Single scattering estimates for the scintillation function of waves in random media

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Abstract

The energy density of high frequency waves propagating in highly oscillatory random media is well approximated by solutions of deterministic kinetic models. The scintillation function determines the statistical instability of the kinetic solution. This paper analyzes the single scattering term in the scintillation function. This is the term of the scintillation function that is linear in the power spectrum of the random fluctuations. We show that the structure of the scintillation function is already quite complicated in this simplified setting. It strongly depends on the singularity of the initial conditions for the wave field and on the correlation properties of the random medium. We obtain limiting expressions for the scintillation function as the correlation length of the random medium tends to zero.

1 Introduction

The energy density of high frequency waves propagating in highly oscillatory random media is well-known to be accurately approximated by the solution to kinetic (radiative transfer) models; see e.g. [2, 3, 7, 11, 12, 18, 22, 23, 24].

It is also well-known that in some weak sense and asymptotically, the approximation holds for all realizations of the random medium, in the sense that the probability that the energy density of the waves not be given by the kinetic model converges to 0 when the correlation length of the random medium converges to 0. This result is referred to as statistical stability or self-averaging [1, 4, 6, 8, 19, 20]. These results show that the scintillation function, which is defined as the correlation function of the wave energy density (in the phase space), tends to 0 in a weak sense. Understanding how fast the scintillation function converges to 0 is a difficult and complicated problem. It is difficult

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because the algebra is never straightforward and it is complicated because it depends in a non-trivial way on quite a few physical parameters such as the regularity of the wave initial condition and on the power spectrum of the random fluctuations in the underlying medium.

This paper provides a careful analysis of the single scattering contribution in the scintillation function. By single scattering, we mean the following. The waves are modeled in this paper by a Schrödinger equation. We consider high frequency waves with correlation length λ much smaller than the distance L over which propagation is observed. In other words $\varepsilon = \frac{\lambda}{L} \ll 1$. The random fluctuations of the underlying medium through which waves propagate are assumed to be stationary. The correlation length of the random fluctuations is also assumed to be of size εL so that both the waves and the random medium oscillate at similar frequencies. The power spectrum of the fluctuations is the Fourier transform of the correlation function of the random medium. The scintillation function, which is the correlation function for the wave energy density, may then formally be written as an infinite expansion with terms corresponding to increasing orders of interaction of the waves with the underlying medium [12, 23, 24]. The first non-trivial term in the scintillation function is the one that is linear in the power spectrum. All other terms are at least quadratic in the power spectrum or, when the random fluctuations are not modeled as Gaussian processes, depend on higher-order statistics of the fluctuations. When scattering is relatively weak, the linear term in the power spectrum will then presumably be the dominant contribution in the scintillation function. We referred to this term as the single scattering contribution to scintillation. A detailed description of the wave equation and the single scattering contribution to the scintillation function is presented in section 2.

A complete description of the single scattering term is not sufficient to fully characterize the scintillation. It is known that in the Itô-Schrödinger regime of wave propagation, which is a simplified model for wave propagation, the single scattering term may not be the leading contribution to scintillation [9]. Moreover, the single scattering contribution to scintillation tends to 0 with ε even in dimension $d = 1$ even though it is known that the kinetic model is not the right limit when $d = 1$ [14, 16] since waves localize (with a random limit) rather than transport.

Nonetheless, the single scattering contribution is sufficiently rich and interesting physically that we want to present it in detail. Our main results on the asymptotic behavior of single scattering scintillation are given in Theorem 3.1 in section 3 below. There, it is shown that the amplitude of scintillation mainly depends on two ingredients: the structure (regularity) of the initial conditions for the wave equation; and the long-range properties of the correlation function of the random fluctuations. Scintillation is also very much a function of the scale at which the energy is observed. Point-wise estimates or estimates at a scale smaller than or equal to ε inevitably yield unstable quantities. The energy density needs to be averaged over a sufficiently large domain in order to be stable. We mainly consider the stability of the energy density averaged over such a sufficiently large domain. We briefly comment on the stability of measurements performed over small domains that are nonetheless of size much larger than ε; see also [8, 9].

The salient features of Theorem 3.1 below are that scintillation is typically larger when the initial conditions for the Schrödinger equation are highly localized in space.

This picture holds when the correlation function of the random medium is integrable. When the correlation function of the random medium decreases very slowly (this is the regime of long-range memory effects), then a different behavior emerges and the maximal scintillation is obtained when the initial condition of the wave equation is equally singular in the space and momentum variables. This peculiar behavior is also dimension dependent. The complex behavior of the scintillation function depends on the detailed structure of the initial conditions for the Schrödinger equation and is explained in detail in section 2.

The proof of the results is based on the analysis of oscillatory integrals, which is conducted by means of estimates of various Fourier transforms. The details of the proof can be found in section 4. It uses standard lemmas of real analysis that are stated without proofs in section 5.

2 Single scattering scintillation

Although we expect the results mentioned below to generalize to other wave equations, we restrict ourselves here to the simplest mathematical model for high frequency waves propagating in random media and define u_ε as the solution to the following random Schrödinger equation:

$$\left(i\varepsilon\frac{\partial}{\partial t}+\frac{\varepsilon^2}{2}\Delta+\sqrt{\varepsilon}V\left(\frac{x}{\varepsilon}\right)\right)u_\varepsilon(t,x)=0,\quad t>0,\quad x\in\mathbb{R}^d,$$

augmented with a *deterministic* initial condition $u_\varepsilon(0,\cdot)$ uniformly bounded in $L^2(\mathbb{R}^d)$ with respect to ε, for $d\geq 1$. Here, V is a mean-zero homogeneous stationary random field with autocorrelation $R(x):=\mathbb{E}V(x+y)V(y)$ and is time-independent. The symbol $\mathbb{E}$ denotes the ensemble average with respect to a given probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which V is defined. The Wigner transform of u_ε is defined as, see [17]:

$$W_\varepsilon(t,x,k):=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ik\cdot y}u_\varepsilon\left(t,x-\frac{\varepsilon y}{2}\right)\bar{u}_\varepsilon\left(t,x+\frac{\varepsilon y}{2}\right)\,\mathrm{d}y,$$

where $\bar{u}_\varepsilon$ is the complex conjugate of u_ε, and W_ε satisfies the stochastic Wigner equation

$$\frac{\partial}{\partial t}W_\varepsilon+k\cdot\nabla_x W_\varepsilon=A_\varepsilon W_\varepsilon, \tag{1}$$

with

$$\begin{aligned}(A_\varepsilon W_\varepsilon)(x,k)&:=\int_{\mathbb{R}^d}f_\varepsilon(x,k-\eta)W_\varepsilon(x,\eta)\,\mathrm{d}\eta,\\ f_\varepsilon(x,\xi)&:=\frac{i}{\sqrt{\varepsilon}\pi^d}\left[\hat{V}(-2\xi)e^{-i2\xi\cdot x/\varepsilon}-\hat{V}(2\xi)e^{i2\xi\cdot x/\varepsilon}\right],\end{aligned}$$

where $\hat{V}$ denotes the Fourier transform of V with the convention

$$\hat{V}(k)=\int_{\mathbb{R}^d}e^{-ik\cdot x}V(x)dx.$$

The initial condition of (1), denoted by $W^0_\varepsilon(x,k)$, is the Wigner transform of $u_\varepsilon(0,\cdot)$. Let $a_\varepsilon := \mathbb{E}W_\varepsilon$ be the ensemble average of W_ε. For sufficiently rapidly decaying correlation function R, a_ε converges in a proper functional setting to the solution a_0 of a radiative transfer equation. Such result has been proved in different frameworks by several authors: see for instance [5, 6, 21] for time-dependent random potentials, or [1, 8, 20] for Itô-Schrödinger equations. For the more difficult case of time-independent potentials, the rigorous convergence is proved in [24, 12] for Gaussian potentials, and a_0 solves

$$\frac{\partial}{\partial t}a_0 + k\cdot\nabla_x a_0 = \int_{\mathbb{R}^d}\sigma(p,k)[a_0(t,x,p)-a_0(t,x,k)]\,\mathrm{d}p,$$

with scattering cross section $\sigma(p,k) = \hat{R}(p-k)\delta(|k|^2-|p|^2)$, where δ is the Dirac distribution and the power spectrum $\hat{R}(k)$ is the Fourier transform of the correlation function $R(x)$. The above radiative transfer is known to hold for wave equations other than the Schrödinger equation [18, 22].

Long range correlations. Here, we are interested in random fields with possibly *long range* interactions, which can be modeled with slowly decaying autocorrelations that do not belong to $L^1(\mathbb{R}^d)$. Assuming $R(x) \sim_{|x|\to\infty} x^{\delta-d}$, with $0<\delta<d$, some simple rescaling arguments show that $\hat{R}$ is singular at the origin and behaves like $|k|^{-\delta}$. This leads us to consider correlation functions with singular Fourier transforms near the origin of the form

$$\hat{R}(k) = \frac{S(k)}{|k|^\delta}, \quad 0<\delta<d, \tag{2}$$

with S bounded and continuous at zero. Since, $0<\delta<d$, $\hat{R}$ is locally integrable. Physically realizable media will also have $\int \hat{R}(k)\,\mathrm{d}k = R(0) < \infty$. Short-range correlations correspond to integrable R. In this case $\hat{R}$ is bounded so we may take $\delta = 0$ in (2).

