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Optical tomography for small volume absorbing inclusions

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Abstract

We present the asymptotic expansion of the solution to a diffusion equation with a finite number of absorbing inclusions of small volume. We use the first few terms in this expansion measured at the domain boundary to reconstruct the absorption parameters of the inclusions and certain geometrical characteristics. We demonstrate theoretically and numerically that the number of inclusions, their location and their capacity can be reconstructed in a stable way even from moderately noisy data. The reconstruction of the absorption parameter, which is important in optical tomography to discriminate between healthy and unhealthy tissues, requires us however to have far less noisy data. Since the reconstruction of absorption maps from boundary measurements is an extremely ill posed problem, the method of asymptotic expansions of small volume inclusions provides a useful framework to decide which information can be reconstructed from boundary measurements with a given noise level.

1. Introduction

Optical tomography, which consists of reconstructing physical properties of human tissues from boundary measurements of near-infrared (NIR) photons, has received considerable recent attention. This non-invasive medical imaging technique has several benefits: it is harmless as NIR photons are low energy and it has very good discrimination properties between healthy and unhealthy tissues. Its drawback is a quite poor spatial resolution because of the high scattering rate of NIR photons. We refer to [4, 22, 32] for recent references on optical tomography.

Mathematically the image is obtained by solving an inverse problem, where the absorption and scattering coefficients of a radiative transfer equation, or the absorption and diffusion coefficients of a diffusion equation, are reconstructed from boundary measurements. Whereas radiative transfer equations [16, 29] offer a better model for photon propagation than diffusion equations, they are much more expensive computationally. We only consider the diffusion approximation in this paper and refer to [5, 10–12, 17, 19, 25, 30] for additional information on the theory and practice of inverse problems using radiative transfer equations.

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The reconstruction of the coefficients of a diffusion equation from boundary measurements has been thoroughly studied in the mathematical literature. We refer for instance to [15, 18, 23, 28]. It has also received a lot of attention in optical tomography [4, 20, 21, 24]. One striking aspect of these inverse problems is their extremely low stability [1, 14], which implies that very different coefficient profiles have very similar signatures at the domain boundary, with variations quite often below any experimentally achievable noise level.

This has naturally incited engineers and mathematicians to simplify the inverse problems by adding *a priori* information on the physical coefficients so as to improve the reconstruction stability. The most straightforward simplification consists of assuming that the coefficients have low contrast. The inverse problem can then be linearized [20, 21, 24]. Another technique consists of determining the asymptotic fluctuations at the boundary of a domain caused by inclusions with physical coefficients of arbitrary contrast with respect to a known background medium but of asymptotically small volume. This technique was pioneered in impedance tomography in [9] and further developed by Vogelius and co-authors [2, 3, 7, 8].

The first correction term that appears in this asymptotic expansion allows us to reconstruct the product of the inclusion volume with a nonlinear function of the background and inclusion conductivities [9]. In the case of absorbing inclusions considered in this paper, the first correction term takes the form of the product of the volume of the inclusion with the variation of absorption coefficient. In practice, however, one is interested in both the absorption fluctuation of the inclusion (which characterizes whether the tissue is healthy or not) and its volume. This paper's objective is to address what type of information may be obtained from asymptotic expansions of well separated inclusions of small volume. We restrict ourselves to the case of constant diffusion coefficients and varying absorption coefficients. The analysis of absorbing inclusions is mathematically simpler, but also quite relevant in practice since the absorption properties of healthy and unhealthy tissues are quite different at the NIR frequencies [27].

Our objectives are twofold. First, we show that the first two terms of the asymptotic expansion allow us to reconstruct both the volume and absorption fluctuations of small inclusions in specific cases. Second, we emphasize that no other type of information than the capacity of the inclusions (the product of their volume with their absorption fluctuation) can be obtained if the noise level is sufficiently large.

The paper is organized as follows. Section 2 applies the techniques of small volume expansions developed in [9] to the case of absorption fluctuations. Section 3 extends the expansion to higher orders and proposes a reconstruction technique for both the volume and absorption coefficients of ellipsoidal inclusions. Section 4 recalls the stability results of the general reconstruction problem and shows that the reconstruction of the location, absorption and volume of inclusions from one boundary measurement is stable. Section 5 proposes some reconstruction techniques and section 6 presents numerical experiments that quantify the theory. Section 7 offers some conclusions.

2. First-order effect of small volume inclusions

The full inverse problem in optical tomography based on the diffusion equation as a model for photon propagation consists of reconstructing the diffusion coefficient $D(\boldsymbol{x})$ and the absorption coefficient $\sigma_a(\boldsymbol{x})$ on a domain Ω from the measurements of $u(\boldsymbol{x};\omega)$ for $\boldsymbol{x} \in \partial\Omega$ and at least two frequencies ω for all possible prescribed currents at the boundary $g(\boldsymbol{x};\omega)$, where $u(\boldsymbol{x};\omega)$

solves the following equation:

$$\begin{aligned}
&\frac{\mathrm{i}\omega}{c}u(\boldsymbol{x};\omega) - \nabla\cdot D(\boldsymbol{x})\nabla u(\boldsymbol{x};\omega) + \sigma_a(\boldsymbol{x})u(\boldsymbol{x};\omega) = 0 \qquad \text{in } \Omega\\
&D(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x};\omega) = g(\boldsymbol{x};\omega) \qquad \text{on } \partial\Omega\\
&u(\boldsymbol{x};\omega) \qquad \text{measured on } \partial\Omega.
\end{aligned} \tag{1}$$

Here, c is the light speed and $\boldsymbol{\nu}(\boldsymbol{x})$ is the outward unit normal to Ω at $\boldsymbol{x} \in \partial\Omega$. At a given frequency ω, only one of the coefficients σ_a or D can be reconstructed [28]. With full measurements at two frequencies (two different values of ω), both coefficients can then be reconstructed [15, 24].

In this paper we simplify the above problem quite substantially and assume that the diffusion coefficient $D(\boldsymbol{x})$ is known. To simplify we set it to $D(\boldsymbol{x}) \equiv 1$. Whereas the reconstruction of $D(\boldsymbol{x})$ is important in optical tomography, the main discrimination between healthy and unhealthy tissues comes from differences in the absorption coefficients [27]. We also make the simplifying assumption that the frequency $\omega = 0$. Generalization of the theory to other frequencies is straightforward and will not be considered here.

