcal{C}^\infty(\overline{B})$ of order $d+2$. Writing $\Psi^\varepsilon(\mathbf{x}) := \sum_{|j|=0}^{d+1} \frac{\varepsilon^{|j|}}{j!} \partial^j V(\mathbf{x}_0) \phi_j^\eta(\mathbf{x})$, with

$$\phi_j^\eta(\mathbf{x}) + \varepsilon^\eta T \phi_j^\eta(\mathbf{x}) = -T\mathbf{x}^j,$$

and following the preceding proof of existence when $d \geq 3$, we verify that $r_V^\varepsilon \in H^1(B)$ with a norm bounded by $C\varepsilon^{d+2}$ and that r^ε and ϕ_j^η are uniquely defined in $H^1(B)$. Also, examining $\|R^\varepsilon\|_{\mathcal{L}(L^2(B))}$ as in the proof of existence, we find that r^ε is bounded in $L^2(B)$ independently of ε when $d=3$, is $\mathcal{O}(\varepsilon^{-\alpha})$ for any $\alpha > 0$ when $d=4$ and $\mathcal{O}(\varepsilon^{-1})$ when $d=5$. When $q_0 \equiv 0$, r^ε is bounded in $H^1(B)$ independently of ε since $\|R^\varepsilon\|_{\mathcal{L}(H^1(B))}$ is uniformly bounded. We thus obtain the expression (24) announced in the proposition for $d \geq 3$. When $d=2$, the equation for r^ε has to be replaced by

$$\begin{aligned} r^\varepsilon + \varepsilon^\eta T r^\varepsilon &= -R^\varepsilon\left(V^\varepsilon + w^*\right) + \frac{\log\varepsilon}{2\pi}\int_B q_1\left(\mathbf{x}\right) v^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x}) d\mathbf{x}, \\ &= -\widetilde{R}^\varepsilon\left(V^\varepsilon + w^*\right) - \left(\frac{\log\varepsilon}{2\pi} + R(\mathbf{x}_0, \mathbf{x}_0 + \varepsilon\cdot)\right) C^\varepsilon, \end{aligned}$$

where C^ε is the same constant as before. When $\eta > 0$, we verify that $r^\varepsilon \in H^1(B)$ with a norm of order $\log\varepsilon$. When $\eta = 0$, assumption **(H-3)** implies $C^\varepsilon = 0$. Since $\widetilde{R}^\varepsilon$ is $\mathcal{O}(\varepsilon)$ in $\mathcal{L}(L^2(B), H^1(B))$, we deduce that r^ε is $\mathcal{O}(\varepsilon)$ in $H^1(B)$ since V is uniformly bounded in B and w^* is bounded in $L^2(B)$ according to (54). $\square$

Proof of Theorem 3.3. We express v^ε in terms of V and the Green function N, to obtain, *a.e.* in Ω:

$$v^\varepsilon(\mathbf{y}) = V(\mathbf{y}) - \varepsilon^{d-2+\eta}\int_B q_1(\mathbf{x})\left(V(\mathbf{x}_0 + \varepsilon\mathbf{x}) + w^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x})\right) N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y}) d\mathbf{x}. \quad (56)$$

Taking the trace of (56) on $\partial\Omega$, which is well defined in $L^2(\partial\Omega)$ and thus almost everywhere, replacing $w^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x})$ by the expression in (24) and Taylor expanding both V and N according to (20), lead to the result. $\square$

Proof of Proposition 3.4. The outline of the proof is as follows: starting from the asymptotic expansion for v^ε in theorem 3.3, our aim is to recover that of u^ε in theorem 2.2 and the expression of the polarization tensor M. This is done in several steps. First, we verify that assumptions **(H-1)**, **(H-2)** and **(H-3)** are satisfied for the particular form (26) of the potential q_1. In a second step, we show that the term f^ε in the expansion of v^ε is of order $\mathcal{O}(\varepsilon^4)$ so that $\varepsilon^{2(d-2)} f^\varepsilon$ is $\mathcal{O}(\varepsilon^{2d})$ and can be treated as a remainder. Then, we show in (27)–(28) that the two first-order terms in the expansion of v^ε are actually of order $\mathcal{O}(\varepsilon^{2d})$ so that they can be neglected and the expansions for v^ε and u^ε have the same leading order $\mathcal{O}(\varepsilon^d)$. Finally, using the particular form of the potential q_1, we perform some transformations in the polarization tensors Q and Q^η for $\eta = 0$ leading to the expression of the polarization tensor M in theorem 2.2.

We will need the following lemma, which is one of the main ingredients to show the equivalence of the tensors:

Lemma 4.1 *Assume $v \in H^1(B)$ verifies in the distribution sense,*

$$-\Delta v + q_1 v = h \quad \text{in } \mathcal{D}'(B), \qquad q_1(\mathbf{x}) = \frac{\Delta\sqrt{D_0 + D_1(\mathbf{x})}}{\sqrt{D_0 + D_1(\mathbf{x})}}, \qquad h \in L^2(B). \tag{57}$$

Then, for all $\varphi \in H^1(B)$ and harmonic in B, we have:

$$\int_B q_1(\mathbf{x})\, v(\mathbf{x})\varphi(\mathbf{x}) d\mathbf{x} = \frac{1}{\sqrt{D_0}} \int_B D_1(\mathbf{x}) \nabla\Big(\frac{v(\mathbf{x})}{\sqrt{D_0 + D_1(\mathbf{x})}}\Big) \cdot \nabla\varphi(\mathbf{x}) d\mathbf{x} - \int_B h\varphi d\mathbf{x}.$$

Proof. Define $D(\mathbf{x}) := D_0 + D_1(\mathbf{x})$. Note that $\partial_{\mathbf{n}} D_1 = 0$ on ∂B since $D_1 \in \mathcal{C}^2(\Omega)$ and D_1 is supported in B. Hence, two successive integrations by parts yield:

$$\int_B \frac{\Delta\sqrt{D(\mathbf{x})}}{\sqrt{D(\mathbf{x})}} v(\mathbf{x})\varphi(\mathbf{x}) d\mathbf{x} = -\int_B \nabla\sqrt{D} \cdot \nabla\Big(\frac{v\varphi}{\sqrt{D}}\Big) d\mathbf{x} = \int_B \Big(\sqrt{D} - \sqrt{D_0}\Big)\Delta\Big(\frac{v\varphi}{\sqrt{D}}\Big) d\mathbf{x}.$$

The above expression makes sense since φ is harmonic and $\Delta v \in L^2(B)$ because of (57). Starting from (57), we verify after some algebra that v solves

$$\nabla \cdot D\nabla\Big(\frac{v}{\sqrt{D}}\Big) = \sqrt{D} h, \quad \text{in } \mathcal{D}'(B), \tag{58}$$

which, since $D > 0$ in $\mathbb{R}^d$, is equivalent to:

$$2\,\nabla\sqrt{D} \cdot \nabla\Big(\frac{v}{\sqrt{D}}\Big) + \sqrt{D}\Delta\Big(\frac{v}{\sqrt{D}}\Big) = h.$$