Scintillation. Very few results exist on the rigorous limit of the random process W_ε. It is proved in [6] for potentials that are Markovian in time and under additional hypotheses on the Wigner transform (essentially that it is square integrable by mixture of states) that W_ε converges weakly and in probability to its average a_ε, that is

$$\mathbb{P}\Big(|\langle W_\varepsilon(t),\varphi\rangle - \langle a_\varepsilon(t),\varphi\rangle| \geq \eta\Big) \to 0, \qquad \text{uniformly on compact intervals.}$$

Above, φ is a test function in the Schwarz space $\mathcal{S}(\mathbb{R}^{2d})$ and $\langle\cdot,\cdot\rangle$ denotes the $\mathcal{S}'-\mathcal{S}$ duality product, where $\mathcal{S}'$ is the space of tempered distributions. The latter result means that the Wigner transform is *self-averaging*. This is an important property for instance in the analysis of the refocusing properties of time-reversed waves [6, 10, 13, 19] for which it is shown that the quality of refocusing is independent of the local fluctuations of the random medium and hence only depends on macroscopic characteristics. Within the Itô-Schrödinger regime, the optimal rate of convergence can be computed and shown to depend on some parameters of the problems such as the size of support of the initial condition of the Schrödinger equation, see [8, 9]. The convenient tool in the analysis is the *scintillation function* J_ε (or covariance function), defined as

$$J_\varepsilon(t,x,k,y,p) = \mathbb{E}W_\varepsilon(t,x,k)W_\varepsilon(t,y,p) - \mathbb{E}W_\varepsilon(t,x,k)\,\mathbb{E}W_\varepsilon(t,y,p), \tag{3}$$

whose weak convergence to zero implies the convergence in probability thanks to the Chebyshev inequality

$$\mathbb{P}\Big(|\langle W_\varepsilon(t),\varphi\rangle - \langle a_\varepsilon(z),\varphi\rangle| \geq \eta\Big) \leq \frac{1}{\eta^2}\langle J_\varepsilon(t),\varphi\otimes\varphi\rangle.$$

Showing that J_ε goes to zero is a difficult task and can be rigorously done within the Itô-Schrödinger regime for short-range correlations since it satisfies a *closed-form* equation, a transport equation with highly oscillating coefficients. In the regime of interest in this paper, the scintillation *does not* satisfy a closed-form equation. We will therefore follow a perturbative approach and only consider the scintillation created by single scattering, that is after only one interaction with the random potential, assuming scattering is weak enough so that multiple interactions can be neglected. Doing so, we can obtain an exact expression of the scintillation and fully characterize its limit. Such expression follows from a multiple scattering expansion of W_ε: introducing first the free transport semigroup J, $Jh(t,x,k) := h(x-tk,k)$, and the operator

$$D^{-1}h(t,x,k) := \int_0^t h(t-s,x-sk,k)\,\mathrm{d}s,$$

then (1) can be recast as the integral equation

$$(I - D^{-1}A_\varepsilon)W_\varepsilon = JW^0_\varepsilon,$$

whose solution can be decomposed formally as the multiple scattering expansion:

$$W_\varepsilon = \sum_{j=0}^{\infty}(D^{-1}A_\varepsilon)^j JW^0_\varepsilon.$$

Retaining only the terms $j \leq 1$ in the latter decomposition, we have

$$\begin{aligned} J_\varepsilon(t,x,k,y,p) &\approx \mathbb{E}\left\{(JW^0_\varepsilon + D^{-1}A_\varepsilon JW^0_\varepsilon)(t,x,k)(JW^0_\varepsilon + D^{-1}A_\varepsilon JW^0_\varepsilon)(t,y,p)\right\} \\ &\quad -\mathbb{E}\left\{(JW^0_\varepsilon + D^{-1}A_\varepsilon JW^0_\varepsilon)(t,x,k)\right\}\mathbb{E}\left\{(JW^0_\varepsilon + D^{-1}A_\varepsilon JW^0_\varepsilon)(t,y,p)\right\}, \\ &= \mathbb{E}\left\{D^{-1}A_\varepsilon JW^0_\varepsilon)(t,x,k)(D^{-1}A_\varepsilon JW^0_\varepsilon)(t,y,p)\right\}, \\ &:= \mathbb{E}\mathcal{W}^\varepsilon_{11}(t,x,k,y,p). \end{aligned}$$

Above, we used the facts that V is mean-zero and the initial condition is assumed to be deterministic.

Initial conditions. In the Itô-Schrödinger regime [8, 9], the scintillation function is known to very much depend on the structure of the initial conditions. Such a statement remains valid here.

Consider first initial conditions $u_\varepsilon(0,\cdot)$ oscillating at frequencies of order ε^{-1} and with a spatial support of size ε^α for $0 \leq \alpha \leq 1$. The parameter α quantifies the macroscopic concentration of the initial condition. The simplest example is a modulated plane wave of the form (or a pure state):

$$u_\varepsilon(0,x) = \frac{1}{\varepsilon^{\frac{d\alpha}{2}}}\chi\Big(\frac{x}{\varepsilon^\alpha}\Big)e^{i\frac{x\cdot q_0}{\varepsilon}}, \tag{4}$$

where $\chi \in \mathcal{S}(\mathbb{R}^d)$. The direction of propagation is given by q_0. Note that the above sequence of initial conditions is uniformly bounded in $L^2(\mathbb{R}^d)$, and that the related Wigner transform reads

$$W^0_\varepsilon(x,q) = \frac{1}{\varepsilon^d}a\left(\frac{x}{\varepsilon^\alpha}, \frac{q-q_0}{\varepsilon^{1-\alpha}}\right), \tag{5}$$

where $a(x,k)$ is the Wigner transform of the rescaled initial condition $u_{\varepsilon=1}$ and is real-valued. We then slightly generalize the latter expression by considering initial condition of the form

$$W^0_\varepsilon(x,q) = \frac{1}{\varepsilon^{d(\alpha+\beta)}}a\left(\frac{x}{\varepsilon^\alpha}, \frac{q-q_0}{\varepsilon^{\beta}}\right). \tag{6}$$

The parameter α measures the concentration of the initial conditions in the spatial variables while β measures that in the momentum variables. We restrict α and β to be less than one to ensure that ε^{-1} is the highest frequency in the problem. Allowing for higher frequencies while still considering a Wigner transform at the frequency ε^{-1} will lead to vanishing limiting Wigner transforms and would be of little interest for then energy is lost when passing to the limit, see e.g. [15, 17].

The most physical case is when $\alpha+\beta=1$ as in (5). This is related to the Heisenberg uncertainty principle, which states that waves cannot be localized both in space and momentum. The case $\alpha+\beta>1$ can be treated mathematically in the same fashion as the physical case and so we present it for completeness. The case $\alpha+\beta<1$ corresponds to mixtures of states and can be obtained by averaging of $a(\cdot,\cdot;\zeta)$ with respect to an additional measure in the ζ variable in order to regularize the initial conditions; see e.g. [6].

Some notations. We denote by $\mathcal{F}f$ the Fourier transform of $f(x,q)$ with respect to both variables x and q. For a function $f(z^1,\cdots,z^n) \in \mathcal{C}^m(\mathbb{R}^{nd})$, $z^j \in \mathbb{R}^d$, $j=1,\cdots,n$ and a multi-index $i=(i_1,\cdots,i_{nd}) \in \mathbb{N}^{nd}$ with $|i| = i_1+\cdots+i_{nd} \le m$, we introduce

$$\partial^{|i|}_{z^1,\cdots,z^n} f := \frac{\partial^{i_1}}{\partial_{z^1_1}}\cdots\frac{\partial^{i_{nd}}}{\partial_{z^n_d}}f.$$

Let as well $\langle x\rangle := (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^d$ and $a\wedge b$ (resp. $a \vee b$) be the minimum (resp. maximum) of a and b. We denote by $a \lesssim b$ the inequality $a \le Cb$, where $C>0$ is some universal constant.

3 Main results

Let $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$ be a real valued test function and let W^ε_{11} be the expectation of $\mathcal{W}^\varepsilon_{11}$, $W^\varepsilon_{11} := \mathbb{E}\mathcal{W}^\varepsilon_{11}$. We denote by w^ε the quantity

$$w^\varepsilon(t) := \int_{\mathbb{R}^{2d}} W^\varepsilon_{11}(t,x,k,y,p)\,\varphi(x,k)\,\varphi(y,p)dxdydkdp.$$

The main result of the paper is Theorem 3.1 below, which states that the scintillation corresponding to single scattering is of order $\varepsilon^{(d-\delta)(1-\alpha)+1-\alpha\wedge\beta}$.

When $\delta = 0$, which corresponds to the case of an integrable correlation function $R(x)$, we find in the physical case $\alpha + \beta = 1$ that scintillation is maximal when $\beta = 0$ and $\alpha = 1$. In this case it is proportional to ε. This corresponds to highly localized initial conditions in space and is consistent with the results obtained in the Itô-Schrödinger regime in [8, 9]. The unphysical case $\alpha = \beta = 1$ predicts stability of order $O(1)$. This is consistent with the results obtained in [1] for initial conditions of the form δ in space and δ in wavenumbers. However, we repeat that the case $\alpha = \beta = 1$ is not a physical description of initial conditions for the Schrödinger equation that are square integrable.