Our objective is to apply the theory of asymptotic expansions developed in impedance tomography [9] to optical tomography. We thus assume that the absorption map is the sum of a background contribution, here a constant σ_0 to simplify, and a finite number of fluctuations of arbitrary size but localized in volumes of small diameter. The diffusion equation (1) with small absorbing inclusions then takes the form

$$\begin{aligned}
&-\Delta u_\varepsilon(\boldsymbol{x}) + \sigma_\varepsilon(\boldsymbol{x})u_\varepsilon(\boldsymbol{x}) = 0, \qquad \Omega\\
&\frac{\partial u_\varepsilon}{\partial \boldsymbol{\nu}} = g, \qquad \partial\Omega,
\end{aligned} \tag{2}$$

where the absorption map is given by

$$\sigma_\varepsilon(\boldsymbol{x}) = \sigma_0 + \sum_{m=1}^{M} \sigma_m \chi_{\boldsymbol{z}_m+\varepsilon B_m}(\boldsymbol{x}). \tag{3}$$

Here, εB_m is the shape of the mth normalized inclusion centred at $\boldsymbol{z}_m$, and $\chi_{\boldsymbol{z}_m+\varepsilon B_m}(\boldsymbol{x}) = 1$ if $\boldsymbol{x} - \boldsymbol{z}_m \in \varepsilon B_m$ and 0 otherwise. The inclusions are centred at $\mathbf{0}$ in the sense that

$$\int_{B_m} \boldsymbol{x}\,\mathrm{d}\boldsymbol{x} = \mathbf{0} \qquad \text{for all } m, \tag{4}$$

and are assumed to be at a distance greater than $d > 0$, independent of ε, from each other and from the boundary $\partial\Omega$.

The parameter ε measures the diameter of the inclusions. Our objective is to derive an asymptotic expansion for u_ε in powers of ε and address which type of information on the inclusions we may deduce from the first few terms of this expansion. This section concentrates on the leading term of the error $u_\varepsilon(\boldsymbol{x}) - U(\boldsymbol{x})$, where $U(\boldsymbol{x})$ is the solution of the homogeneous problem defined in (6) below.

We denote by $G(\boldsymbol{x};\boldsymbol{z})$ the Green function of the corresponding homogeneous problem

$$\begin{aligned}
&-\Delta G(\boldsymbol{x};\boldsymbol{z}) + \sigma_0 G(\boldsymbol{x};\boldsymbol{z}) = \delta(\boldsymbol{x}-\boldsymbol{z}), \qquad \Omega\\
&\frac{\partial G}{\partial \boldsymbol{\nu}}(\boldsymbol{x};\boldsymbol{z}) = 0, \qquad \partial\Omega,
\end{aligned} \tag{5}$$

and by $U(\boldsymbol{x})$ the homogeneous-domain solution of

$$\begin{aligned}
&-\Delta U(\boldsymbol{x}) + \sigma_0 U(\boldsymbol{x}) = 0, \qquad \Omega\\
&\frac{\partial U}{\partial \boldsymbol{\nu}}(\boldsymbol{x}) = g(\boldsymbol{x}), \qquad \partial\Omega.
\end{aligned} \tag{6}$$

As $\varepsilon \to 0$, the volume of the inclusions tends to zero and u_ε converges to U. To show this, we multiply (5) by u_ε and integrate by parts to obtain

$$u_\varepsilon(\boldsymbol{z}) = \int_{\partial\Omega} g(\boldsymbol{x})G(\boldsymbol{x};\boldsymbol{z})\,\mathrm{d}\sigma(\boldsymbol{x}) - \sum_{m=1}^{M}\int_{\boldsymbol{z}_m+\varepsilon B_m} \sigma_m G(\boldsymbol{x};\boldsymbol{z})u_\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Using the same procedure for $U(\boldsymbol{x})$, we obtain

$$u_\varepsilon(\boldsymbol{z}) = U(\boldsymbol{z}) - \sum_{m=1}^{M}\int_{\boldsymbol{z}_m+\varepsilon B_m} \sigma_m G(\boldsymbol{x};\boldsymbol{z})u_\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{7}$$

In three space dimensions, the Green function is given by

$$G(\boldsymbol{x};\boldsymbol{z}) = \frac{\mathrm{e}^{-\sqrt{\sigma_0}|\boldsymbol{z}-\boldsymbol{x}|}}{4\pi|\boldsymbol{z}-\boldsymbol{x}|} + w(\boldsymbol{x};\boldsymbol{z}), \tag{8}$$

where $w(\boldsymbol{x};\boldsymbol{z})$ is a smooth function (because it solves (6) with smooth boundary conditions) provided that $\partial\Omega$ is smooth. For $\boldsymbol{z}$ at a distance greater than $d > 0$ away from the inclusions $\boldsymbol{x}_m + \varepsilon B_m$, we then deduce from the L^∞ bound on u_ε (because g and $\partial\Omega$ are assumed to be sufficiently regular [13]) that

$$u_\varepsilon(\boldsymbol{z}) = U(\boldsymbol{z}) + \mathrm{O}(\varepsilon^3).$$

In the vicinity of the inclusions, we deduce from the relation

$$\int_{\boldsymbol{z}_m+\varepsilon B_m} G(\boldsymbol{x};\boldsymbol{z})\,\mathrm{d}\boldsymbol{x} = \mathrm{O}(\varepsilon^2), \qquad \boldsymbol{z}-\boldsymbol{z}_m \in \varepsilon B_m,$$

that $u_\varepsilon(\boldsymbol{z}) - U(\boldsymbol{z})$ is of order ε when $\boldsymbol{z}$ is sufficiently close to an inclusion. Notice that the above relation holds in any space dimension except $n = 2$, where ε^2 should be replaced by $\varepsilon^2|\ln\varepsilon|$. Although we concentrate here on the case of three space dimensions, the generalization to arbitrary space dimensions is straightforward. This also shows that the operator

$$K_\varepsilon u_\varepsilon(\boldsymbol{z}) = -\sum_{m=1}^{M}\int_{\boldsymbol{z}_m+\varepsilon B_m} \sigma_m G(\boldsymbol{x};\boldsymbol{z})u_\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{9}$$

is a bounded linear operator in $\mathcal{L}(L^\infty(\Omega))$ with a norm of order ε. This implies that for sufficiently small values of ε, we can write

$$u_\varepsilon(\boldsymbol{z}) = \sum_{k=0}^{\infty} K_\varepsilon^k U(\boldsymbol{z}). \tag{10}$$

The above series converges fast when ε is small. Notice however that the series does not converge as fast as ε^3, the volume of the inclusions, because of the singular behaviour of the Green function $G(\boldsymbol{x};\boldsymbol{z})$ when $\boldsymbol{x}$ is close to $\boldsymbol{z}$.