Since $D = D_0$ on ∂B and is constant, it follows from the above equation and another integration by parts that:

$$2\int_{\partial B} \frac{\partial v}{\partial \mathbf{n}} \varphi d\sigma - \int_B \sqrt{D}\Delta\Big(\frac{v}{\sqrt{D}}\Big)\varphi d\mathbf{x} - 2\int_B \sqrt{D}\nabla\Big(\frac{v}{\sqrt{D}}\Big) \cdot \nabla\varphi d\mathbf{x} = \int_B h\varphi d\mathbf{x}. \tag{59}$$

Here, σ is the surface measure on ∂B and the boundary term above has to be understood as the $H^{-\frac{1}{2}}(\partial B)$–$H^{\frac{1}{2}}(\partial B)$ duality product since $\partial_{\mathbf{n}} v \in H^{-\frac{1}{2}}(\partial B)$ because $v \in H^1(B)$ and $\Delta v \in L^2(B)$ thanks to (57). Using the fact that φ is harmonic in B, that $D = D_0$ on ∂B, and using equation (59), we find:

$$\begin{aligned}
& \int_B \Big(\sqrt{D} - \sqrt{D_0}\Big)\Delta\Big(\frac{v\varphi}{\sqrt{D}}\Big) d\mathbf{x} \\
= \;& \int_B \Big(\sqrt{D} - \sqrt{D_0}\Big)\Delta\Big(\frac{v}{\sqrt{D}}\Big)\varphi\, d\mathbf{x} + 2\int_B \Big(\sqrt{D} - \sqrt{D_0}\Big)\nabla\Big(\frac{v}{\sqrt{D}}\Big) \cdot \nabla\varphi\, d\mathbf{x}, \\
= \;& -\int_B \sqrt{D_0}\,\Delta\Big(\frac{v}{\sqrt{D}}\Big)\varphi\, d\mathbf{x} - 2\int_B \sqrt{D_0}\,\nabla\Big(\frac{v}{\sqrt{D}}\Big) \cdot \nabla\varphi\, d\mathbf{x} + 2\int_{\partial B} \frac{\partial v}{\partial \mathbf{n}}\varphi d\sigma - \int_B h\varphi d\mathbf{x}, \\
= \;& -\int_B \sqrt{D_0}\,\nabla\Big(\frac{v}{\sqrt{D}}\Big) \cdot \nabla\varphi\, d\mathbf{x} + \int_{\partial B} \frac{\partial v}{\partial \mathbf{n}}\varphi d\sigma - \int_B h\varphi d\mathbf{x}.
\end{aligned}$$

To conclude, we just need to remark that, thanks to (58),

$$\int_{\partial B} \frac{\partial v}{\partial \mathbf{n}}\varphi d\sigma = \frac{1}{\sqrt{D_0}} \int_{\partial B} D \frac{\partial}{\partial \mathbf{n}}\Big(\frac{v}{\sqrt{D}}\Big)\varphi d\sigma = \frac{1}{\sqrt{D_0}} \int_B D\nabla\Big(\frac{v}{\sqrt{D}}\Big) \cdot \nabla\varphi d\mathbf{x}.$$

□

Coming back to the proof of proposition 3.4, we first verify that assumptions **(H-1)**, **(H-2)** and **(H-3)** are satisfied. Since $q_0 = 0$, **(H-1)** trivially holds because of the compatibility conditions (17). The same is true for **(H-3)**. Regarding **(H-2)**, we have to show that if

$$\varphi + T\varphi = 0, \qquad \forall \varphi \in L^2(B), \tag{60}$$

then $\varphi = 0$. To this aim, we first remark that T maps $L^2(B)$ to $H^1(B)$, so that every φ verifying (60) belongs to $H^1(B)$. Now, φ can be extended to $\mathbb{R}^d$ to a function $\varphi^* \in H^1_{\text{loc}}(\mathbb{R}^d)$ by the relation:

$$\begin{cases} \varphi^*(\mathbf{y}) = \varphi(\mathbf{y}), & \mathbf{y} \in B, \\ \varphi^*(\mathbf{y}) = -T\varphi(\mathbf{y}), & \text{otherwise.} \end{cases}$$

Moreover, when (60) holds, then so does the following in the distributional sense:

$$-\Delta\varphi^* + q_1\varphi^* = 0, \quad \text{in } \mathcal{D}'(\Omega'), \tag{61}$$

for any bounded set $\Omega' \subset \mathbb{R}^d$. Consider $\mathbf{y} \in \mathbb{R}^d \backslash \overline{B}$. Then $\Gamma(\mathbf{x}-\mathbf{y})$ is harmonic for $\mathbf{x} \in B$. We then apply lemma 4.1 with $h = 0$ to find, uniformly in $\mathbf{y}$:

$$\begin{aligned} \varphi^*(\mathbf{y}) &= -\int_B q_1(\mathbf{x})\,\varphi^*(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y}, \\ &= -\frac{1}{\sqrt{D_0}}\int_B D_1(\mathbf{x})\nabla\left(\frac{\varphi^*(\mathbf{x})}{\sqrt{D_0 + D_1(\mathbf{x})}}\right)\cdot\nabla\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}. \end{aligned}$$

We thus deduce from the above equation for $d \geq 2$ the following behavior at infinity:

$$\varphi^*(\mathbf{y}) = \mathcal{O}(|\mathbf{y}|^{1-d}), \qquad \nabla\varphi^*(\mathbf{y}) = \mathcal{O}(|\mathbf{y}|^{-d}). \tag{62}$$

Besides, equation (61) can be reformulated as:

$$\nabla\cdot(D_0 + D_1)\nabla\left(\frac{\varphi^*}{\sqrt{D_0 + D_1}}\right) = 0, \quad \text{in } \mathcal{D}'(\Omega'). \tag{63}$$

After multiplication by $\overline{\varphi^*}(D_0 + D_1)^{-\frac{1}{2}}$ in $H^1_{\text{loc}}(\mathbb{R}^d)$, and an integration on the sphere $B_R \supset\supset B$ of radius R and boundary S_R, we find:

$$\int_{B_R}(D_0 + D_1)\left|\nabla\left(\frac{\varphi^*}{\sqrt{D_0 + D_1}}\right)\right|^2 d\mathbf{x} - \int_{S_R}\frac{\partial\varphi^*}{\partial\mathbf{n}}\overline{\varphi^*}d\sigma = 0.$$

Letting $R \to \infty$ leads, together with (62), to $\varphi = 0$ so that assumption **(H-2)** is satisfied.