When long range correlations are present, the structure of scintillation is modified. When δ is close to d, which corresponds to the strongest possible long range interactions as the correlation function barely decays, the largest scintillation (in the physical case $\alpha + \beta = 1$) is obtained for $\alpha = \beta = \frac{1}{2}$ and thus gives a scintillation of order close to $\varepsilon^{\frac{1}{2}}$.

These results show that the single scattering contribution of scintillation converges to zero and this is consistent with the fact that the Wigner function is self-averaging. Note however, that in dimension $d = 1$, the above results also predict self-averaging of the Wigner transform since scintillation is always smaller than $O(\varepsilon^{\frac{1}{2}})$. Yet it is known that waves localize in dimension $d = 1$ and that the deterministic radiative transfer model is replaced by a stochastic limit [14, 16]. In dimension $d = 1$, it turns out that there are larger contributions to scintillation than that given by single scattering. The single scattering contribution is however dominant in certain regimes and its asymptotic limit is characterized in detail in the following result.

Theorem 3.1 *Assume the initial condition W_ε^0 has the form (6) with $a \in \mathcal{S}(\mathbb{R}^{2d})$ and that the scattering cross section is given by (2). Then $\varepsilon^{-(d-\delta)(1-\alpha)-1+\alpha\wedge\beta} w^\varepsilon(t)$ is bounded in $L^\infty((0,T))$, converges pointwise on $(0,T]$ and uniformly on $[t_0,T]$ to $w(t)$, for any $t_0 > 0$ independent of ε, where:*

- *if $\alpha = 0$, $0 < \beta \le 1$:*

$$\begin{aligned} w(t) &= \frac{1}{2^{d-\delta}} \int_{\mathbb{R}^d} \frac{S(0)}{|k|^\delta} \left| \int_0^t e^{-isq_0\cdot k} \mathcal{F}(fa)(k,0)\, ds \right|^2 dk, \\ f &:= f(t,s,x,k) = k \cdot ((t-s)\nabla_x \varphi(x+tq_0,q_0) + \nabla_q \varphi(x+tq_0,q_0)), \end{aligned}$$

- *if $\beta = 0$, $0 < \alpha < 1$:*

$$\begin{aligned} w(t) &= \frac{1}{2^{d-\delta}} \int_{\mathbb{R}^d} \frac{S(0)}{|k|^\delta} \left| \int_0^\infty e^{-isq_0\cdot k} \mathcal{F}(fa)(k,sk)\, ds \right|^2 dk, \\ f &:= f(t,q,k) = k \cdot (t\nabla_x \varphi(t(q+q_0),q+q_0) + \nabla_q \varphi(t(q+q_0),q+q_0)), \end{aligned}$$

- *if $\beta = 0$, $\alpha = 1$:*

$$\begin{aligned} w(t) &= \frac{1}{2^{d-\delta}} \int_{\mathbb{R}^d} \frac{S(k)}{|k|^\delta} \left| \int_0^\infty e^{-isq_0\cdot k} \mathcal{F}(fa)(k,sk)\, ds \right|^2 dk, \\ f &:= f(t,s,q,k) = \sum_{\pm} \pm\varphi(t(q+q_0) \pm (t-s)k, q+q_0 \pm k), \end{aligned}$$

- *if* $0 < \beta < \alpha < 1$*:*

$$\begin{aligned} w(t) &= \frac{2\pi}{2^{d-\delta}|q_0|}\int_{\{q_0\cdot k=0\}}\int_0^\infty \frac{S(k)}{|k|^\delta}|\mathcal{F}(fa)(k,sk)|^2\,ds\,dk,\\ f &:= f(t,k) = k\cdot(t\nabla_x\varphi(tq_0,q_0)+\nabla_q\varphi(tq_0,q_0)), \end{aligned}$$

- *if* $0 < \alpha \le \beta < 1$*:*

$$\begin{aligned} w(t) &= \frac{2\pi}{2^{d-\delta}|q_0|}\int_{\{q_0\cdot k=0\}}\int_0^t \frac{S(k)}{|k|^\delta}|\mathcal{F}(fa)(k,K)|^2\,ds\,dk,\\ f &:= f(t,k) = k\cdot((t-s)\nabla_x\varphi(tq_0,q_0)+\nabla_q\varphi(tq_0,q_0)),\\ K &= sk \quad \textit{if} \quad \alpha=\beta \quad \textit{and zero otherwise}, \end{aligned}$$

- *if* $\alpha = \beta = 1$*:*

$$\begin{aligned} w(t) &= \frac{2\pi}{2^{d-\delta}|q_0|}\int_{\{q_0\cdot k=0\}}\int_0^t \frac{S(k)}{|k|^\delta}|\mathcal{F}(fa)(k,sk)|^2\,ds\,dk,\\ f(t,s,k) &= \sum_\pm \pm\varphi(tq_0\pm(t-s)k,q_0\pm k). \end{aligned}$$

Above $\mathcal{F}(fa)$ denotes the Fourier transform of the product fa with respect to the variables x and q.

All the integrals appearing in the above theorem are finite since the product fa belongs to $\mathcal{S}(\mathbb{R}^{2d})$ and so does $\mathcal{F}(fa)$. Such a regularity on the initial condition a is not used in the analysis and only simplifies the calculations. The hypotheses $a\in\mathcal{S}(\mathbb{R}^{2d})$ could thus be relaxed to a large extent. Note also that the above integrals do not vanish for generic choices of a and φ. The above terms indeed characterize the limiting scintillation functions.

We repeat that the physical case of a pure state initial condition (4) yields initial conditions for the Wigner transform of the form (5), i.e., with $\alpha+\beta=1$. The case $\alpha=\beta=1$ is therefore presented for completeness only as such initial conditions cannot be obtained from taking the Wigner transform of solutions to the Schrödinger equation.

The results of the theorem can be straightforwardly generalized to some particular cases. For instance, when $\alpha=\beta=0$, which corresponds to choosing smooth initial conditions for the Wigner equation, we can actually consider test functions of the form

$$\frac{1}{\varepsilon^{d(\gamma_1+\gamma_2)}}\varphi\left(\frac{x}{\varepsilon^{\gamma_1}},\frac{q-q_0}{\varepsilon^{\gamma_2}}\right), \tag{7}$$

and simple calculations show that the roles of (α,β) and (γ_1,γ_2) are symmetrical. As another example, when $\alpha=\gamma_2=0$, the theorem applies with minor changes with α replaced by γ_1. More precisely, we have the following proposition:

Proposition 3.2 *Assume the initial condition W^0_ε has the form (6) with $a\in\mathcal{S}(\mathbb{R}^{2d})$ and that the scattering cross section is given by (2). Assume moreover that the test* *function φ in the definition of w^ε is replaced by* (7). *Then, there exist two non-identically zero continuous functions w_1 and w_2 on $[0,T]$, such that, if $\alpha=\beta=0$,*

$$\varepsilon^{-(d-\delta)(1-\gamma_1)-1+\gamma_1\wedge\gamma_2} w^\varepsilon(t) \to w_1(t),$$

or if $\alpha=\gamma_2=0$,

$$\varepsilon^{-(d-\delta)(1-\gamma_1)-1+\gamma_1\wedge\beta} w^\varepsilon(t) \to w_2(t),$$

pointwise on $(0,T]$ and uniformly on $[t_0,T]$, for any $t_0>0$ independent of ε.

According to the proposition, when $\alpha=0$ and $\beta=1$, the scintillation is of order $\mathcal{O}(\varepsilon^{(d-\delta)(1-\gamma_1)})$, so that statistical stability occurs as soon as $\gamma_1<1$, i.e., as soon as the array of detectors is large compared to the wavelength, as we expect physically. When $\gamma=1$, scintillation is an $\mathcal{O}(1)$. These results are consistent with the ones obtained in [8]. The proof of the proposition is postponed to section 4.8.

4 Proofs

The proof is done by deriving an exact expression for w^ε and by passing to the limit in it. The general equation for w^ε is obtained in section 4.1. We then treat the different cases in the following sections according to α and β. The schemes of the proofs are all essentially the same: we perform Taylor expansions of the function f^ε defined in (10) and carefully estimate the growth of the remainders according to the different variables. This allows to recast w^ε as a leading term and a negligible one and the different expressions of the limiting w^ε follow from a passage to the limit in the leading order.