Let us now use that

$$\begin{aligned} u_\varepsilon(\boldsymbol{z}) = U(\boldsymbol{z}) &- \sum_{m=1}^{M}\int_{\boldsymbol{z}_m+\varepsilon B_m} \sigma_m G(\boldsymbol{x};\boldsymbol{z})U(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &+ \sum_{m=1}^{M}\sum_{n=1}^{M}\int_{\boldsymbol{z}_m+\varepsilon B_m}\int_{\boldsymbol{z}_n+\varepsilon B_n} \sigma_m\sigma_n G(\boldsymbol{x};\boldsymbol{z})G(\boldsymbol{y};\boldsymbol{x})u_\varepsilon(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{11}$$

For the same reasons as above, the last term is of order ε^5, and expanding smooth solutions $U(\boldsymbol{x})$ and $G(\boldsymbol{x};\boldsymbol{z})$ inside inclusions of diameter ε, we obtain that

$$u_\varepsilon(\boldsymbol{x}) = U(\boldsymbol{x}) - \sum_{m=1}^{M} G(\boldsymbol{z};\boldsymbol{z}_m)C_m^{(1)}U(\boldsymbol{z}_m) + \mathrm{O}(\varepsilon^5), \tag{12}$$

where the *capacity* $C_m^{(1)}$ is given by

$$C_m^{(1)} = \varepsilon^3 |B_m| \sigma_m. \tag{13}$$

The reason why we obtain a correction term of order ε^5 in (12) comes from the fact that (4) holds so that the terms of order ε^4, proportional to $\boldsymbol{x} \cdot \nabla U$ or $\boldsymbol{x} \cdot \nabla G$, vanish.

This result is the analogue of the formulae obtained in [9] in the case of small volume inclusions with different diffusive properties, with the notable difference that (13) is linear in σ_m.

The main drawback of the formula (13) is that it only gives information about the product of the volume of the inclusion $\varepsilon^3|B_m|$ and the contrast σ_m. Since one of the main objectives in optical imaging is precisely to obtain an accurate estimate of σ_m so as to know which type of inclusions we deal with, getting some information about $C_m^{(1)}$ is not sufficient in practice. As the same time, the above asymptotic analysis shows that if one is not capable of getting measurements of order ε^5, the next-order term in (11), one should *not* expect to reconstruct more that $C_m^{(1)}$. One should then rather look for additional information about the volume of the inclusions, for instance with another imaging technique, in order to determine their absorbing properties.

3. Contrast reconstruction

The goal of this section is precisely to push the asymptotic expansion of the preceding section to higher orders of accuracy and get a reconstruction formula for the absorption fluctuations σ_m. We use (7) one more time to obtain that

$$\begin{aligned} u_\varepsilon(\boldsymbol{z}) = U(\boldsymbol{z}) &- \sum_{m=1}^{M} \int_{\boldsymbol{z}_m+\varepsilon B_m} \sigma_m G(\boldsymbol{x};\boldsymbol{z}) U(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &+ \sum_{m=1}^{M}\sum_{n=1}^{M} \int_{\boldsymbol{z}_m+\varepsilon B_m}\int_{\boldsymbol{z}_n+\varepsilon B_n} \sigma_m\sigma_n G(\boldsymbol{x};\boldsymbol{z}) G(\boldsymbol{y};\boldsymbol{x}) U(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &- \sum_{m=1}^{M}\sum_{n=1}^{M}\sum_{l=1}^{M} \int_{\boldsymbol{z}_m+\varepsilon B_m}\int_{\boldsymbol{z}_n+\varepsilon B_n}\int_{\boldsymbol{z}_l+\varepsilon B_l} \sigma_m\sigma_n\sigma_l G(\boldsymbol{x};\boldsymbol{z}) G(\boldsymbol{y};\boldsymbol{x}) G(\boldsymbol{p};\boldsymbol{y}) \\ &\times u_\varepsilon(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Again, we obtain that the last term is of order ε^7. The cross-terms of the second term, corresponding to values $m \neq n$, contribute to a term of order ε^6. Using the same asymptotic expansions as above, we thus obtain that

$$\begin{aligned} u_\varepsilon(\boldsymbol{z}) = U(\boldsymbol{z}) &- \sum_{m=1}^{M} \int_{\boldsymbol{z}_m+\varepsilon B_m} \sigma_m G(\boldsymbol{x};\boldsymbol{z}) U(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &+ \sum_{m=1}^{M} G(\boldsymbol{z}_m;\boldsymbol{z}) \left[\int_{\boldsymbol{z}_m+\varepsilon B_m}\int_{\boldsymbol{z}_m+\varepsilon B_m} \sigma_m^2 G(\boldsymbol{y};\boldsymbol{x})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\right] U(\boldsymbol{z}_m) + \mathrm{O}(\varepsilon^6). \end{aligned} \tag{14}$$

Expanding again $G(\boldsymbol{x};\boldsymbol{z})U(\boldsymbol{x})$ about the points $\boldsymbol{z}_m$, we get that

$$u_\varepsilon(\boldsymbol{z}) = U(\boldsymbol{z}) - \sum_{m=1}^{M} C_m G(\boldsymbol{z}_m;\boldsymbol{z}) U(\boldsymbol{z}_m) - \varepsilon^5 \sigma_m \nabla \cdot \alpha^m \nabla (G(\boldsymbol{z}_m;\boldsymbol{z}) U(\boldsymbol{z}_m)) + \mathrm{O}(\varepsilon^6). \tag{15}$$

Here, we have defined

$$
\begin{aligned}
C_m &= \varepsilon^3 |B_m| \sigma_m - \sigma_m^2 \int_{z_m+\varepsilon B_m} \int_{z_m+\varepsilon B_m} G(\boldsymbol{y};\boldsymbol{x})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\
\alpha^m &= (\alpha^m_{kl})_{k,l}, \qquad \alpha^m_{kl} = \tfrac{1}{2}\int_{B_m} x_k x_l \,\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{16}
$$