We now show the equivalence of the tensors. First, the term f^ε given in the expansion of theorem 3.3 is of order $\mathcal{O}(\varepsilon^4)$, which is not obvious at first sight. Consequently, $\varepsilon^{2(d-2)}f(\varepsilon)$ is of order $\mathcal{O}(\varepsilon^{2d})$ and can treated as a remainder in the expansion. To prove this, we apply lemma 4.1 to f^ε and need to estimate r^ε. Let us recall the equation verified by $r^\varepsilon \in H^1(B)$ given in proposition 3.2:

$$r^\varepsilon(\mathbf{y}) + Tr^\varepsilon(\mathbf{y}) = \int_B q_1(\mathbf{x})\,v^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x})R(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{x}_0 + \varepsilon\mathbf{y})d\mathbf{x}. \tag{64}$$

When $d = 2$, we use the fact that assumption **(H-3)** is satisfied since $q_0 = 0$ so that the term involving $\log\varepsilon$ in the equation of proposition 3.2 vanishes. Since v^ε verifies (57) with $h = 0$, and R verifies (21) with $q_0 = 0$ so that we have $\Delta_{\mathbf{y}} R(\mathbf{x}, \mathbf{y}) = \Delta_{\mathbf{y}} R(\mathbf{y}, \mathbf{x}) = 0$ since R is symmetric in its arguments and is thus harmonic, we apply lemma 4.1 to find:

$$\int_B q_1(\mathbf{x})\, v^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x}) R(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{x}_0 + \varepsilon\mathbf{y}) d\mathbf{x}$$
$$= \frac{\varepsilon}{\sqrt{D_0}} \int_B D_1(\mathbf{x}) \nabla \left(\frac{v^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{x})}{\sqrt{D_0 + D_1(\mathbf{x})}} \right) \cdot \nabla_{\mathbf{x}} R(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{x}_0 + \varepsilon\mathbf{y}) d\mathbf{x}.$$

Moreover, we show that

$$\left\| \nabla \left(\frac{v^\varepsilon(\mathbf{x}_0 + \varepsilon\cdot)}{\sqrt{D_0 + D_1}} \right) \right\|_{L^2(B)} = \mathcal{O}(\varepsilon), \tag{65}$$

so that the left hand side of (64) is of order $\mathcal{O}(\varepsilon^2)$. This is obtained by proving that the leading term in the above expression vanishes. That is to say, thanks to the decomposition given theorem 3.3, $v^\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{y}) = V(\mathbf{x}_0 + \varepsilon\mathbf{y}) + \sum_{|j|=0}^{d+1} \frac{\varepsilon^{|j|}}{j!} \partial^j V(\mathbf{x}_0)\phi_j(\mathbf{y}) + \varepsilon^{d-2} r^\varepsilon(\mathbf{y}) + \mathcal{O}(\varepsilon^{d+2})$, $\mathbf{y}$ *a.e.* in B, that

$$\nabla \Big(\frac{V(\mathbf{x}_0)(1 + \phi_0(\mathbf{x}))}{\sqrt{D_0 + D_1(\mathbf{x})}} \Big) = 0. \tag{66}$$

The argument is very similar to that in the verification of assumption **(H-2)** and so we just sketch the proof. Since ϕ_0 verifies $\phi_0 + T\phi_0 = -T1$, it can be extended to $\mathbb{R}^d$ to $\phi_0^* \in H^1_{\text{loc}}(\mathbb{R}^d)$ which admits the behavior at infinity given in (62). We also have, for any bounded set $\Omega' \subset \mathbb{R}^d$,

$$-\Delta(\phi_0^* + 1) + q_1(\phi_0^* + 1) = 0, \quad \text{in } \mathcal{D}'(\Omega'), \tag{67}$$

so that, still denoting by B_R the sphere of radius R,

$$\int_{B_R} (D_0 + D_1) \Big| \nabla \Big(\frac{\phi_0^* + 1}{\sqrt{D_0 + D_1}} \Big) \Big|^2 d\mathbf{x} - \int_{S_R} \frac{\partial \phi_0^*}{\partial \mathbf{n}} (\overline{\varphi_0^*} + 1) d\sigma = 0.$$

Sending R to infinity then gives the result thanks to the decay of $\nabla\phi_0^*$ at infinity. Owing to this result, the decomposition (24), the fact that ϕ_j and r^ε belong to $H^1(B)$, and r^ε is at least an $\mathcal{O}(\varepsilon)$ when $d = 2$ as mentioned in theorem (3.3), we get that (65) holds. Furthermore, using again the fact that R is harmonic, we verify from (64) that r^ε solves in the distribution sense:

$$-\Delta r^\varepsilon + q_1 r^\varepsilon = 0, \quad \text{in } \mathcal{D}'(B).$$

We cannot apply lemma 4.1 directly to (64) since for $(\mathbf{x}, \mathbf{y}) \in B \times B$, we have

$$-\Delta_{\mathbf{x}} \Gamma(\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad \text{in } \mathcal{D}'(B),$$

and Γ is not harmonic. Nevertheless, the lemma can easily be adapted to this special case so that, $\mathbf{y}$ *a.e.* in B, we have

$$\begin{aligned}\int_B q_1(\mathbf{x})\, r^\varepsilon(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x} &= \frac{1}{\sqrt{D_0}}\int_B D_1(\mathbf{x})\nabla\Big(\frac{r^\varepsilon(\mathbf{x})}{\sqrt{D_0+D_1(\mathbf{x})}}\Big)\cdot\nabla\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}\\ &\quad -\frac{\sqrt{D_0+D_1(\mathbf{y})}-\sqrt{D_0}}{\sqrt{D_0+D_1(\mathbf{y})}}r^\varepsilon(\mathbf{y}).\end{aligned}$$

Plugging the above expression into (64), we finally find the following equation for $r^\varepsilon \in H^1(B)$, $\mathbf{y}$ *a.e.* in B:

$$\begin{aligned}&\frac{r^\varepsilon(\mathbf{y})}{\sqrt{D_0+D_1(\mathbf{y})}}+\frac{1}{D_0}\int_B D_1(\mathbf{x})\nabla\Big(\frac{r^\varepsilon(\mathbf{x})}{\sqrt{D_0+D_1(\mathbf{x})}}\Big)\cdot\nabla\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}\\ &=\frac{\varepsilon}{D_0}\int_B D_1(\mathbf{x})\nabla\Big(\frac{v^\varepsilon(\mathbf{x})}{\sqrt{D_0+D_1(\mathbf{x})}}\Big)\cdot\nabla_{\mathbf{x}}R(\mathbf{x}_0+\varepsilon\mathbf{x},\mathbf{x}_0+\varepsilon\mathbf{y})d\mathbf{x}.\end{aligned}$$