4.1 Equation for Scintillation.

Here we derive an equation for the lowest order scintillation term W_{11}^ε, and an expression for its integral against a pair of test functions of $\mathcal{S}(\mathbb{R}^{2d})$. When there is no confusion, we do not precise the domains of integration to simplify the presentation. We have, for the first order scintillation:

$$\begin{aligned}\mathcal{W}_{11}^\varepsilon(t,x_1,q_1,x_2,q_2) &= (D^{-1}A_\varepsilon JW_\varepsilon^0)(t,x_1,q_1)(D^{-1}A_\varepsilon JW_\varepsilon^0)(t,x_2,q_2)\\ &= \int_0^t\int_0^t\int\int f_\varepsilon(x_1-s_1q_1,q_1-\eta_1)f_\varepsilon(x_2-s_2q_2,q_2-\eta_2)\\ &\quad\times JW_0(t-s_1,x_1-s_1q_1,\eta_1)JW_0(t-s_2,x_2-s_2q_2,\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}s_1\,\mathrm{d}s_2.\end{aligned}$$

Using the fact that

$$\mathbb{E}\hat{V}(\xi)\hat{V}(\nu) = (2\pi)^d\hat{R}(\xi)\delta(\xi+\nu),$$

where δ is the Dirac distribution, we obtain

$$\begin{aligned}&\mathbb{E}f_\varepsilon(x_1-s_1q_1,q_1-\eta_1)f_\varepsilon(x_2-s_2q_2,q_2-\eta_2)\\ &\quad=\frac{2}{\varepsilon}\cos\left\{\frac{2}{\varepsilon}(q_1-\eta_1)\cdot(x_1-s_1q_1-x_2+s_2q_2)\right\}\\ &\qquad\times\hat{R}(2(q_1-\eta_1))\Big[\delta(q_1-\eta_1+q_2-\eta_2)-\delta(q_1-\eta_1-(q_2-\eta_2))\Big].\end{aligned}$$

So

$$
\begin{aligned}
W_{11}^{\varepsilon} &= \mathbb{E}\mathcal{W}_{11}^{\varepsilon} \\
&= \frac{2}{\varepsilon}\int_0^t \int_0^t \int \cos\left\{\frac{2}{3}(q_1-\eta_1)\cdot(x_1-s_1q_1-x_2+s_2q_2)\right\}\hat{R}(2(q_1-\eta_1)) \\
&\quad \times JW_{\varepsilon}^0(t-s_1,x_1-s_1q_1,\eta_1)\Big[JW_{\varepsilon}^0(t-s_2,x_2-s_2q_2,q_2+(q_1-\eta_1)) \\
&\qquad - JW_{\varepsilon}^0(t-s_2,x_2-s_2q_2,q_2-(q_1-\eta_1))\Big]\,\mathrm{d}\eta_1\,\mathrm{d}s_1\,\mathrm{d}s_2.
\end{aligned}
$$

Now make the substitution $k = q_1 - \eta_1$. Since $\hat{R}$ and cos are even functions, only the even part of the remaining terms survives. We thereby conclude

$$
\begin{aligned}
W_{11}^{\varepsilon} &= \frac{1}{\varepsilon}\int_0^t \int_0^t \int \cos\left\{\frac{2}{\varepsilon}k\cdot(x_1-s_1q_1-x_2+s_2q_2)\right\}\hat{R}(2k) \\
&\quad \times \Big[JW_{\varepsilon}^0(t-s_1,x_1-s_1q_1,q_1+k) - JW_{\varepsilon}^0(t-s_1,x_1-s_1q_1,q_1-k)\Big] \\
&\quad \times \Big[JW_{\varepsilon}^0(t-s_2,x_2-s_2q_2,q_2-k) - JW_{\varepsilon}^0(t-s_2,x_2-s_2q_2,q_2-k)\Big]\,\mathrm{d}k\,\mathrm{d}s_1\,\mathrm{d}s_2, \\
&= \frac{1}{\varepsilon}\int_0^t \int \hat{R}(2k)\left(\prod_{j=1}^{2}\int_0^t \exp\left\{(-1)^{j-1}i\frac{2k}{\varepsilon}\cdot(x_j-s_jq_j)\right\}\right. \\
&\quad \left. \times \left[\sum_{\pm} \pm JW_{\varepsilon}^0(t-s_j,x_j-s_jq_j,q_j\pm k)\right]\mathrm{d}s_j\right)\mathrm{d}k,
\end{aligned}
$$

where, since the integrand is even, we have replaced $\cos\theta$ by $\exp[i\theta]$. The moment may now be written

$$
\begin{aligned}
&\int W_{11}^{\varepsilon}(t,x_1,q_1,x_2,q_2)\varphi(x_1,q_1)\varphi(x_2,q_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}q_1\,\mathrm{d}q_2 \\
&= \frac{1}{\varepsilon}\int \hat{R}(2k)\left(\int \prod_{j=1}^{2}\int_0^t \exp\left\{(-1)^{j-1}i\frac{2k}{\varepsilon}\cdot(x_j-s_jq_j)\right\}\varphi(x_j,q_j)\right. \\
&\quad \left. \times \left[\sum_{\pm} \pm JW_{\varepsilon}^0(t-s_j,x_j-s_jq_j,q_j\pm k)\right]\mathrm{d}s_j\,\mathrm{d}x_j\,\mathrm{d}q_j\right)\mathrm{d}k.
\end{aligned}
$$

Now change $k \mapsto \varepsilon k$ to get

$$
\begin{aligned}
w^{\varepsilon}(t) &= \varepsilon^{d-1}\int \hat{R}(2k\varepsilon)\left(\int \prod_{j=1}^{2}\int_0^t \exp\left\{(-1)^{j-1}i2k\cdot(x_j-s_jq_j)\right\}\varphi(x_j,q_j)\right. \\
&\quad \left. \times \left[\sum_{\pm} \pm JW_{\varepsilon}^0(t-s_j,x_j-s_jq_j,q_j\pm \varepsilon k)\right]\mathrm{d}s_j\,\mathrm{d}x_j\,\mathrm{d}q_j\right)\mathrm{d}k.
\end{aligned}
$$

Substituting for JW^0_ε using (6) and the expression of the free transport semigroup J, and get

$$w^\varepsilon(t) = \varepsilon^{d-1-d(\alpha+\beta)} \int \hat{R}(2k\varepsilon) \left(\int \prod_{j=1}^{2} \int_0^t \exp\left\{(-1)^{j-1} i2k\cdot(x_j - s_j q_j)\right\} \varphi(x_j,q_j) \right.$$
$$\left. \times \left[\sum_{\pm} \pm a\left(\frac{x_j - s_j q_j - (t-s_j)(q_j \pm \varepsilon k)}{\varepsilon^\alpha}, \, \frac{q_j \pm \varepsilon k - q_0}{\varepsilon^\beta} \right) \right] \mathrm{d}s_j \, \mathrm{d}x_j \, \mathrm{d}q_j \right) \mathrm{d}k. \tag{8}$$

Changing variables, in order, $x \mapsto x + s_j q_j + (t - s_j)(q_j \pm \varepsilon k)$, $q \mapsto q \mp \varepsilon k$, $x \mapsto x\varepsilon^\alpha$, $q \mapsto q_0 + q\varepsilon^\beta$, $2k \mapsto \varepsilon^{-\alpha} k$ and $s \mapsto t - s$, we obtain the final expression:

$$w^\varepsilon(t) \quad = \quad \frac{\varepsilon^{(d-\delta)(1-\alpha)-1}}{2^{d-\delta}} \int \mathrm{d}k \, \frac{S(k\varepsilon^{1-\alpha})}{|k|^\delta}$$
$$\times \left| \int_0^t e^{-i\varepsilon^{-\alpha} s q_0 \cdot k} \mathcal{F}(f^\varepsilon_s a)(k, \varepsilon^{\beta-\alpha} sk, \varepsilon^{1-\alpha} k) \, \mathrm{d}s \right|^2, \tag{9}$$

$$f^\varepsilon_s(x,q,k) \quad := \quad \sum_{\pm} \pm \varphi(x\varepsilon^\alpha + tq\varepsilon^\beta + tq_0 \pm (t-s)k/2, \, q\varepsilon^\beta + q_0 \pm k/2). \tag{10}$$

Above, we recall that $\mathcal{F}(f^\varepsilon_s a)(\xi,\nu,k)$ denotes the Fourier transform of $f^\varepsilon_s(x,q,k)a(x,q)$ with respect to x and q. Equations (9) and (10) are the starting points of the proof of the theorem. We treat now the different cases separately.

4.2 The case $\alpha = 0$, $0 < \beta \leq 1$.

For such a configuration, we have:

$$w^\varepsilon(t) = \frac{\varepsilon^{d-\delta-1}}{2^{d-\delta}} \int \frac{S(k\varepsilon)}{|k|^\delta} \left| \int_0^t e^{-isq_0\cdot k} \mathcal{F}(f^\varepsilon_s a)(k, \varepsilon^\beta sk, \varepsilon k) \, \mathrm{d}s \right|^2 \mathrm{d}k. \tag{11}$$

We single out the leading order in w^ε by applying lemma 5.1 to f^ε_s with $y = x + tq_0$, $\alpha_1 = \alpha_2 = \beta$, $y_1 = tq$, $\beta_1 = \beta_2 = 1$, $y_2 = (t-s)k/2$, $p = q_0$, $p_1 = q$, $p_2 = k/2$. It comes:

$$\begin{aligned}
f^\varepsilon_s(x,q,\varepsilon k) \quad &= \quad \varepsilon f_s(x,k) + \varepsilon^{1+\beta} r^\varepsilon_s(x,q,k), \\
f_s(x,k) \quad &= \quad k\cdot((t-s)\nabla_x \varphi(x+tq_0,q_0) + \nabla_q \varphi(x+tq_0,q_0)), \\
r^\varepsilon_s(x,q,k) \quad &= \quad (\varepsilon^{1-\beta} r^\varepsilon_1 + r^\varepsilon_2 + r^\varepsilon_3)(x+tq_0, tq, (t-s)k/2, q_0, q, k/2),
\end{aligned}$$

where, for any multi-index λ, and $t \leq T$,

$$|\partial^{|\lambda|}_x f_s(x,k)| \lesssim |k| \qquad ; \qquad |\partial^{|\lambda|}_{x,q} r^\varepsilon_s(x,q,k)| \lesssim |k|(|k| + |q|). \tag{12}$$