Using different values of $G(\boldsymbol{z}_m;\boldsymbol{z})U(\boldsymbol{z}_m)$ and $\nabla_{z_m}\otimes\nabla_{z_m}G(\boldsymbol{z}_m;\boldsymbol{z})U(\boldsymbol{z}_m)$, we can have access to C_m and $\sigma_m\alpha^m$. Since the second contribution in C_m is of order ε^5, the ratio of $\sigma_m\alpha^m$ and C_m gives an approximation of order ε^2 of

$$\rho^m = |B_m|^{-1}\alpha^m.$$

It is still not possible to obtain the volume of the inclusion from the tensor ρ^m. To do so, we need to assume that the inclusion has a *specific geometric structure*. Assuming for instance that the inclusion B_m is an *ellipsoid*, we can then deduce from the tensor ρ^m the volume $|B_m|$ of the inclusions. This in turn gives us the constant σ_m since the capacity C_m is also known. For instance in the case of a ball B_m of radius R, so that the radius of the real inclusion is $R\varepsilon$, we obtain that

$$\rho^m = \frac{R^2}{10} I_3, \qquad \text{or} \qquad \mathrm{Tr}\,\rho^m = \frac{3R^2}{10}.$$

Once R is known, we can obtain the volume of the ball B_m. In the case of a cubic inclusion, we would obtain that $\mathrm{Tr}\,\rho^m = R^2/2$. The volumes of spherical and cubic inclusions would then be given by

$$V_S = \frac{4\pi}{3}\left(\frac{10}{3}\right)^{3/2} (\mathrm{Tr}\,\rho^m)^{3/2}, \qquad \text{and} \qquad V_C = 8\,2^{3/2}(\mathrm{Tr}\,\rho^m)^{3/2},$$

respectively. The ratio $V_S/V_C \approx 1.1266$. So using the ball as a model for a cube would overestimate the real volume of the inclusion by approximately 12%, hence underestimating the absorption coefficient by roughly the same amount. This is actually not that big and it shows a certain robustness of the reconstruction with respect to the geometrical assumptions.

Notice that we could push the asymptotic expansion to higher orders. However, in three space dimensions, the next term, of order ε^6, involves interactions between the inclusions. We are no longer in the regime where the total effect of the fluctuations is the sum of the contributions of each fluctuation. Moreover we shall see that the terms of order ε^5 turn out to be quite small already for not-so-small inclusions. It is therefore unclear whether the terms of smaller order may be used in practice to improve the reconstruction of well separated small inclusions.

4. Uniqueness and stability results

The reconstruction of arbitrary absorption coefficients $\sigma(\boldsymbol{x}) \in L^\infty(\Omega)$ ($\sigma_\varepsilon(\boldsymbol{x})$ in (2)) is known to be uniquely determined by boundary measurements. More precisely, $\sigma(\boldsymbol{x})$ is characterized by the Neumann-to-Dirichlet map Λ, which associates $\Lambda g = (u_\varepsilon)|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ to Neumann conditions $g \in H^{-1/2}(\partial\Omega)$. We refer to [14, 28] for such results.

The stability of the reconstruction is however extremely poor. A result expected to be optimal obtained in [1] shows that

$$\|\sigma(\boldsymbol{x}) - \sigma'(\boldsymbol{x})\|_{L^\infty(\Omega)} \leqslant C|\log\|\Lambda - \Lambda'\|_X|^{-\delta}. \tag{17}$$

Here Λ and Λ' are the maps corresponding to the diffusion equations with absorption coefficients σ and σ', respectively, $X = \mathcal{L}(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))$ and $\delta \in (0, 1)$ is a constant that only depends on the spatial dimension (>3). This formula implies that even very small errors of measurements may have quite large effects on the reconstructed absorption coefficient.

The role of the *a priori* constraint (3) on the absorption coefficient is precisely to improve the stability (17). Actually, in the asymptotic limit we consider, where only a finite number of coefficients are unknown, we can show that these coefficients can be reconstructed in a stable way from the boundary measurements of a single experiment (i.e. we do not need the full Neumann-to-Dirichlet map Λ). Our stability results are very similar to those presented in [9]. To simplify the presentation we assume that all inclusions are balls, so that the formula (15) can be recast as

$$u_\varepsilon(z) = U(z) - \sum_{m=1}^{M} C_m(G(z_m; z)U(z_m)) - D_m \Delta_{z_m}(G(z_m; z)U(z_m)) + \mathrm{O}(\varepsilon^6). \tag{18}$$

We denote by u_ε and u'_ε the solution of two problems with absorption coefficients σ_ε and σ'_ε of the form (3). Using (18), we obtain that

$$u_\varepsilon(z) - u'_\varepsilon(z) = F(z) + \mathrm{O}(\varepsilon^6),$$

with

$$\begin{aligned} F(z) = -\sum_{m=1}^{M}(C_m(G(z_m; z)U(z_m)) - C'_m(G(z'_m; z)U(z'_m))) \\ + \sum_{m=1}^{M}(D_m \Delta_{z_m}(G(z_m; z)U(z_m)) - D'_m \Delta_{z'_m}(G(z'_m; z)U(z'_m))). \end{aligned} \tag{19}$$

Here we use $M = \max(M, M')$ with a small abuse of notation; we will see shortly that $M = M'$. The function $F(z)$ satisfies the homogeneous equation $-\Delta F + \sigma_0 F = 0$ on Ω except at the points z_m and z'_m. Moreover, we have that $\frac{\partial F}{\partial \nu} = 0$ at $\partial\Omega$. If $F = 0$ on $\partial\Omega$, we deduce from the uniqueness of the Cauchy problem for the operator $-\Delta + \sigma_0$ that $F \equiv 0$ in Ω. As $\varepsilon \to 0$ and $u_\varepsilon - u'_\varepsilon \to 0$, we deduce that $F(z)$ becomes small not only at $\partial\Omega$ but also inside Ω (the continuation of F from $\partial\Omega$ to $\Omega\backslash\{z_m \cup z'_m\}$ is independent of ε). However, the functions $G(z_m; z)U(z_m)$ and $\Delta_{z_m}(G(z_m; z)U(z_m))$ clearly form an independent family. Each term must therefore be compensated by a term from the sum over the *prime* coefficients. We thus obtain that $M = M'$ and that

$$\begin{aligned} |C_m(G(z_m; z)U(z_m)) - C'_m(G(z'_m; z)U(z'_m))| + |D_m \Delta_{z_m}(G(z_m; z)U(z_m)) \\ - D'_m \Delta_{z'_m}(G(z'_m; z)U(z'_m))| \leqslant C(\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial\Omega)} + \mathrm{O}(\varepsilon^6)). \end{aligned}$$

The first term can be recast as

$$(C_m - C'_m)G(z_m; z)U(z_m) + C'_m(z_m - z'_m)\partial_{z_m}(G(\bar{z}_m; z)U(\bar{z}_m))$$

where $\bar{z}_m = \theta z_m + (1-\theta)z'_m$ for some $\theta \in (0, 1)$. Again these two functions are linearly independent so we deduce that

$$|C_m - C'_m| + |C'_m||z_m - z'_m| \leqslant C(\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial\Omega)} + \mathrm{O}(\varepsilon^6)).$$

We obtain the same result for $|D_m - D'_m|$.