Identifying the right hand side of the latter equation with $S^\varepsilon(\varepsilon\mathbf{x})$ and $(D_0+D_1)^{-\frac{1}{2}}r^\varepsilon$ with $r_1^\varepsilon(\mathbf{x})+S^\varepsilon(\varepsilon\mathbf{x})$ in the proof of theorem 2.2, we see that $(D_0+D_1)^{-\frac{1}{2}}r^\varepsilon$ and $r_1^\varepsilon(\mathbf{x})+S^\varepsilon(\varepsilon\mathbf{x})$ satisfy similar equations so that the same technique yield

$$\Big\|\nabla\Big(\frac{r^\varepsilon}{\sqrt{D_0+D_1}}\Big)\Big\|_{L^2(B)} \le C\varepsilon^2\Big\|\nabla\Big(\frac{v^\varepsilon}{\sqrt{D_0+D_1}}\Big)\Big\|_{L^2(B)}\|\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}R\|_{L^\infty(B_0\times B_0)}.$$

From proposition 3.1, $R\in\mathcal{C}^\infty(\Omega\times\Omega)$. Together with (65), this finally gives that:

$$\Big\|\nabla\Big(\frac{r^\varepsilon}{\sqrt{D_0+D_1}}\Big)\Big\|_{L^2(B)} = \mathcal{O}(\varepsilon^3). \tag{68}$$

We conclude by applying once again lemma 4.1 to obtain

$$\begin{aligned}\|f^\varepsilon\|_{L^2(\partial\Omega)} &= \Big\|\int_B q_1(\mathbf{x})r^\varepsilon(\mathbf{x})N(\mathbf{x}_0+\varepsilon\mathbf{x},\cdot)d\mathbf{x}\Big\|_{L^2(\partial\Omega)}\\ &= \frac{\varepsilon}{\sqrt{D_0}}\Big\|\int_B D_1(\mathbf{x})\nabla\Big(\frac{r^\varepsilon(\mathbf{x})}{\sqrt{D_0+D_1(\mathbf{x})}}\Big)\cdot\nabla_{\mathbf{x}}N(\mathbf{x}_0+\varepsilon\mathbf{x},\cdot)d\mathbf{x}\Big\|_{L^2(\partial\Omega)} = \mathcal{O}(\varepsilon^4),\end{aligned}$$

thanks to (68).

We now prove (27) and (28) so that the leading order in the expansion of theorem 2.2 is $\mathcal{O}(\varepsilon^d)$ as in the case of the diffusion equation. We remark that, for $\eta=0$,

$$\begin{aligned}\sum_{j=0}^{d+1}\frac{\varepsilon^{|j|}}{j!}\partial^j V(\mathbf{x}_0)\Big(Q_{0j}+Q_{0j}^0\Big) &= \sum_{j=0}^{d+1}\frac{\varepsilon^{|j|}}{j!}\partial^j V(\mathbf{x}_0)\int_B q_1(\mathbf{x})\Big(\phi_j^\eta(\mathbf{x})+\mathbf{x}^j\Big)d\mathbf{x},\\ &= \int_B q_1(\mathbf{x})\Big(\Psi^\varepsilon(\mathbf{x})+V(\mathbf{x}_0+\varepsilon\mathbf{x})-R_V^\varepsilon(\mathbf{x})\Big)d\mathbf{x},\end{aligned}$$

where Ψ^ε is given in the theorem and R_V^ε is the remainder of the Taylor expansion of $V(\mathbf{x}_0+\varepsilon\mathbf{x})$ at the order $d+2$ and is thus of order $\mathcal{O}(\varepsilon^{d+2})$. In order to apply lemma 4.1, we verify from (25) that $\Psi^\varepsilon\in H^1(B)$ solves,

$$-\Delta\Psi^\varepsilon+q_1\Psi^\varepsilon=-q_1\Big(V(\mathbf{x}_0+\varepsilon\mathbf{x})-R_V^\varepsilon(\mathbf{x})\Big)\quad\text{in }\mathcal{D}'(B).$$

Setting $v(\mathbf{x}) = \Psi^\varepsilon(\mathbf{x}) + V(\mathbf{x}_0 + \varepsilon\mathbf{x})$, $h = q_1 R_V^\varepsilon$ and $\varphi = 1$ in lemma 4.1 yields (27). Regarding (28), we write, for $\mathbf{y} \in \partial\Omega$,

$$\sum_{i=0}^{d+1} \frac{\varepsilon^{|i|}}{i!}\partial^i N(\mathbf{x}_0, \mathbf{y})\Big(Q_{i0} + Q_{i0}^0\Big) = \int_B q_1(\mathbf{x})\Big(1 + \phi_0(\mathbf{x})\Big)(N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y}) - R_N^\varepsilon(\mathbf{x}, \mathbf{y}))d\mathbf{x},$$

where R_N^ε is the remainder of the $d+2$ order Taylor expansion of $N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y})$ with respect to $\mathbf{x}$ and is thus of order $\mathcal{O}(\varepsilon^{d+2})$. Since $N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y})$ is harmonic when $\mathbf{x} \in B$ and $\mathbf{y} \in \partial\Omega$, we apply lemma 4.1 thanks to (67) to find:

$$\begin{aligned}\sum_{i=0}^{d+1} \frac{\varepsilon^{|i|}}{i!}\partial^i N(\mathbf{x}_0, \mathbf{y})\Big(Q_{i0} + Q_{i0}^0\Big) &= \frac{1}{\sqrt{D_0}}\int_B D_1 \nabla\Big(\frac{(1 + \phi_0(\mathbf{x}))}{\sqrt{D_0 + D_1(\mathbf{x})}}\Big) \cdot \nabla_{\mathbf{x}} N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y})d\mathbf{x} \\ &\quad + \mathcal{O}(\varepsilon^{d+2}) = \mathcal{O}(\varepsilon^{d+2}),\end{aligned}$$

since the above integral vanishes thanks to (66).

At this point of the proof, we have thus shown that v^ε satisfies, *a.e.* on $\partial\Omega$, that

$$v^\varepsilon(\mathbf{y})|_{\partial\Omega} = V(\mathbf{y})|_{\partial\Omega} - \sum_{|j|=1}^{d+1}\sum_{|i|=1}^{d+1} \frac{\varepsilon^{d-2+|i|+|j|}}{i!j!}\left(Q_{ij} + Q_{ij}^0\right)\partial^j V(\mathbf{x}_0)\,\partial^i N(\mathbf{x}_0, \mathbf{y})\Big|_{\partial\Omega} + \mathcal{O}(\varepsilon^{2d}).$$

Setting $v^\varepsilon(\mathbf{y}) := u^\varepsilon(\mathbf{y})\sqrt{D_0 + D_1(\frac{\mathbf{y} - \mathbf{x}_0}{\varepsilon})}$, $V := \sqrt{D_0}U$, we verify that u^ε and U are solutions to (1) and (4), respectively, with the boundary term g multiplied by $\sqrt{D_0}$. It thus remains to show that (29) and (30) hold to recover the asymptotic expansion for u^ε of theorem 2.2. Since $\mathbf{x}^j + \phi_j$ satisfies (57) when $|j| = 1$ and $\mathbf{x}^i$ is harmonic when $|i| = 1$, we have, for $|i| = |j| = 1$:

$$\begin{aligned} Q_{ij} + Q_{ij}^0 &= \int_B q_1(\mathbf{x})\Big(\mathbf{x}^j + \phi_j(\mathbf{x})\Big)\mathbf{x}^i d\mathbf{x}, \\ &= \frac{1}{\sqrt{D_0}}\int_B D_1 \nabla\Big(\frac{\mathbf{x}^j + \phi_j(\mathbf{x})}{\sqrt{D_0 + D_1(\mathbf{x})}}\Big) \cdot \nabla\mathbf{x}^i d\mathbf{x}. \end{aligned} \tag{69}$$