It thus follows that

$$\left| \int_0^t e^{-isq_0\cdot k} \mathcal{F}(f^\varepsilon_s a)(k, \varepsilon^\beta sk, \varepsilon k) \, \mathrm{d}s \right|^2 \quad = \quad \varepsilon^2 |L^\varepsilon(t,k)|^2 + \varepsilon^{2+\beta} R^\varepsilon(t,k),$$

where

$$
\begin{aligned}
L^\varepsilon(t,k) &= \int_0^t e^{-isq_0\cdot k}\mathcal{F}(f_s a)(k,\varepsilon^\beta sk,\varepsilon k)\,\mathrm{d}s,\\
R^\varepsilon(t,k) &= \varepsilon^\beta\left|\int_0^t e^{-isq_0\cdot k}\mathcal{F}(r_s^\varepsilon a)(k,\varepsilon^\beta sk,\varepsilon k)\,\mathrm{d}s\right|^2\\
&\quad +2\mathrm{Re}\int_0^t e^{-isq_0\cdot k}\mathcal{F}(f_s a)(k,\varepsilon^\beta sk,\varepsilon k)\,\mathrm{d}s\int_0^t e^{iuq_0\cdot k}\overline{\mathcal{F}(r_u^\varepsilon a)}(k,\varepsilon^\beta uk,\varepsilon k)\,\mathrm{d}u.
\end{aligned}
$$

As a belongs to $\mathcal{S}(\mathbb{R}^{2d})$, so do the quantities $\mathcal{F}(f_s a)$ and $\mathcal{F}(r^\varepsilon a)$. In particular, using (12) and lemma 5.2 with $f(x,q,k)=r_s^\varepsilon(x,q,k)$ and $g=a$, it follows that R^ε decays fast enough with respect to k so that $|k|^{-\delta}S(k\varepsilon)R^\varepsilon(t,k)$ is integrable in k (recall that S is bounded) with a bound independent of ε, for any $0\le t\le T$. As a result, the contribution to w^ε from the term involving R^ε is of order $\varepsilon^{d-\delta+1+\beta}$, uniformly in time. In the same way, (12) together with lemma 5.2 gives for instance, $\forall (t,s,k)\in[0,T]^2\times\mathbb{R}^d$,

$$
|\mathcal{F}(f_s a)(k,\varepsilon^\beta sk,\varepsilon k)|\lesssim\frac{|k|}{(1+|k|^{d+1})(1+(\varepsilon^\beta s|k|)^{d+1})}\lesssim\frac{1}{1+|k|^d},
$$

which allows first to pass to the limit pointwise in k and uniformly for $t\in[0,T]$ in $L^\varepsilon(t,k)$ using dominated convergence and then in (11) uniformly for $t\in[0,T]$ to obtain the announced result.

4.3 The case $\beta=0$, $0<\alpha<1$.

In this case, after the change of variable $s\mapsto s\varepsilon^\alpha$, we find:

$$
w^\varepsilon(t)=\frac{\varepsilon^{(d-\delta)(1-\alpha)-1+2\alpha}}{2^{d-\delta}}\int\frac{S(k\varepsilon^{1-\alpha})}{|k|^\delta}\left|\int_0^{t\varepsilon^{-\alpha}} e^{-isq_0\cdot k}\mathcal{F}(f^\varepsilon_{s\varepsilon^\alpha}a)(k,sk,\varepsilon^{1-\alpha}k)\,\mathrm{d}s\right|^2\mathrm{d}k. \tag{13}
$$

Applying lemma 5.1 to f_s^ε with $y=t(q+q_0)$, $\alpha_1=\alpha$, $y_1=x$, $\beta_1=\beta_2=1-\alpha$, $y_2=(t-s\varepsilon^\alpha)k/2$, $p=q+q_0$, $p_1=0$, $p_2=k/2$, we have

$$
\begin{aligned}
f^\varepsilon_{s\varepsilon^\alpha}(x,q,\varepsilon k) &= \varepsilon^{1-\alpha}f_{s\varepsilon^\alpha}(q,k)+\varepsilon^{1-\alpha+\alpha\wedge(1-\alpha)}r_s^\varepsilon(x,q,k),\\
f_{s\varepsilon^\alpha}(q,k) &:= k\cdot((t-s\varepsilon^\alpha)\nabla_x\varphi(t(q+q_0),q+q_0)+\nabla_q\varphi(t(q+q_0),q+q_0)),\\
r_s^\varepsilon(x,q,k) &:= (\varepsilon^{0\vee(1-2\alpha)}r_1^\varepsilon+\varepsilon^{0\vee(2\alpha-1)}r_2^\varepsilon)(t(q+q_0),x,(t-s\varepsilon^\alpha)k/2,q+q_0,0,k/2).
\end{aligned}
$$

We have, for any multi-index λ,

$$
|\partial_q^{|\lambda|}f_{s\varepsilon^\alpha}(q,k)|\lesssim|k|\qquad;\qquad|\partial_{x,q}^{|\lambda|}r_s^\varepsilon(x,q,k)|\lesssim|k|(|k|+|x|). \tag{14}
$$

Note that we used above the fact that $0\le s\varepsilon^\alpha\le t$ and $0\le t\le T$. We thus can write:

$$
\left|\int_0^{t\varepsilon^{-\alpha}} e^{-isq_0\cdot k}\mathcal{F}(f^\varepsilon_{s\varepsilon^\alpha}a)(k,sk,\varepsilon^{1-\alpha}k)\,\mathrm{d}s\right|^2 = \varepsilon^{2(1-\alpha)}|L^\varepsilon(t,k)|^2+\varepsilon^{2(1-\alpha)+\alpha\wedge(1-\alpha)}R^\varepsilon(t,k),
$$

where, with $\hat{k} := \frac{k}{|k|}$,

$$\begin{aligned}
L^\varepsilon(t,k) &= \int_0^{t\varepsilon^{-\alpha}} e^{-isq_0\cdot k}\mathcal{F}(f_{s\varepsilon^\alpha}a)(k,sk,\varepsilon^{1-\alpha}k)\,\mathrm{d}s,\\
&= \frac{1}{|k|}\int_0^{t\varepsilon^{-\alpha}|k|} e^{-isq_0\cdot\hat{k}}\mathcal{F}(f_{s\varepsilon^\alpha}a)(k,s\hat{k},\varepsilon^{1-\alpha}k)\,\mathrm{d}s,\\
R^\varepsilon(t,k) &= \varepsilon^{\alpha\wedge(1-\alpha)}\left|\int_0^{t\varepsilon^{-\alpha}} e^{-isq_0\cdot k}\mathcal{F}(r_s^\varepsilon a)(k,sk,\varepsilon^{1-\alpha}k)\,\mathrm{d}s\right|^2\\
&\quad +2\mathrm{Re}\int_0^{t\varepsilon^{-\alpha}} e^{-isq_0\cdot k}\mathcal{F}(f_{s\varepsilon^\alpha}a)(k,sk,\varepsilon^{1-\alpha}k)\,\mathrm{d}s\\
&\qquad \times\int_0^{t\varepsilon^{-\alpha}} e^{iuq_0\cdot k}\overline{\mathcal{F}(r_u^\varepsilon a)}(k,uk,\varepsilon^{1-\alpha}k)\,\mathrm{d}u.
\end{aligned}$$

Using (12) and lemma 5.2 with $f(x,q,k) = f_{s\varepsilon^\alpha}(q,k), r_s^\varepsilon(x,q,k)$ and $g = a$ yields for instance

$$|\mathcal{F}(f_{s\varepsilon^\alpha}a)(k,s\hat{k},\varepsilon^{1-\alpha}k)| + |\mathcal{F}(r_s^\varepsilon a)(k,s\hat{k},\varepsilon^{1-\alpha}k)| \lesssim \frac{|k|}{(1+|k|^{d+1})(1+s^{d+1})}. \quad (15)$$

Consequently,

$$|L^\varepsilon(t,k)| \le \frac{1}{1+|k|^{d+1}}\int_0^\infty \frac{\mathrm{d}s}{1+s^{d+1}}, \quad (16)$$

with equivalent relation for R^ε. This shows first that $|k|^{-\delta}S(k\varepsilon^{1-\alpha})R^\varepsilon(t,k)$ is integrable in k with a bound independent of ε, for any $0 \le t \le T$, so that the remainder is of order $\varepsilon^{(d-\delta)(1-\alpha)+1+\alpha\wedge(1-\alpha)}$ uniformly in time.