Using (15) and (16), we then obtain, assuming that $\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial\Omega)} \approx \varepsilon^6$, that

$$\begin{aligned} &|B_m\sigma_m - B'_m\sigma'_m| + |z_m - z'_m| \leqslant C\varepsilon^{-3}\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial\Omega)} \\ &|\sigma_m\alpha^m - \sigma'_m\alpha^{m'}| \leqslant C\varepsilon^{-5}\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial\Omega)}. \end{aligned} \tag{20}$$

Assuming that the accuracy of the measured data is compatible with the expansion (15), i.e. that the u_ε is known on $\partial\Omega$ up to an error term of order ε^6, we can then reconstruct the location z_m

of the heterogeneities up to an error of order ε^3. The *product* of the volume of the inclusion and the absorption fluctuation is also known with the same accuracy. The tensor α^m, which allows us to discriminate between the volume and the absorption fluctuations of the inclusion, is known with a precision of order ε.

We thus obtain that the reconstruction of the location, capacity and tensor α^m from boundary measurements is a stable process. However, the above estimates show that there is little hope of obtaining the tensor α^m when the accuracy of the data is not better than ε^5. All that can be obtained from the data is then the *location* of the inclusions and their *capacity* $|B_m|\sigma_m$. The absorption fluctuation σ_m, of highest interest in practice, must be obtained by other means.

5. Techniques of reconstruction

The uniqueness and stability results of the last section did not provide any constructive formulae to recover the characteristics of the inclusions. We now describe explicit techniques to reconstruct the location, volume and absorption fluctuation of the inclusions based on the asymptotic expansions of the form (15).

Let w be a test function whose sole requirement is to solve the background equation

$$-\Delta w + \sigma_0 w = 0, \qquad \Omega.$$

We do not specify the boundary conditions. We then define the quantity

$$\Gamma_\varepsilon = \int_{\partial\Omega} \left(g w - \frac{\partial w}{\partial \boldsymbol{\nu}} u_\varepsilon \right) \mathrm{d}\sigma. \tag{21}$$

Since w is known and u_ε is measured on $\partial\Omega$, the quantity Γ_ε is known. Moreover, by integration by parts, we obtain that

$$\Gamma_\varepsilon = \sum_{m=1}^{M} \int_{z_m+\varepsilon B_m} \sigma_m w(\boldsymbol{x}) u_\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{22}$$

From (11), we deduce that

$$\begin{aligned}\Gamma_\varepsilon = {} & \sum_{m=1}^{M} \int_{z_m+\varepsilon B_m} \sigma_m w(\boldsymbol{x}) U(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ & - \sum_{m=1}^{M}\sum_{n=1}^{M} \int_{z_m+\varepsilon B_m}\int_{z_n+\varepsilon B_n} \sigma_m\sigma_n w(\boldsymbol{x}) G(\boldsymbol{x};\,\boldsymbol{y}) U(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} + \mathrm{O}(\varepsilon^8).\end{aligned} \tag{23}$$

In the right-hand side of the above expression, the first term is of order ε^3 and the second term of order ε^5. The non-diagonal contribution (when $m \neq n$) is of order ε^6 and can therefore be neglected. Following the same calculations as in the preceding section, we deduce that

$$\Gamma_\varepsilon = \sum_{m=1}^{M} C_m (wU)(\boldsymbol{z}_m) + \varepsilon^5 \sigma_m \boldsymbol{\nabla} \cdot \alpha^m \boldsymbol{\nabla}(wU)(\boldsymbol{z}_m) + \mathrm{O}(\varepsilon^6). \tag{24}$$

We now have complete choice of the background function U and the test function w so long as they are in the kernel of $-\Delta + \sigma_0$ on Ω. We can then use Calderòn's formula to obtain an explicit reconstruction formula [6]. Let $\boldsymbol{\eta} \in \mathbb{R}^3$ be given. We define by $\boldsymbol{\eta}^\perp$ a vector orthogonal to $\boldsymbol{\eta}$ and of the same length. Let us choose

$$U(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}(\boldsymbol{\eta}+\mathrm{i}\gamma\boldsymbol{\eta}^\perp)\cdot\boldsymbol{x}}, \qquad w(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}(\boldsymbol{\eta}-\mathrm{i}\gamma\boldsymbol{\eta}^\perp)\cdot\boldsymbol{x}},$$

where γ is such that

$$\gamma^2 = 1 + \frac{\sigma_0}{|\boldsymbol{\eta}|^2}. \tag{25}$$

The choice of the sign of γ does not matter and can be absorbed in the choice of $\boldsymbol{\eta}^{\perp}$. We verify that $U(\boldsymbol{x})$ and $w(\boldsymbol{x})$ are in the kernel of $-\Delta + \sigma_0$. Straightforward calculations show that

$$\Gamma_\varepsilon = \Gamma_\varepsilon(\boldsymbol{\eta}) = \sum_{m=1}^{M} (C_m - 4\varepsilon^5 \sigma_m \boldsymbol{\eta} \cdot \alpha^m \boldsymbol{\eta}) \mathrm{e}^{2\mathrm{i}\boldsymbol{\eta}\cdot \boldsymbol{z}_m} + \mathrm{O}(\varepsilon^6). \tag{26}$$

Upon taking the inverse Fourier transform of the preceding expression, we obtain delta functions and second-order differential of delta functions with support at the points $\boldsymbol{z}_m$. The more singular part can be used to recover the tensors α^m. The less singular part is used to reconstruct the coefficients C_m. So, at least theoretically, we obtain from Calderòn's formula that the boundary measurements are sufficient to reconstruct the location and the coefficients α^m and C_m of each inclusion. The Calderòn formula was used in [2] to reconstruct the diffusion coefficient of small volume inclusions from boundary measurements.