We introduce the following extension to ϕ_j on $\mathbb{R}^d$:

$$\begin{cases} \phi_j^*(\mathbf{y}) = \phi_j\Big(\mathbf{y}\Big), & \mathbf{y} \in B, \\ \phi_j^*(\mathbf{y}) = -T\phi_j(\mathbf{y}) - T\mathbf{x}^j, & \text{otherwise}, \end{cases}$$

which thus satisfies the conditions at infinity in (62). We recall that ϕ_{j0}^0, the function introduced in theorem 2.2 to define the polarization tensor M, is the unique weak solution in the space $H^1_{\text{loc}}(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d \backslash \overline{B})$ to the following system posed in $\mathbb{R}^d$:

$$\nabla \cdot \Big(D_0 + D_1(\mathbf{x})\Big)\nabla\phi_{j0}^0 = -\nabla \cdot \Big(D_1(\mathbf{x})\nabla\mathbf{x}^j\Big), \tag{70}$$

$$\phi_{j0}^0(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-d}) \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{71}$$

When $|j| = 1$, notice that ϕ_{j0}^0 is given by

$$\phi_{j0}^0(\mathbf{x}) = \Big(\frac{\sqrt{D_0 + D_1}}{\sqrt{D_0}} - 1\Big)\mathbf{x}^j + \frac{\sqrt{D_0 + D_1}}{\sqrt{D_0}}\phi_j^*(\mathbf{x}),$$

so that (29) is proved using (69). To prove (30), we need to sum over i and j to be able to use lemma 4.1 since $\mathbf{x}^i$ is not harmonic for $|i| \geq 2$ and $\mathbf{x}^j + \phi_j(\mathbf{x})$ satisfies (57) with a negligible left- hand side h of order $\mathcal{O}(\varepsilon^{d+2})$ only after summation. We thus write, using the same arguments as for the proof of (27) and (28), for $\mathbf{y} \in \partial\Omega$:

$$
\begin{aligned}
&\sum_{|j|=1}^{d+1} \sum_{|i|=1}^{d+1} \frac{\varepsilon^{|i|+|j|}}{i!j!} \Big(Q_{ij} + Q_{ij}^0 \Big) \partial^i N(\mathbf{x}_0, \mathbf{y}) \partial^j V(\mathbf{x}_0) \\
&= \sum_{|j|=1}^{d+1} \sum_{|i|=1}^{d+1} \frac{\varepsilon^{|i|+|j|}}{i!j!} \partial^i N(\mathbf{x}_0, \mathbf{y}) \partial^j V(\mathbf{x}_0) \int_B q_1(\mathbf{x}) \Big(\mathbf{x}^j + \phi_j(\mathbf{x}) \Big) \mathbf{x}^i d\mathbf{x}, \\
&= \int_B q_1(\mathbf{x}) \big(V(\mathbf{x}_0 + \varepsilon\mathbf{x}) - R_V^\varepsilon(\mathbf{x}) + \Psi^\varepsilon(\mathbf{x}) \big) (N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y}) - R_N^\varepsilon(\mathbf{x}, \mathbf{y})) d\mathbf{x} + \mathcal{O}(\varepsilon^{d+2}), \\
&= \frac{\varepsilon}{\sqrt{D_0}} \int_B D_1 \nabla \Big(\frac{V(\mathbf{x}_0 + \varepsilon\mathbf{x}) + \Psi^\varepsilon(\mathbf{x})}{\sqrt{D_0 + D_1(\mathbf{x})}} \Big) \cdot \nabla_{\mathbf{x}} N(\mathbf{x}_0 + \varepsilon\mathbf{x}, \mathbf{y}) d\mathbf{x} + \mathcal{O}(\varepsilon^{d+2}), \\
&= \sum_{|j|=1}^{d+1} \sum_{|i|=1}^{d+1} \frac{\varepsilon^{|i|+|j|}}{i!j!} \partial^i N(\mathbf{x}_0, \mathbf{y}) \partial^j V(\mathbf{x}_0) \int_B D_1(\mathbf{x}) \nabla \Big(\frac{\mathbf{x}^j + \phi_j(\mathbf{x})}{\sqrt{D_0 + D_1(\mathbf{x})}} \Big) \cdot \nabla \mathbf{x}^i d\mathbf{x} + \mathcal{O}(\varepsilon^{d+2}).
\end{aligned}
$$

It remains to relate the latter sum to M. For that, let f_j be defined as:

$$
f_j(\mathbf{y}) = \Big(\frac{\sqrt{D_0 + D_1}}{\sqrt{D_0}} - 1 \Big) \mathbf{x}^j + \frac{\sqrt{D_0 + D_1}}{\sqrt{D_0}} \phi_j^*(\mathbf{y}) - \phi_0^0(\mathbf{y}).
$$

Then f_j belongs to $H^1_{\text{loc}}(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d \backslash \overline{B})$ by construction and solves:

$$
\begin{aligned}
\nabla \cdot (D_0 + D_1(\mathbf{x})) \nabla f_j &= -\mathbb{1}_B(\mathbf{x}) \sqrt{D_0 + D_1(\mathbf{x})}\, \Delta \mathbf{x}^j, \quad \mathbf{x} \in \mathbb{R}^d, && (72) \\
f_j(\mathbf{y}) &= \mathcal{O}(|\mathbf{y}|^{1-d}) \qquad \text{as} \quad |\mathbf{y}| \to \infty. && (73)
\end{aligned}
$$

Here, $\mathbb{1}_B$ is the characteristic function of the set B and ϕ_j^* is the extension of ϕ_j to $\mathbb{R}^d$. Note that $f_j = 0$ when $|j| = 1$ so that we recover the preceding relationship between ϕ_j^* and ϕ_0^0. To conclude the proof, it suffices to show that an appropriate linear combination of the terms f_j is of order $\mathcal{O}(\varepsilon^{d+2})$. Let:

$$
T_V^\varepsilon(\mathbf{x}) := \sum_{|j|=1}^{d+1} \frac{\varepsilon^{|j|}}{j!} \partial^j V(\mathbf{x}_0) \Delta \mathbf{x}^j, \quad F^\varepsilon(\mathbf{x}) := \sum_{|j|=1}^{d+1} \frac{\varepsilon^{|j|}}{j!} \partial^j V(\mathbf{x}_0) f_j(\mathbf{x}),
$$

so that since $\Delta V(\mathbf{x}_0 + \varepsilon\mathbf{x}) = 0$, for all $\mathbf{x} \in B$, we have $T_V^\varepsilon(\mathbf{x}) = \mathcal{O}(\varepsilon^{d+2})$ uniformly in B and $F^\varepsilon \in H^1_{\text{loc}}(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d \backslash \overline{B})$ solves

$$
\begin{aligned}
\nabla \cdot (D_0 + D_1(\mathbf{x})) \nabla F^\varepsilon &= -\mathbb{1}_B(\mathbf{x}) \sqrt{D_0 + D_1(\mathbf{x})}\, T_V^\varepsilon, \quad \mathbf{x} \in \mathbb{R}^d, \\
F^\varepsilon(\mathbf{y}) &= \mathcal{O}(|\mathbf{y}|^{1-d}) \qquad \text{as} \quad |\mathbf{y}| \to \infty.
\end{aligned}
$$