Regarding the leading term, we first fix $|k| > 0$, and $0 < t \le T$. Using dominated convergence, thanks to estimates (15) above, we have,

$$L^\varepsilon(t,k) \to \frac{1}{|k|}\int_0^\infty e^{-isq_0\cdot\hat{k}}\mathcal{F}(f_0a)(k,s\hat{k},0)\,\mathrm{d}s, \quad \varepsilon\to 0.$$

If $t > t_0$, where t_0 is independent of ε, then the convergence is uniform with respect to t. We then pass to the limit in (13) using dominated convergence and (16). I.e.

$$\begin{aligned}
\varepsilon^{-(d-\delta)(1-\alpha)-1}w^\varepsilon &\to \frac{1}{2^{d-\delta}}\int \frac{S(0)}{|k|^\delta}\lim_{\varepsilon\to 0} L^\varepsilon(t,k)\,\mathrm{d}k\\
&= \frac{1}{2^{d-\delta}}\int \frac{S(0)}{|k|^\delta}\left|\int_0^\infty e^{-isq_0\cdot k}\mathcal{F}(f_0a)(k,sk,0)\,\mathrm{d}s\right|^2\,\mathrm{d}k.
\end{aligned}$$

Therefore, $\varepsilon^{-(d-\delta)(1-\alpha)-1}w^\varepsilon(t)$ converges pointwise $(0,T]$ and uniformly on $[t_0,T]$ to the expression above.

4.4 The case $\beta = 0$, $\alpha = 1$.

Since this case is very similar to the previous one, we only underline the main differences. Directly starting from (13) with $\alpha = 1$, we use here the expansion

$$\begin{aligned}
f^\varepsilon_{s\varepsilon}(x,q,k) &= \sum_\pm \pm\varphi(x\varepsilon + t(q+q_0) \pm (t-s\varepsilon)k/2, q+q_0 \pm k/2) \\
&= \sum_\pm \pm\varphi(t(q+q_0) \pm (t-s\varepsilon)k/2, q+q_0 \pm k/2) \\
&\qquad + \varepsilon \sum_\pm \pm \int_0^1 x\cdot\nabla_x\varphi(x\varepsilon\tau + t(q+q_0) \pm (t-s)k/2, q+q_0 \pm k/2)\,\mathrm{d}\tau, \\
&:= f_{s\varepsilon}(q,k) + \varepsilon\, r^\varepsilon_s(x,q,k),
\end{aligned}$$

and have consequently the following estimates, valid for any multi-index λ.

$$|\partial_q^{|\lambda|} f_{s\varepsilon}(q,k)| \lesssim |k| \qquad ; \qquad |\partial_{x,q}^{|\lambda|} r^\varepsilon_s(x,q,k)| \lesssim |x||k|. \tag{17}$$

We then obtain the following decomposition into leading and negligible terms:

$$\left|\int_0^{t\varepsilon^{-1}} e^{-isq_0\cdot k}\mathcal{F}(f^\varepsilon_{s\varepsilon}a)(k,sk,k)\,\mathrm{d}s\right|^2 \;=\; |L^\varepsilon(t,k)|^2 + \varepsilon R^\varepsilon(t,k),$$

where

$$\begin{aligned}
L^\varepsilon(t,k) &= \int_0^{t\varepsilon^{-1}} e^{-isq_0\cdot k}\mathcal{F}(f_{s\varepsilon}a)(k,sk,k)\,\mathrm{d}s, \\
R^\varepsilon(t,k) &= \varepsilon\left|\int_0^{t\varepsilon^{-1}} e^{-isq_0\cdot k}\mathcal{F}(r^\varepsilon_s a)(k,sk,k)\,\mathrm{d}s\right|^2 \\
&\quad + 2\mathrm{Re}\int_0^{t\varepsilon^{-1}} e^{-isq_0\cdot k}\mathcal{F}(f_{s\varepsilon}a)(k,sk,k)\,\mathrm{d}s \int_0^{t\varepsilon^{-1}} e^{iuq_0\cdot k}\overline{\mathcal{F}(r^\varepsilon_u a)}(k,uk,k)\,\mathrm{d}u.
\end{aligned}$$

As before, (17) and lemma 5.2 give majorizing functions to apply the Lebesgue dominated convergence theorem. It follows that the remainder involving R^ε is of overall order ε^2 and can be neglected. Passing to the limit in the leading term then gives, pointwise in $(0,T]$ and uniformly on $[t_0,T]$,

$$\varepsilon^{-1}w^\varepsilon \to \frac{1}{2^{d-\delta}}\int \frac{S(k)}{|k|^\delta}\left|\int_0^\infty e^{-isq_0\cdot k}\mathcal{F}(f_{s\varepsilon}a)(k,sk,k)\,\mathrm{d}s\right|^2 \mathrm{d}k.$$

4.5 The case $0 < \beta < \alpha < 1$.

After the change of variable $s \mapsto s\varepsilon^{\alpha-\beta}$, (9) becomes:

$$\begin{aligned}
w^\varepsilon(t) = \frac{\varepsilon^{(d-\delta)(1-\alpha)-1+2(\alpha-\beta)}}{2^{d-\delta}} \int \frac{S(k\varepsilon^{1-\alpha})}{|k|^\delta} \\
\times\left|\int_0^{t\varepsilon^{\beta-\alpha}} e^{-i\varepsilon^{-\beta}sq_0\cdot k}\mathcal{F}(f^\varepsilon_{s\varepsilon^{\alpha-\beta}}a)(k,sk,\varepsilon^{1-\alpha}k)\,\mathrm{d}s\right|^2 \mathrm{d}k.
\end{aligned} \tag{18}$$

Applying lemma 5.1 to $f^\varepsilon_{s\varepsilon^{\alpha-\beta}}$ with $y = tq_0$, $\alpha_1 = \alpha_2 = \beta$, $y_1 = tq + x\varepsilon^{\alpha-\beta}$, $\beta_1 = \beta_2 = 1-\alpha$, $y_2 = (t - s\varepsilon^{\alpha-\beta})k/2$, $p = q_0$, $p_1 = q$, $p_2 = k/2$, we find:

$$
\begin{aligned}
f^\varepsilon_{s\varepsilon^{\alpha-\beta}}(x,q,k) &= \varepsilon^{1-\alpha} f_{s\varepsilon^{\alpha-\beta}}(k) + \varepsilon^{1-\alpha+\beta\wedge(1-\alpha)} r^\varepsilon_s(x,q,k),\\
f_{s\varepsilon^{\alpha-\beta}}(k) &= k\cdot((t-s\varepsilon^{\alpha-\beta})\nabla_x\varphi(tq_0,q_0) + \nabla_q\varphi(tq_0,q_0)),\\
r^\varepsilon_s(x,q,k) &= (\varepsilon^{0\vee(1-\alpha-\beta)} r^\varepsilon_1 + (\varepsilon^{0\vee(\alpha+\beta-1)}(r^\varepsilon_2 + r^\varepsilon_3))\\
&\qquad\qquad (tq_0, tq + x\varepsilon^{\alpha-\beta}, (t-s\varepsilon^{\alpha-\beta})k/2, q_0, q, k/2),
\end{aligned}
$$

where, for any multi-index λ,

$$
|f_{s\varepsilon^{\alpha-\beta}}(k)| \lesssim |k| \qquad ; \qquad |\partial^{|\lambda|}_{x,q} r^\varepsilon_s(x,q,k)| \lesssim |k|(|k|+|x|+|q|). \tag{19}
$$

Note that we used above the facts that $s\varepsilon^{\alpha-\beta} \leq t$ and $0 \leq t \leq T$. We now decompose k on an orthonormal basis of $\mathbb{R}^d$ such that $k = k_\parallel \hat{q_0} + k_\perp$, $k_\parallel \in \mathbb{R}$, $k_\perp \in \mathbb{R}^d$, with $q_0 \cdot k_\perp = 0$ and denote by $k = (k_\parallel, k^2_\perp, \cdots, k^d_\perp)$ its corresponding components. Performing the change of variable $k_\parallel \mapsto \varepsilon^\beta k_\parallel$ and defining $k_\varepsilon := (\varepsilon^\beta k_\parallel, k^2_\perp, \cdots, k^d_\perp)$, (18) becomes