In practice however, the Calderòn formula is not very useful because a new experiment needs to be carried out for each value of $\boldsymbol{\eta}$. Another possible choice for U and w is

$$U(\boldsymbol{x}) = \mathrm{e}^{-\sqrt{\sigma_0}\boldsymbol{x}\cdot\hat{\boldsymbol{k}}}, \qquad w(\boldsymbol{x}) = \mathrm{e}^{-\sqrt{\sigma_0}\boldsymbol{x}\cdot\hat{\boldsymbol{p}}}, \tag{27}$$

for any unit vectors $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{p}}$. We then obtain that

$$\Gamma_\varepsilon(\hat{\boldsymbol{k}}, \hat{\boldsymbol{p}}) = \sum_{m=1}^{M} (C_m + \varepsilon^5 \sigma_m \sigma_0 (\hat{\boldsymbol{k}} + \hat{\boldsymbol{p}}) \cdot \alpha^m (\hat{\boldsymbol{k}} + \hat{\boldsymbol{p}})) \mathrm{e}^{-\sqrt{\sigma_0}\boldsymbol{z}_m\cdot(\hat{\boldsymbol{k}}+\hat{\boldsymbol{p}})}.$$

The advantage of such a method is that we can have as many measurements as we choose values of $\hat{\boldsymbol{k}}$ for each physical experiment $\hat{\boldsymbol{p}}$. The reconstruction of the locations $\boldsymbol{z}_m$ and the parameters C_m and α^m cannot be shown rigorously as with Calderòn's technique because we can no longer take inverse Fourier transforms. The numerical simulations of the next section are based on a similar reconstruction algorithm that requires us to solve only a few forward problems.

6. Numerical simulations

6.1. Minimization problem

The setting of the numerical simulations is the following. The domain Ω is the unit cube $[0, 1]^3$ and the inclusions are balls of radius R_m and absorption fluctuation σ_m. We consider the following approximation of the inverse problem. The 'exact' model is given by

$$u_\varepsilon(\boldsymbol{z}) = U(\boldsymbol{z}) - \sum_{m=1}^{M} \int_{\boldsymbol{z}_m+\varepsilon B_m} \sigma_m G(\boldsymbol{x}; \boldsymbol{z}) U(\boldsymbol{z}), \tag{28}$$

where

$$G(\boldsymbol{x}; \boldsymbol{z}) = \frac{\mathrm{e}^{-\sqrt{\sigma_0}|\boldsymbol{x}-\boldsymbol{z}|}}{|\boldsymbol{x}-\boldsymbol{z}|}, \tag{29}$$

which is the Green function of the full domain $\mathbb{R}^3$ instead of the cube Ω. Replacing $G(\boldsymbol{x}; \boldsymbol{z})$ by the real Green function on Ω would make the numerical analysis more complicated because we would no longer have the analytic expression (29) but would not modify the complexity of the inverse problem. What is missing in (28) are terms of order ε^6 and higher. We shall see later on that adding the only contribution of order ε^6, namely the extra-diagonal terms on the second line in (11), to the measured data does not substantially change the reconstruction.

The asymptotic approximation our reconstruction is based on then takes the form

$$u_\varepsilon(\boldsymbol{z}) = U(\boldsymbol{z}) - \sum_{m=1}^{M} C_m G(\boldsymbol{z}_m; \boldsymbol{z}) U(\boldsymbol{z}_m) - D_m \Delta_{\boldsymbol{z}_m} (G(\boldsymbol{z}_m; \boldsymbol{z}) U(\boldsymbol{z}_m)) + \mathrm{O}(\varepsilon^6), \tag{30}$$

where

$$C_m = \varepsilon^3 \sigma_m \frac{4\pi R_m^3}{3}, \qquad D_m = \varepsilon^5 \sigma_m \frac{2\pi R_m^5}{15}. \tag{31}$$

The homogeneous solution $U(\boldsymbol{x})$ is chosen to be of the form

$$U(\boldsymbol{x}) = \exp\big(-\sqrt{\sigma_0}\boldsymbol{k}\cdot\boldsymbol{x}\big), \tag{32}$$

for some unitary vector $\boldsymbol{k} \in S^2$. In practice this requires that we are able to impose a current of the form

$$\frac{\partial U}{\partial \boldsymbol{\nu}}(\boldsymbol{x}) = -\sqrt{\sigma_0}\boldsymbol{k}\cdot\boldsymbol{\nu}\exp\big(-\sqrt{\sigma_0}\boldsymbol{k}\cdot\boldsymbol{x}\big)$$

at the boundary of the domain Ω. Measurements are made on three faces of the cube, the forward face ($x = 1$), the East face ($y = 1$), and the North face ($z = 1$). Each face has 5×5 measurements at the points of an equidistant two-dimensional grid. There is therefore a total of $N_m = 75$ measurements per experiment (choice of $\boldsymbol{k}$). The number of 'physical' experiments N_e has been chosen to be three, and corresponds to values of $\boldsymbol{k}$ in (32) of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The 'data' collected from (28) are called d_n^p for $1 \leqslant p \leqslant N_e$ and $1 \leqslant n \leqslant N_m$. This set of data is equivalent in complexity to choosing N_m independent test functions w in (27). In both cases, we do not have any proof that the reconstruction is possible from such measurements.

The location and characteristics of the inclusions are obtained by least-squares minimization:

$$\min_{\boldsymbol{z}_m, \sigma_m, R_m} \sum_{p=1}^{N_e} \sum_{n=1}^{N_m} (d_n^p - u_n^p)^2, \tag{33}$$

where u_n^p is obtained from (30). The minimization is performed by a Newton method [26, 31]. The gradient of the cost function (33) is calculated analytically since the functions U and G in (30) are known exactly. We do not have any theoretical convergence result for the Newton method. We found numerically that the Newton algorithm was much more stable with three experiments ($N_e = 3$) than with only one experiment ($N_e = 1$). Starting from a relatively poor initial guess, the algorithm often converges to a good reconstruction with three experiments and often diverges with only one experiment. In the absence of noise, the reconstructions with one and three experiments are comparable when the initial guess is sufficiently close to the exact solution. In the numerical experiments presented below, the initial guess was chosen with an error on the location of order 0.1 and with vanishing initial coefficients C_m and D_m.

6.2. Reconstruction of similar inclusions

We now present two numerical experiments with three inclusions. In the first experiment, all inclusions have very similar sizes. The background absorption is $\sigma_0 = 1$ and the characteristics of the inclusions are given in table 1. Using the minimization algorithm presented above with exact data given by (28) and approximated data given by (30), we obtain the reconstruction presented in table 2. The reconstruction is almost perfect and relatively stable. The initial conditions for the reconstruction were $\boldsymbol{z}_1 = (0.4, 0.3, 0.4)$, $\boldsymbol{z}_2 = (0.4, 0.6, 0.6)$ and $\boldsymbol{z}_3 = (0.6, 0.4, 0.6)$, and the constants C_m and D_m were set to zero.