The above equation is very similar to (45) at the end of proof of proposition 2.8 and a similar analysis yields

$$
\|\nabla F^\varepsilon\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon^{d+2}).
$$

We conclude the proof by calculating that

$$\sum_{|j|=1}^{d+1}\sum_{|i|=1}^{d+1}\frac{\varepsilon^{|i|+|j|}}{i!j!}\partial^i N(\mathbf{x}_0,\mathbf{y})\partial^j V(\mathbf{x}_0)\int_B D_1(\mathbf{x})\nabla\Big(\frac{\mathbf{x}^j+\phi_j(\mathbf{x})}{\sqrt{D_0+D_1(\mathbf{x})}}\Big)\cdot\nabla\mathbf{x}^i d\mathbf{x},$$
$$=\sum_{|j|=1}^{d+1}\sum_{|i|=1}^{d+1}\frac{\varepsilon^{|i|+|j|}}{i!j!}\partial^i N(\mathbf{x}_0,\mathbf{y})\partial^j V(\mathbf{x}_0)\frac{1}{\sqrt{D_0}}\int_B D_1(\mathbf{x})\nabla\Big(\mathbf{x}^j+\psi_j+f_j\Big)\cdot\nabla\mathbf{x}^i d\mathbf{x},$$
$$=\frac{1}{\sqrt{D_0}}\sum_{|j|=1}^{d+1}\sum_{|i|=1}^{d+1}\frac{\varepsilon^{|i|+|j|}}{i!j!}\partial^i N(\mathbf{x}_0,\mathbf{y})\partial^j V(\mathbf{x}_0)M_{ij}+\mathcal{O}(\varepsilon^{d+2}).$$

□

4.3 Appendix

This appendix states several lemmas that were needed in the preceding analyses.

Lemma 4.2 *Let $\mathbf{F}\in(L^2(\mathbb{R}^d))^d$ and $D_1\in W^{1,\infty}(\mathbb{R}^d)$ compactly supported in a bounded domain B, and D_0 a strictly positive constant. Assume moreover that $D_0+D_1(\mathbf{x})\geq C_0>0$ a.e. in $\mathbb{R}^d$. Then, the following problem (P):*

$$\begin{aligned}\nabla\cdot(D_0+D_1(\mathbf{x}))\nabla\phi &= \nabla\cdot\mathbf{F} \qquad in\ \mathcal{D}'(\mathbb{R}^d),\\ \phi(\mathbf{x}) &= \mathcal{O}(|\mathbf{x}|^{1-d}) \quad as \quad |\mathbf{x}|\to\infty,\end{aligned}$$

admits unique solution in $H^1_{loc}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$. Moreover, ϕ satisfies the estimates, for any bounded set $A\subset\mathbb{R}^d$,

$$\|\nabla\phi\|_{L^2(\mathbb{R}^d)}\leq C_0^{-1}\|\mathbf{F}\|_{(L^2(B))^d},\quad \|\phi\|_{L^2(A)}\leq C\|\mathbf{F}\|_{(L^2(B))^d}\left(1+\|D_1\|_{L^\infty(\mathbb{R}^d)}\right),\qquad (74)$$

and is the unique solution, a.e. on every bounded set of $\mathbb{R}^d$, to the integral equation

$$D_0\,\phi(\mathbf{y}) \;=\; -\int_B D_1(\mathbf{x})\nabla\phi(\mathbf{x})\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}+\int_B \mathbf{F}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{x}.\quad (75)$$

Proof. We show that (P) is equivalent to a problem posed on a bounded domain that can be solved with the Lax-Milgram lemma. To do so, let B_R be the sphere of radius R with $B\subset\subset B_R$ and denote by S_R its boundary. Consider the solution ϕ to (P) with the announced regularity. Since both D_1 and $\mathbf{F}$ are supported in B, the function ϕ is harmonic in $\mathbb{R}^d\backslash\overline{B}$ and in particular in $\mathbb{R}^d\backslash\overline{B_R}$. Denoting by $\Lambda: H^{\frac{1}{2}}(S_R)\to H^{-\frac{1}{2}}(S_R)$ the exterior Dirichlet-Neumann operator on the sphere S_R, we then have the standard relation

$$\frac{\partial\phi}{\partial\mathbf{n}}=\Lambda\,\phi|_{S_R},$$

where $\frac{\partial\phi}{\partial\mathbf{n}}$ is the outer normal derivative of ϕ on S_R and $\phi|_{S_R}$ its outer trace. Since ϕ is harmonic in $\mathbb{R}^d\backslash\overline{B}$ and is thus of class $\mathcal{C}^\infty$ on this set, $\frac{\partial\phi}{\partial\mathbf{n}}$ and $\phi|_{S_R}$ are continuous across S_R. Using this fact and integrating (P) against a test function $v\in\mathcal{C}^\infty(\overline{B_R})$, we find

$$\int_{B_R}(D_0+D_1)\nabla\phi\cdot\nabla v\,d\mathbf{x}-D_0\langle\Lambda\,\phi|_{S_R},v|_{S_R}\rangle=\int_B\mathbf{F}\cdot\nabla v\,d\mathbf{x},$$

where $\langle\cdot,\cdot\rangle$ denotes the $H^{\frac{1}{2}}(S_R)-H^{-\frac{1}{2}}(S_R)$ duality product. The restriction of ϕ to B_R is therefore a solution to the following variational problem (P2): Find $u \in H^1(B_R)$ such that

$$a(u,v) = l(v), \qquad \forall v \in H^1(B_R),$$

with obvious notation for the bilinear form a and the linear form l. Let us assume for the moment the existence of a unique solution u to (P2). That solution can be extended to a function u^* solution to (P). Let indeed u^* be defined as:

$$\begin{cases} u^* = u, & \text{in } B_R, \\ u^* = U, & \text{in } \mathbb{R}^d \backslash \overline{B_R}, \end{cases}$$

where U is the solution to

$$\begin{aligned} \Delta U &= 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \backslash \overline{B_R}), \\ U|_{S_R} &= u|_{S_R}, \qquad U(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty. \end{aligned}$$