$$
\begin{aligned}
w^\varepsilon(t) &= \frac{\varepsilon^{(d-\delta)(1-\alpha)-1+2(\alpha-\beta)+\beta}}{2^{d-\delta}}\\
&\qquad \times \int \frac{S(k_\varepsilon \varepsilon^{1-\alpha})}{|k_\varepsilon|^\delta} \left| \int_0^{t\varepsilon^{\beta-\alpha}} e^{-is|q_0|k_\parallel} \mathcal{F}(f^\varepsilon_{s\varepsilon^{\alpha-\beta}} a)(k_\varepsilon, sk_\varepsilon, \varepsilon^{1-\alpha}k_\varepsilon)\,\mathrm{d}s \right|^2 \mathrm{d}k,\\
&= \frac{\varepsilon^{(d-\delta)(1-\alpha)+1-\beta}}{2^{d-\delta}} \int \frac{S(k_\varepsilon \varepsilon^{1-\alpha})}{|k_\varepsilon|^\delta} \left(|L^\varepsilon(t,k_\varepsilon)|^2 + \varepsilon^{\beta\wedge(1-\alpha)} R^\varepsilon(t,k_\varepsilon)\right) dk,\\
L^\varepsilon(t,k_\varepsilon) &= \int_0^{t\varepsilon^{\beta-\alpha}} e^{-is|q_0|k_\parallel} \mathcal{F}(f_{s\varepsilon^{\alpha-\beta}} a)(k_\varepsilon, sk_\varepsilon, \varepsilon^{1-\alpha}k_\varepsilon)\,\mathrm{d}s,\\
&= \frac{1}{|k_\varepsilon|} \int_0^{t\varepsilon^{\beta-\alpha}|k^\varepsilon|} e^{-is|q_0|k_\parallel|k^\varepsilon|^{-1}} \mathcal{F}(f_{s\varepsilon^{\alpha-\beta}} a)(k_\varepsilon, s\hat{k}_\varepsilon, \varepsilon^{1-\alpha}k_\varepsilon)\,\mathrm{d}s,\\
R^\varepsilon(t,k_\varepsilon) &= \varepsilon^{\beta\wedge(1-\alpha)} \left| \int_0^{t\varepsilon^{\beta-\alpha}} e^{-is|q_0|k_\parallel} \mathcal{F}(r^\varepsilon_s a)(k_\varepsilon, sk_\varepsilon, \varepsilon^{1-\alpha}k_\varepsilon)\,\mathrm{d}s \right|^2\\
&\quad + 2\mathrm{Re} \int_0^{t\varepsilon^{\beta-\alpha}} e^{-is|q_0|k_\parallel} \mathcal{F}(f_{s\varepsilon^{\alpha-\beta}} a)(k_\varepsilon, sk_\varepsilon, \varepsilon^{1-\alpha}k_\varepsilon)\,\mathrm{d}s\\
&\qquad \times \int_0^{t\varepsilon^{\beta-\alpha}} e^{iu|q_0|k_\parallel} \overline{\mathcal{F}(r^\varepsilon_u a)}(k_\varepsilon, uk_\varepsilon, \varepsilon^{1-\alpha}k_\varepsilon)\,\mathrm{d}u.
\end{aligned}
$$

Using (19) and lemma 5.2 with $f(x,q,k) = f_{s\varepsilon^{\alpha-\beta}}(k), r^\varepsilon_s(x,q,k)$ and $g = a$ gives the estimates, for $\lambda = 0, 1$:

$$
\begin{aligned}
|\partial^\lambda_s \mathcal{F}(f_{s\varepsilon^{\alpha-\beta}} a)(k_\varepsilon, s\hat{k}_\varepsilon, \varepsilon^{1-\alpha}k_\varepsilon)| &\lesssim \frac{|k_\varepsilon|}{(1+|k_\varepsilon|^{d+2})(1+s^{d+2})}, \qquad (20)\\
|\partial^\lambda_s \mathcal{F}(r^\varepsilon_s a)(k_\varepsilon, s\hat{k}_\varepsilon, \varepsilon^{1-\alpha}k_\varepsilon)| &\lesssim \frac{|k_\varepsilon|}{(1+|k_\varepsilon|^{d+2})(1+s^{d+2})}.
\end{aligned}
$$

Using the latter relations and the fact that, for any $f \in \mathcal{C}^1(\mathbb{R})$,

$$\begin{aligned}
&\frac{k_\parallel}{|k_\varepsilon|}\int_0^{t\varepsilon^{\beta-\alpha}/|k_\varepsilon|} e^{-is|q_0|k_\parallel/|k_\varepsilon|} f(s)ds \\
&\quad = \frac{i}{|q_0|}\left(f(t\varepsilon^{\beta-\alpha}|k_\varepsilon|)e^{-it\varepsilon^{\beta-\alpha}|q_0|k_\parallel/|k_\varepsilon|} - f(0) - \int_0^{t\varepsilon^{\beta-\alpha}/|k_\varepsilon|} e^{-is|q_0|k_\parallel/|k_\varepsilon|} f'(s)ds\right),
\end{aligned}$$

we find the uniform estimate, after the change of variable $s \mapsto s|k_\varepsilon|^{-1}$ in R^ε:

$$|L^\varepsilon(t,k_\varepsilon)|^2 + |R^\varepsilon(t,k_\varepsilon)| \lesssim \frac{1}{(1+|k_\parallel|^2)(1+|k_\perp|^{2d})}.$$

The end of the proof is now identical to that of section 4.3: using dominated convergence and the estimates above, we pass first to the limit in L^ε and then in w^ε and obtain the convergence of $\varepsilon^{-(d-\delta)(1-\alpha)-1+\beta} w^\varepsilon$ pointwise on $(0,T]$ and uniformly on $[t_0,T]$. Using the Fourier Plancherel equality and denoting by $k_0 := (0, k_\perp^2, \cdots, k_\perp^d)$ and $dk_\perp := dk_\perp^2 \cdots dk_\perp^n$:

$$\begin{aligned}
&\frac{1}{2^{d-\delta}}\int \frac{S(k_0)}{|k_0|^\delta}\left|\int_0^\infty e^{-is|q_0|\cdot k_\parallel}\mathcal{F}(f_0 a)(k_0, sk_0, 0)\,\mathrm{d}s\right|^2 \mathrm{d}k \\
&\qquad = \frac{2\pi}{2^{d-\delta}|q_0|}\int_{\mathbb{R}^{d-1}}\int_0^\infty \frac{S(k_0)}{|k_0|^\delta}|\mathcal{F}(f_0 a)(k_0, sk_0, 0)|^2\,\mathrm{d}s\,\mathrm{d}k_\perp, \\
&\qquad = \frac{2\pi}{2^{d-\delta}|q_0|}\int_{\{q_0\cdot k=0\}}\int_0^\infty \frac{S(k)}{|k|^\delta}|\mathcal{F}(f_0 a)(k, sk, 0)|^2\,\mathrm{d}s\,\mathrm{d}k.
\end{aligned}$$

4.6 The case $0 < \alpha \le \beta < 1$.

That configuration is very close to the previous one, so that we leave some details to the reader. We start from expression (9) and apply lemma 5.1 to f_s^ε with $y = tq_0$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $y_1 = tq\varepsilon^{\beta-\alpha} + x$, $\beta_1 = \beta_2 = 1-\alpha$, $y_2 = (t-s)k/2$, $p = q_0$, $p_1 = q$, $p_2 = k/2$ and find:

$$\begin{aligned}
f_s^\varepsilon(x,q,k) &= \varepsilon^{1-\alpha} f_s(k) + \varepsilon^{1-\alpha+\alpha\wedge(1-\alpha)} r_s^\varepsilon(x,q,k), \\
f_s(k) &= k\cdot((t-s)\nabla_x\varphi(tq_0,q_0) + \nabla_q\varphi(tq_0,q_0)), \\
r_s^\varepsilon(x,q,k) &= (\varepsilon^{0\vee(1-2\alpha)} r_1^\varepsilon + \varepsilon^{0\vee(2\alpha-1)} r_2^\varepsilon + \varepsilon^{\beta-\alpha\wedge(1-\alpha)} r_3^\varepsilon) \\
&\qquad\qquad (tq_0, tq\varepsilon^{\beta-\alpha} + x, (t-s)k/2, q_0, q, k/2),
\end{aligned}$$

where, for any multi-index λ,

$$|f_s(k)| \lesssim |k| \qquad ; \qquad |\partial_{x,q}^{|\lambda|} r_s^\varepsilon(x,q,k)| \lesssim |k|(|k| + |x| + |q|).$$

We introduce as well $k = k_\parallel \hat{q}_0 + k_\perp$, $k_\parallel \in \mathbb{R}$, $k_\perp \in \mathbb{R}^d$, with $q_0 \cdot k_\perp = 0$. Performing the change of variable $k_\parallel \mapsto \varepsilon^\alpha k_\parallel$ and denoting by $k_\varepsilon = (\varepsilon^\alpha k_\parallel, k_\perp^2, \cdots, k_\perp^d)$, (9) becomes

$$\begin{aligned}
w^\varepsilon(t) &= \frac{\varepsilon^{(d-\delta)(1-\alpha)-1+\alpha}}{2^{d-\delta}} \\
&\qquad \times \int \frac{S(k_\varepsilon \varepsilon^{1-\alpha})}{|k_\varepsilon|^\delta} \left| \int_0^t e^{-is|q_0|k_\parallel} \mathcal{F}(f_s^\varepsilon a)(k_\varepsilon, \varepsilon^{\beta-\alpha} s k_\varepsilon, \varepsilon^{1-\alpha} k_\varepsilon) \,\mathrm{d}s \right|^2 \mathrm{d}k, \\
&= \frac{\varepsilon^{(d-\delta)(1-\alpha)+1-\alpha}}{2^{d-\delta}} \int \frac{S(k_\varepsilon \varepsilon^{1-\alpha})}{|k_\varepsilon|^\delta} \left(|L^\varepsilon(t, k_\varepsilon)|^2 + \varepsilon^{(1-\alpha)\wedge\alpha} R^\varepsilon(t, k_\varepsilon) \right) dk, \\
L^\varepsilon(t, k_\varepsilon) &= \int_0^t e^{-is|q_0|k_\parallel} \mathcal{F}(f_s a)(k_\varepsilon, \varepsilon^{\beta-\alpha} s k_\varepsilon, \varepsilon^{1-\alpha} k_\varepsilon) \,\mathrm{d}s, \\
R^\varepsilon(t, k_\varepsilon) &= \varepsilon^{(1-\alpha)\wedge\alpha} \left| \int_0^t e^{-is|q_0|k_\parallel} \mathcal{F}(r_s^\varepsilon a)(k_\varepsilon, \varepsilon^{\beta-\alpha} s k_\varepsilon, \varepsilon^{1-\alpha} k_\varepsilon) \,\mathrm{d}s \right|^2 \\
&\quad + 2\mathrm{Re} \int_0^t e^{-is|q_0|k_\parallel} \mathcal{F}(f_s a)(k_\varepsilon, \varepsilon^{\beta-\alpha} s k_\varepsilon, \varepsilon^{1-\alpha} k_\varepsilon) \,\mathrm{d}s \\
&\qquad \times \int_0^t e^{iu|q_0|k_\parallel} \overline{\mathcal{F}(r_u^\varepsilon a)}(k_\varepsilon, \varepsilon^{\beta-\alpha} u k_\varepsilon, \varepsilon^{1-\alpha} k_\varepsilon) \,\mathrm{d}u.
\end{aligned}$$