The recorded values d_n^p range between 1.5×10^{-4} and 7.5×10^{-4}. We now consider noisy data with an additive noise term uniformly distributed on $(-\varepsilon, \varepsilon)$. For $\varepsilon = 10^{-7}$, which corresponds to a noise of at most 0.1%, we show in table 3 the reconstruction for a 'typical' realization. Whereas the locations $\boldsymbol{z}_m$ and coefficients C_m are still perfectly reconstructed,

Table 1. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3000, 0.2000, 0.3000)	0.1000	0.5000	2.0944	2.09
2	(0.4000, 0.7000, 0.7000)	0.0900	1.0000	3.0536	2.47
3	(0.7500, 0.2500, 0.7500)	0.1000	0.7500	3.1416	3.14

Table 2. Characteristics of the reconstructed absorbing inclusions without noise. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3004, 0.2002, 0.3004)	0.1011	0.4844	2.0951	2.14
2	(0.4000, 0.6999, 0.7000)	0.0902	0.9932	3.0539	2.49
3	(0.7501, 0.2500, 0.7501)	0.0998	0.7537	3.1403	3.13

Table 3. Characteristics of the reconstructed absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$. The noise level is $\varepsilon = 10^{-7}$, which corresponds to up to 0.1% of the exact data $u_\varepsilon - U$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3002, 0.2002, #0.3004)	0.0966	0.5547	2.0955	1.96
2	(0.4000, 0.7000, 0.7000)	0.0918	0.9414	3.0538	2.58
3	(0.7501, 0.2500, 0.7501)	0.0999	0.7523	3.1401	3.13

the coefficients D_m are up to 10% inaccurate, which propagates in the reconstruction of σ_m and R_m.

If we now increase the noise level to $\varepsilon = 10^{-6}$, which corresponds to up to 1% of the measured data, we show in table 4 the reconstruction for a 'typical' realization. Whereas the reconstruction of the locations z_m and capacities C_m is still roughly 1% accurate, the reconstruction of the coefficients D_m is no longer possible. We cannot reconstruct both the absorption and the radius of the inclusions with data with roughly 1% of noise. If we further increase the noise level to $\varepsilon = 10^{-5}$, which corresponds to data with up to 10% of noise, we also obtain a reconstruction of the location and capacity of the inclusions with an error of roughly 10% as can be observed in table 5. This is consistent with the stability estimates given in the previous sections with $\varepsilon \approx 0.1$.

The above results are based on using (28) as a model for the exact measurements. Two approximations have been made in this model. First the capacity (16) has been replaced by (13). Accounting for this modification in the capacity is straightforward since the difference between (16) and (13) is an explicit function of the radius and the absorption parameters of the inclusions. It does not substantially modify the reconstruction algorithm. The second assumption is the removal of the higher-order terms in (11). The terms that have been neglected are asymptotically of order $O(\varepsilon^6)$ and the leading term is given by

$$\sum_{m=1}^{M}\sum_{n=1}^{M} C_m C_n G(z_m; z) G(z_n; z_m) U(z_n), \tag{34}$$

with an accuracy of order $O(\varepsilon^7)$. This is the leading term in the manifestation that the total influence of the inclusions on the boundary measurements is not the sum of their individual influences. The maximal value taken by this nonlinear term is roughly 2.7×10^{-7}. We have added this term to the solution given by (28). In the absence of noise, the results of the

Table 4. Characteristics of the reconstructed absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$. The noise level is $\varepsilon = 10^{-6}$, which corresponds to up to 1% of the exact data $u_\varepsilon - U$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.2955, 0.1946, 0.2941)	0.0175	92.468	2.0843	0.06
2	(0.4002, 0.6998, 0.6996)	0.1007	0.7171	3.0631	3.10
3	(0.7496, 0.2501, 0.7497)	0.1026	0.6957	3.1513	3.32

Table 5. Characteristics of the reconstructed absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$. The noise level is $\varepsilon = 10^{-5}$, which corresponds to up to 10% of the exact data $u_\varepsilon - U$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.2852, 0.1732, 0.2685)	0.1351	0.2111	2.1824	3.99
2	(0.4013, 0.6998, 0.7012)	0.1584	0.1833	3.0537	7.66
3	(0.7489, 0.2532, 0.7522)	0.1484	0.2291	3.1370	6.91

Table 6. Characteristics of the reconstructed absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$. An approximation (up to terms of order ε^8) of the extra-diagonal terms on the second line in (11) has been added to the data given by (28).

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3003, 0.2001, 0.3003)	0.1027	0.4614	2.0942	2.21
2	(0.4000, 0.7000, 0.7000)	0.0907	0.9777	3.0527	2.51
3	(0.7501, 0.2500, 0.7501)	0.1002	0.7458	3.1391	3.15

reconstruction are given in table 6. The increase in the error obtained by adding these nonlinear terms is relatively small for inclusions of radius of order 0.1, which are already quite large for practical purposes. A more troublesome difficulty in using the asymptotic model (30) to reconstruct both the absorption and geometric properties of the inclusions is to assume that we 'know' something about the geometry of the inclusions (for instance that they are balls) and that we have sufficiently accurate data. We have seen that misinterpreting cubic inclusions as spherical inclusions only has the effect of underestimating the absorption coefficient by a little more than 10%. At the same time, a noise level of 1% is already too large in this numerical setting to allow for the reconstruction of the coefficients D_m, although the location and the capacity of the inclusions are still very well reconstructed.

6.3. Reconstruction of different size inclusions

An additional difficulty arises when the inclusions have quite different sizes. The characteristics of the inclusions of the second numerical experiment are given in table 7. The background absorption is $\sigma_0 = 1$ and we still aim at reconstructing three spherical inclusions. The main difference from the previous experiment is that the second inclusion is now much bigger than the two other ones, with coefficients C_2 and D_2 one order of magnitude larger than C_i and D_i for $i = 1$ and 3.