By construction, the trace of u^* is continuous across S_R. Since U is harmonic in $\mathbb{R}^d \backslash \overline{B_R}$ and vanishes at infinity, it also verifies: $\frac{\partial U}{\partial \mathbf{n}} = \Lambda\, U|_{S_R} = \Lambda\, u|_{S_R}$. It then suffices to integrate the equation solved by U against a test function $v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and to consider (P2) to find

$$\int_{\mathbb{R}^d} (D_0 + D_1)\nabla u^* \cdot \nabla v \, d\mathbf{x} = \int_B \mathbf{F} \cdot \nabla v \, d\mathbf{x}, \qquad \forall v \in \mathcal{C}_0^\infty(\mathbb{R}^d),$$

so that u^* solves (P). The above equation also implies that u^* is harmonic in $\mathbb{R}^d \backslash \overline{B}$ and is thus of class $\mathcal{C}^\infty$ on this set. It remains to verify the behavior at the infinity, which stems from the fact that $\mathbf{F}$ has compact support in B_R. Setting $v = 1$ in (P2) yields $\langle \Lambda\, u|_{S_R}, 1\rangle = 0$. Getting back to U, since its trace and its normal derivative are known and given by $u|_{S_R}$ and $\Lambda\, u|_{S_R}$, respectively, it admits the following representation formula, for $\mathbf{x} \in \mathbb{R}^d \backslash \overline{B_R}$:

$$U(\mathbf{x}) = \int_{S_R} u|_{S_R}(\mathbf{y}) \frac{\partial \Gamma(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n_y}} d\sigma(\mathbf{y}) - \langle \Lambda\, u|_{S_R}, \Gamma(\mathbf{x} - \cdot)\rangle,$$

where Γ is the fundamental solution of the Laplacian in (6) and σ is the surface measure on S_R. We conclude by noticing that, as $|\mathbf{x}| \to \infty$:

$$\langle \Lambda\, u|_{S_R}, \Gamma(\mathbf{x} - \cdot)\rangle = \langle \Lambda\, u|_{S_R}, \Gamma(\mathbf{x} - \cdot) - \Gamma(\mathbf{x})\rangle = \mathcal{O}(|\mathbf{x}|^{1-d}).$$

It remains to show the existence of a unique solution to (P2). This is a consequence of the Lax-Milgram lemma: a and l are both continuous in $H^1(B_R)$ and the coercivity follows from the Poincaré-type inequality:

$$\|u\|_{L^2(B_R)} \leq C \left(\|\nabla u\|_{L^2(B_R)} + \|u\|_{L^2(S_R)} \right), \qquad \forall u \in H^1(B_R),$$

and the relation

$$C\|u\|^2_{L^2(S_R)} \leq -\langle \Lambda\, u|_{S_R}, u|_{S_R}\rangle, \qquad \forall u \in H^{\frac{1}{2}}(S_R).$$

We now prove the first estimate in (74). Let $v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that

$$\frac{1}{R^d}\|v\|_{L^1(S_R)} \to 0 \text{ as } R \to \infty. \tag{76}$$

Integrating (P) against v yields

$$\int_{B_R} (D_0 + D_1)\nabla\phi \cdot \nabla v \, d\mathbf{x} - D_0 \int_{S_R} \frac{\partial \phi}{\partial \mathbf{n}} v d\sigma = \int_B \mathbf{F} \cdot \nabla v \, d\mathbf{x}.$$

Since $\nabla\phi(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-d})$ as $\mathbf{x}$ tends to infinity, it belongs to $L^p(\mathbb{R}^d \backslash \overline{B_\rho})$ for some $p > 1$ and a ball of radius ρ with $B \subset\subset B_\rho$. The above equality also holds by density for all $v \in V_\rho$, the space of functions v such that $v \in H^1_{\text{loc}}(\mathbb{R}^d)$, v verifies (76) and $\nabla v \in L^{p'}(\mathbb{R}^d \backslash \overline{B_\rho})$ for $\frac{1}{p'} + \frac{1}{p} = 1$. Since $\frac{\partial\phi}{\partial\mathbf{n}} = \mathcal{O}(R^{-d})$, sending R to infinity implies, together with (76), that the boundary term goes to zero. On the other hand, the function $\nabla\phi \cdot \nabla v$ is integrable on $\mathbb{R}^d$ for $v \in V_\rho$, which allows us to use the Lebesgue dominated convergence theorem and obtain as $R \to \infty$:

$$\int_{\mathbb{R}^d} (D_0 + D_1)\nabla\phi \cdot \nabla v \, d\mathbf{x} = \int_B \mathbf{F} \cdot \nabla v \, d\mathbf{x}, \tag{77}$$

for all $v \in V_\rho$. Since $\phi \in V_\rho$ for any $d \geq 2$, we obtain the left estimate of (74).

Let us now consider the integral equation (75) and show that the solution to (P) verifies (75). For $\psi \in L^2(B_R)$, let $v(\mathbf{x}) = \int_{B_R} \Gamma(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})d\mathbf{y}$ for a given ball B_R. Since $\Gamma \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$, it follows from the Young inequality that $v \in H^1_{\text{loc}}(\mathbb{R}^d)$. Set $\mathbf{x} \in \mathbb{R}^d \backslash \overline{B_{R'}}$ with $B_R \subset\subset B_{R'}$. Then $\nabla\Gamma(\cdot - \mathbf{y}) \in L^p(\mathbb{R}^d \backslash \overline{B_{R'}})$ for $p > \frac{d}{d-1}$ and $\mathbf{y} \in B_R$. Such a function v also satisfies (76) for $d \geq 2$ since $\Gamma(\mathbf{x} - \mathbf{y})$ grows at worst as $\log|\mathbf{x}|$ for $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^d \backslash \overline{B_{R'}} \times B_R$. We can thus use v as a test function in (77). In order to use the Fubini theorem, we notice that the function $\nabla v(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})\mathbb{1}_{B_R}(\mathbf{y})$ belongs to $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ thanks to the Sobolev inequality [11] recalled in lemma 4.3 in the appendix since $\nabla v \in L^2(\mathbb{R}^d)$ and $\psi \in L^2(B_R)$. Indeed, since $R < \infty$, we bound the $L^q(\mathbb{R}^d)$ norm of $\psi(\mathbf{y})\mathbb{1}_{B_R}(\mathbf{y})$ by the $L^2(B_R)$ norm of ψ for $q = \frac{2d}{d+2} \leq 2$. Then choose $p = 2$ and $\lambda = d - 1$ in lemma 4.3.