Passing to the limit, then gives, uniformly for $t \in [0, T]$, if $\beta > \alpha$, with $k_0 := (0, k_\perp^2, \cdots, k_\perp^d)$ and $dk_\perp := dk_\perp^2 \cdots dk_\perp^n$:

$$\begin{aligned}
\varepsilon^{-(d-\delta)(1-\alpha)-1+\alpha} w^\varepsilon \to \frac{1}{2^{d-\delta}} \int \frac{S(k_0)}{|k_0|^\delta} \left| \int_0^t e^{-is|q_0|k_\parallel} \mathcal{F}(f_s a)(k, 0, 0) \,\mathrm{d}s \right|^2 \mathrm{d}k \\
= \frac{2\pi}{2^{d-\delta}|q_0|} \int_{\mathbb{R}^{d-1}} \int_0^t \frac{S(k_0)}{|k_0|^\delta} |\mathcal{F}(f_s a)(k_0, 0, 0)|^2 \,\mathrm{d}s \,\mathrm{d}k_\perp, \\
= \frac{2\pi}{2^{d-\delta}|q_0|} \int_{\{q_0 \cdot k = 0\}} \int_0^t \frac{S(k)}{|k|^\delta} |\mathcal{F}(f_s a)(k, 0, 0)|^2 \,\mathrm{d}s \,\mathrm{d}k,
\end{aligned}$$

and, if $\alpha = \beta$,

$$\varepsilon^{-(d-\delta)(1-\alpha)-1+\alpha} w^\varepsilon \to \frac{2\pi}{2^{d-\delta}|q_0|} \int_{\{q_0 \cdot k = 0\}} \int_0^t \frac{S(k)}{|k|^\delta} |\mathcal{F}(f_s a)(k, sk, 0)|^2 \,\mathrm{d}s \,\mathrm{d}k.$$

4.7 The case $\alpha = \beta = 1$.

The proof is identical to the previous case with $\alpha = \beta$, only the expansion of f_s^ε changes. We obtain

$$w^\varepsilon \to \frac{2\pi}{2^{d-\delta}|q_0|} \int_{\{q_0 \cdot k = 0\}} \int_0^t \frac{S(k)}{|k|^\delta} |\mathcal{F}(f_s a)(k, sk, k)|^2 \,\mathrm{d}s \,\mathrm{d}k,$$

where $f_s(k) = \sum_\pm \pm\varphi(tq_0 \pm (t-s)k, q_0 \pm k)$.

4.8 Proof of proposition 3.2.

When $\alpha = \beta = 0$ in (6), and considering test functions of the form

$$\frac{1}{\varepsilon^{d(\gamma_1+\gamma_2)}} \varphi\left(\frac{x}{\varepsilon^{\gamma_1}}, \frac{q - q_0}{\varepsilon^{\gamma_2}}\right),$$

it follows from (8) that

$$\begin{aligned} w^\varepsilon(t) &= \frac{\varepsilon^{(d-\delta)(1-\gamma_1)-1}}{2^{d-\delta}} \int \mathrm{d}k\, \frac{S(k\varepsilon^{1-\gamma_1})}{|k|^\delta} \\ &\qquad \times \left| \int_0^t e^{-i\varepsilon^{-\gamma_1} s q_0\cdot k} \mathcal{F}(\tilde{f}_s^\varepsilon \varphi)(-k, \varepsilon^{\gamma_2-\gamma_1} sk, \varepsilon^{1-\gamma_1}k)\,\mathrm{d}s \right|^2, \\ \tilde{f}_s^\varepsilon(x,q,k) &= \sum_{\pm} \pm a(x\varepsilon^{\gamma_1} - tq\varepsilon^{\gamma_2} - tq_0 \mp (t-s)k/2,\, q\varepsilon^{\gamma_2} + q_0 \pm k/2). \end{aligned}$$

By identification, we get the same expression as (9)-(10) with α replaced by γ_1, β by γ_2 and $f_s^\varepsilon a$ by $\tilde{f}_s^\varepsilon \varphi$. It then suffices to follow the same analysis as that of the theorem to obtain the order of the scintillation. When $\alpha = \gamma_2 = 0$, we find:

$$\begin{aligned} w^\varepsilon(t) &= \frac{\varepsilon^{(d-\delta)(1-\gamma_1)-1}}{2^{d-\delta}} \int \mathrm{d}k\, \frac{S(k\varepsilon^{1-\gamma_1})}{|k|^\delta} \\ &\qquad \times \left| \int_0^t e^{-i\varepsilon^{-\gamma_1} s q_0\cdot k} \mathcal{F}(g_s^\varepsilon)(-k, \varepsilon^{\beta-\gamma_1} sk, \varepsilon^{1-\gamma_1}k)\,\mathrm{d}s \right|^2, \\ g_s^\varepsilon(x,q,k) &= \sum_{\pm} \pm a(x\varepsilon^{\gamma_1} - tq\varepsilon^{\beta} - tq_0 \mp (t-s)k/2,\, q)\varphi(x, q\varepsilon^{\beta} + q_0 \mp k/2), \end{aligned}$$

and here again we proceed by identification.

5 Some technical lemmas

The following two lemmas are extensively used in the proof. The first one stems from a simple application of the Taylor formula and the second one from standard properties of the Fourier transform of functions in $\mathcal{S}(\mathbb{R}^{2d})$.

Lemma 5.1 *Let $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$, $z = (y, y_1, y_2, p, p_1, p_2) \in \mathbb{R}^{6d}$ and $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{R}^4$. Then, we have:*

$$\begin{aligned} &\varphi(y + \varepsilon^{\alpha_1} y_1 + \varepsilon^{\beta_1} y_2, p + \varepsilon^{\alpha_2} p_1 + \varepsilon^{\beta_2} p_2) - \varphi(y + \varepsilon^{\alpha_1} y_1 - \varepsilon^{\beta_1} y_2, p + \varepsilon^{\alpha_2} p_1 - \varepsilon^{\beta_2} p_2) \\ &= 2\varepsilon^{\beta_1} y_2 \cdot \nabla_y \varphi(y,p) + 2\varepsilon^{\beta_2} p_2 \cdot \nabla_p \varphi(y,p) + \varepsilon^{2\beta_1\wedge\beta_2} r_1^\varepsilon + \varepsilon^{\alpha_1+\beta_1\wedge\beta_2} r_2^\varepsilon + \varepsilon^{\alpha_2+\beta_1\wedge\beta_2} r_3^\varepsilon, \end{aligned}$$

where $r_i^\varepsilon := r_i^\varepsilon(z) \in \mathcal{C}^\infty(\mathbb{R}^{6d})$, $i = 1, 2, 3$, and satisfies, for any multi-index λ and any $z \in \mathbb{R}^{6d}$,

$$\begin{aligned} |\partial_z^{|\lambda|} r_1^\varepsilon(z)| &\lesssim |y_2|^2 + |p_2|^2, \\ |\partial_z^{|\lambda|} r_2^\varepsilon(z)| &\lesssim |y_1|(|y_2| + |p_2|), \\ |\partial_z^{|\lambda|} r_3^\varepsilon(z)| &\lesssim |p_1|(|y_2| + |p_2|). \end{aligned}$$

Below we use $f_\eta \lesssim g$ to denote inequality up to a constant independent of the parameter η. Our application will see η as some combination of ε and all other variables not explicitly written into the right hand side of the inequality.

Lemma 5.2 *Suppose $f_\eta(x,q,z) \in \mathcal{C}^\infty(\mathbb{R}^{(2+n)d})$, with $z \in \mathbb{R}^{nd}$, satisfies, for any $(x,q,z) \in \mathbb{R}^{(2+n)d}$ and any multi-indexes λ and μ, $|\partial_x^{|\lambda|}\partial_q^{|\mu|} f_\eta(x,q,z)| \lesssim |z|^\gamma(1+|x|^{\sigma_1}+|q|^{\sigma_2})$, for some σ_1, σ_2 and γ positive. Let $g(x,q) \in \mathcal{S}(\mathbb{R}^{2d})$. Then with $\mathcal{F}(f_\eta g)(\xi,\nu,z)$ denoting the Fourier transform of the product with respect to x and q, for any multi-indexes i and j and any $k > 0$,*

$$|\partial_\xi^{|i|}\partial_\nu^{|j|}\mathcal{F}(f_\eta g)(\xi,\nu,z)| \lesssim |z|^\gamma \langle\xi\rangle^{-k}\,\langle\nu\rangle^{-k}.$$

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