The results of the numerical experiments are the following. The reconstruction obtained without the nonlinear terms and without noise is given in table 8. These results show that the large inclusion is perfectly reconstructed, but not the other ones. The reason is that higher-order terms in the asymptotic expansion for u_ε coming from the second inclusion are no longer

Table 7. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3000, 0.2000, 0.3000)	0.1000	0.5000	2.0944	2.09
2	(0.4000, 0.7000, 0.7000)	0.1500	1.0000	14.137	31.81
3	(0.7500, 0.2500, 0.7500)	0.0800	0.7500	1.6085	1.03

Table 8. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3009, 0.1996, 0.3004)	0.1034	0.4511	2.0891	2.23
2	(0.4000, 0.6999, 0.6999)	0.1498	1.0047	14.140	31.7
3	(0.7499, 0.2501, 0.7499)	0.0755	0.8934	1.6094	0.92

Table 9. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3005, 0.1996, 0.3004)	0.1074	0.4019	2.0871	2.41
2	(0.4000, 0.7000, 0.6999)	0.1499	1.0014	14.135	31.8
3	(0.7499, 0.2501, 0.7499)	0.0775	0.8235	1.6077	0.97

Table 10. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$. The noise level is $\varepsilon = 3 \times 10^{-6}$, which corresponds to up to 1% of the exact data $u_\varepsilon - U$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3035, 0.1959, 0.3003)	0.1074	0.4019	2.0909	0.24
2	(0.3999, 0.6999, 0.6999)	0.1499	1.0014	14.142	32.0
3	(0.7512, 0.2496, 0.7510)	0.0775	0.8235	1.5976	1.47

negligible. Whereas they do not affect the reconstruction of the second inclusion, they have an impact on the reconstruction of the other ones.

When the nonlinear term (34) is added to the measured data, the reconstruction is given in table 9. The error term is roughly comparable to the case of table 8 so that we can again neglect this term in the reconstruction.

The results with roughly 1% of noise (the noise level is 1% of the smallest measurement at the boundary) are shown in table 10. Here we see that 1% of noise still allows us to reconstruct the large inclusion number 2 quite accurately. However the error on the coefficient D_m of the other inclusions is now of order O(1). The capacity of these inclusions as well as their location is still well reconstructed. Yet the data are no longer sufficiently accurate to image both the volume and the absorption of these smaller inclusions.

6.4. Reconstructing the number of inclusions

In the preceding sections the number of reconstructed inclusions was equal to the physical number of inclusions. We deal here with the case where the number of physical inclusions is smaller. Let us consider the case of one inclusion as described in table 11. The locations

Table 11. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3000, 0.2000, 0.3000)	0.0000	0.5000	0.0000	0.00
2	(0.4000, 0.7000, 0.7000)	0.1500	1.0000	14.137	31.81
3	(0.7500, 0.2500, 0.7500)	0.0000	0.7500	0.0000	0.00

Table 12. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$. The noise level is roughly 1% of the exact data.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.3245, 0.1974, 0.3112)	1.5521	0.0000	0.0015	0.35
2	(0.4000, 0.7000, 0.7000)	0.1488	1.0239	14.1368	31.31
3	(0.8512, 0.4847, 0.8298)	0.8327	0.0000	−0.0004	0.03

Table 13. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$. The noise level is roughly 3% of the exact data.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.5480, 0.9000, 0.9000)	1.0910	0.0000	−0.0025	0.30
2	(0.4015, 0.7015, 0.7014)	0.1522	0.9524	14.124	32.56
3	(0.8970, 0.3094, 0.7928)	0.9977	0.000	0.0414	4.12

of the first and third inclusions are used as initializations in the optimization problem. They obviously have no physical meaning.

The results of the reconstruction are the following. In the absence of noise, the location of inclusions 1 and 3 fluctuates significantly from one iteration to the next since their capacity is negligible. Their location was actually constrained to staying within the cube $[0.1, 0.9]^3$ during the Newton iterations. The reconstruction of the second inclusion is quite good; see table 12. The reconstructed capacity gives us an estimator of the number of inclusions. The capacity of the first and third inclusions is here roughly 10^4 times smaller than that of the second inclusion.

In the presence of roughly 3% of noise, the reconstruction of the second inclusion and the estimate of the number of inclusions is still acceptable since the capacity of inclusions 1 and 3 is at least 300 times smaller than that of the second inclusion; see table 13.

When almost 15% is added to the measured data, the reconstruction becomes more delicate. The numerical results are presented in table 14. The location and capacity of the second inclusions are relatively well reconstructed knowing that the noise level is 15%. However the estimation of the number of particles is much less accurate since the capacity of the first inclusion now is only one-third of that of the second inclusion. The algorithm has in effect created two particles to compensate for the noise. Notice that at this noise level only the capacity is well reconstructed, not the individual radius and absorption coefficients.

Similar results can be obtained for two physical inclusions. The reconstruction is quite satisfactory in the presence of moderate noise. The reconstructed capacities give a good estimate of the number of physical inclusions. When the noise level reaches a large fraction of the exact data, even the number of physical inclusions becomes more difficult to assess. But then the separate reconstruction of the radius and absorption of the inclusions is also no longer possible.

Table 14. Characteristics of absorbing inclusions. The physical absorption of the inclusion m is given by $\sigma_0 + \sigma_m$. The noise level is roughly 15% of the exact data.

Inclusion, m	z_m	R_m	σ_m	$10^3 C_m$	$10^6 D_m$
1	(0.5571, 0.7455, 0.7643)	0.4409	−0.0143	−5.1440	−100.00
2	(0.4341, 0.7263, 0.7128)	0.2319	0.3562	18.5994	100.00
3	(0.8306, 0.3765, 0.6591)	1.2022	0.0001	0.5550	−80.21

7. Conclusions

We have presented the asymptotic expansion of the solution to a diffusion equation with absorbing inclusions of small volume. We have shown that the first term in the expansion, after the solution to the homogeneous problem is subtracted, allows us to reconstruct the number of isolated inclusions and their capacity. The capacity of an inclusion is defined as the product of its volume with the variation of absorption between the inclusion and the known background. We have shown that the next-order term of this asymptotic expansion allows us to gain additional information about the inclusions. Assuming that the inclusion is an ellipsoid, we are then able to reconstruct both the absorption of the inclusion and the geometrical parameters of the ellipsoid. This is important in practice since healthy and unhealthy tissues have different absorption properties.

We have then presented some numerical simulations showing that the reconstruction of the parameters of spherical inclusions from boundary measurements is quite stable. However, we have seen that relatively small amounts of noise in the data prevent us from reconstructing both the absorption parameter and the radius of the inclusions, whereas the capacity and the location (centre of mass) of the inclusions are still very well reconstructed. When the number of inclusions is unknown, we also obtain a good estimate of that number when the noise level is sufficiently small.

Asymptotic expansions are of somewhat limited applicability since they assume that most of the domain (the background) is known. However, they give quite satisfactory results for inclusions of relatively large size for practical purposes and provide a good framework to understand what can and what cannot possibly be reconstructed from boundary measurements with a given noise level. In a sense, they provide an analytical tool to face the daunting stability estimate provided by (17).

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