The same conclusion holds for $\mathbf{F}(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})\mathbb{1}_{B_R}(\mathbf{y})$ so that we obtain from (77):

$$\begin{aligned} &D_0 \int_{B_R} \Big(\int_{\mathbb{R}^d} \nabla\phi(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x} - \mathbf{y}) \, d\mathbf{x} \Big) \psi(\mathbf{y}) d\mathbf{y} \\ &= -\int_{B_R} \Big(\int_{\mathbb{R}^d} D_1(\mathbf{x})\nabla\phi(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x} - \mathbf{y}) - \mathbf{F}(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x} - \mathbf{y}) \Big) \psi(\mathbf{y}) d\mathbf{y}. \end{aligned} \tag{78}$$

It thus only remains to show that $\int_{\mathbb{R}^d} \nabla\phi(\mathbf{x}) \cdot \nabla\Gamma(\mathbf{x} - \mathbf{y}) \, d\mathbf{x} = \phi(\mathbf{y})$ *a.e.* on B_R to conclude. To this aim, consider a sequence ϕ^n of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ functions such that $\nabla\phi^n \to \nabla\phi$ in $L^2(\mathbb{R}^d)$ and $\phi^n \to \phi$ in $L^2(A)$ for any bounded set A. Since $-\Delta_{\mathbf{x}}\Gamma(\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ in the distribution sense, we have, for any $\mathbf{y} \in \mathbb{R}^d$:

$$\lim_{\varepsilon \to 0} \int_{|\mathbf{x}-\mathbf{y}| > \varepsilon} \Gamma(\mathbf{x} - \mathbf{y}) \Delta\phi^n(\mathbf{x}) d\mathbf{x} = -\phi^n(\mathbf{y}).$$

The Lebesgue dominated convergence theorem yields consequently:

$$\lim_{\varepsilon\to 0}\int_{B_R}\left(\int_{|\mathbf{x}-\mathbf{y}|>\varepsilon}\Gamma(\mathbf{x}-\mathbf{y})\Delta\phi^n(\mathbf{x})d\mathbf{x}\right)\psi(\mathbf{y})d\mathbf{y} = -\int_{B_R}\phi^n(\mathbf{y})\psi(\mathbf{y})d\mathbf{y}.$$

An integration by parts then gives:

$$\begin{aligned}\int_{|\mathbf{x}-\mathbf{y}|>\varepsilon}\Gamma(\mathbf{x}-\mathbf{y})\Delta\phi^n(\mathbf{x})d\mathbf{x} &= \int_{|\mathbf{x}-\mathbf{y}|=\varepsilon}\frac{\partial\phi^n(\mathbf{x})}{\partial\mathbf{n}}\Gamma(\mathbf{x}-\mathbf{y})d\sigma(\mathbf{x})\\ &\quad -\int_{|\mathbf{x}-\mathbf{y}|>\varepsilon}\nabla\Gamma(\mathbf{x}-\mathbf{y})\cdot\nabla\phi^n(\mathbf{x})d\mathbf{x}.\end{aligned}$$

The boundary integral goes to zero with ε. For the other term, we remark that the function $\mathbb{1}_{|\mathbf{x}-\mathbf{y}|>\varepsilon}\mathbb{1}_{B_R}\nabla\Gamma(\mathbf{x}-\mathbf{y})\cdot\nabla\phi^n(\mathbf{x})\psi(\mathbf{y})$ converges *a.e.* in $\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{1}_{B_R}\nabla\Gamma(\mathbf{x}-\mathbf{y})\cdot\nabla\phi^n(\mathbf{x})\psi(\mathbf{y})$ which belongs to $L^1(\mathbb{R}^d\times\mathbb{R}^d)$ thanks to the Sobolev inequality. Applying again the Lebesgue dominated convergence theorem yields

$$\int_{B_R}\left(\int_{\mathbb{R}^d}\nabla\phi^n(\mathbf{x})\cdot\nabla\Gamma(\mathbf{x}-\mathbf{y})\,d\mathbf{x}\right)\psi(\mathbf{y})d\mathbf{y} = \int_{B_R}\phi^n(\mathbf{y})\psi(\mathbf{y})d\mathbf{y},$$

and it suffices to pass to the limit in the sequence ϕ^n to conclude. This proves that the solution to (P) satisfies (75). Conversely, considering a solution of (75) in $H^1_{\mathrm{loc}}(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d\backslash\overline{B})$, we verify using the same techniques as above that this solution also satisfies (P), which we know admits a unique solution. Therefore, the integral equation (75) also admits a unique solution. The second estimate of (74) follows from (75), the Young inequality and the first estimate of (74). □

Lemma 4.3 *Sobolev inequality (see e.g. [11]). Let $f\in L^p(\mathbb{R}^d)$, $g\in L^q(\mathbb{R}^d)$, $1<p,q<\infty$, $0<\lambda<d$ with the relation $\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{d}=2$. Then:*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(\mathbf{x})g(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\lambda}d\mathbf{x}d\mathbf{y}\le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

The following lemma, which is a standard variational formulation of the Fredholm alternative, is used several times in the paper.

Lemma 4.4 *Let H be a Hilbert space and let $a(\cdot,\cdot)$ be a bilinear form on a $H\times H$ such that $a(\cdot,\cdot)=a_0(\cdot,\cdot)+a_1(\cdot,\cdot)$, where both a_0 and a_1 are continuous in H and a_0 is H-coercive. Assume moreover, that for two sequences u_n and v_n weakly converging in H to u and v, we have*

$$a_1(u_n,v_n)\to a_1(u,v).$$

Then, if the following assertion is verified

$$(a(u,v)=0,\quad \forall v\in H)\Longrightarrow u=0,$$

for all f in H', there exists a unique $u\in H$ which satisfies

$$a(u,v)=\langle f,v\rangle,\qquad \forall v\in H.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the H'-H duality product. Moreover, u verifies the estimate, for some positive constant C:

$$\|u\|_H\le C\|f\|_{H'}.$$

Proof. We sketch a proof for completeness. Since a_0 is coercive, we know from the Lax-Milgram theory the existence of a bounded and boundedly invertible operator S on H such that $a_0(u, v) = (S^{-1}u, v)$, where $(\cdot, \cdot)$ is the inner product on H. By the Riesz representation theorem, we similarly know the existence of a bounded operator A_1 such that $a_1(u, v) = (A_1 u, v)$. The hypotheses on a_1 imply that A_1 is compact on H. Indeed, choose $u_n \rightharpoonup u$ and define $v_n = A_1 u_n - A_1 u$. We verify that $v_n \rightharpoonup 0$ and that $\|A_1 u_n - A_1 u\|^2 = (A_1 u_n, v_n) - (A_1 u, v_n)$ converges to 0 by the above hypothesis on a_1 so that A_1 maps weakly converging sequences to strongly converging sequences and is thus compact.

Now by the Riesz representation theorem, there exists $\tilde{f} \in H$ such that $\langle f, v \rangle = (\tilde{f}, v)$, for all $v \in H$, so that $a(u, v) = \langle f, v \rangle$ is equivalent to $(S^{-1} + A_1)u = \tilde{f}$ and thus equivalent to $(I + SA_1)u = S\tilde{f}$, which admits a unique solution if and only if -1 is not an eigenvalue of the compact operator SA_1, which is equivalent to the fact that $a(u, v) = 0$ for all $v \in H$ implies that $u = 0$. $\square$

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