Combined source and attenuation reconstructions in SPECT

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ABSTRACT. We consider the simultaneous reconstruction of the absorption coefficient and the source term in a linear transport equation from available boundary measurements. This problem finds applications in SPECT, a medical imaging modality. When the absorption coefficient is known, recently derived inversion formulas for the attenuated Radon transform can be used to reconstruct the source term. Moreover, the measurements needs to satisfy some compatibility conditions, which fully characterize the range of the attenuated Radon transform. In this paper, we explore this compatibility condition to obtain information about the absorption coefficient. We consider a linearization of the compatibility condition and show that the absorption term is uniquely determined, partially determined, or fully undetermined, depending on the structure of the source term.

1. Introduction

Single Photon Emission Computerized Tomography (SPECT) is an important medical imaging modality. Radioactive markers that specifically attach to certain molecules we are interested in imaging are injected into tissues. They emit γ particles by radioactive decay, which may be partially absorbed by the underlying tissues or escape the domain through its boundary where they are measured by γ-cameras. The phase-space (position and direction) density of γ particles is modeled by the solution $u(x,\theta)$ of the following linear transport equation with absorption:

$$(1.1) \qquad \theta \cdot \nabla_x u + a(x)u = f(x), \qquad x \in X \subset \mathbb{R}^2, \quad \theta \in \mathbb{S}^1,$$

where f is the source of γ particles and a is absorption. The measurements are then given by $u(x,\theta)$ for x at the boundary ∂X of the domain $X \subset \mathbb{R}^2$ and all $\theta \in \mathbb{S}^1$.

The available data are therefore given by the attenuated Radon transform (AtRT) of f. When a is known, an explicit reconstruction formula for f was recently derived in [**9, 10**] and can also be deduced from the work in [**2, 4**]. This formula is accompanied by a compatibility condition that the AtRT needs to satisfy. This paper aims to exploit this compatibility condition and the explicit expression for f when a is known to obtain two independent equations coupling a and f. In [**10**], it is shown that the latter compatibility condition is a necessary and sufficient condition in an appropriate functional setting. The latter two equations are therefore all that can be learned from the available boundary measurements.

Several results exist on the combined reconstruction of the absorption and source terms in SPECT. In [**8**], range conditions for the attenuated Radon transform are also used to obtain constraints that the absorption term needs to satisfy. Reconstructions are then presented assuming that f is a finite sum of Dirac measures. Accurate numerical reconstructions of both coefficients are also presented in [**12**]. Several results consider the case of constant attenuation and obtain uniqueness and non-uniqueness results for the Exponential Radon transform (ERT) which is a slightly different problem than that of the AtRT [**5, 6, 13**] (results in [**5**] are based on the range characterization of the ERT [**1, 7**]). To the best of our knowledge, the use of the natural compatibility condition obtained in [**10**] (see also [**3**] for compatibility conditions of more general sources) to obtain information about the source term has not been considered before.

The rest of the paper is structured as follows. Section 2 presents the compatibility condition for the AtRT. This provides a nonlinear integral equation for the absorption coefficient that is difficult to analyze. We thus linearize the nonlinear functional for the absorption coefficient in the vicinity of a vanishing absorption. The resulting operator is bilinear in the source term f and the absorption term a. This equation should be coupled with the reconstruction formula providing f when a is known. We thus obtain a system of two equations for two unknown. It turns out that the equation for a coming from the linearization of the compatibility condition does not always uniquely characterize a. In some cases, we show that all of a may be reconstructed. In other cases, we show that only part of a can be reconstructed. For certain sources f, we show that arbitrary compactly supported a makes the whole bilinear functional (a linear operator in a for a fixed f) vanish so that it provides no information about a whatsoever. The non-uniqueness results are presented in section 3 while the uniqueness results are given in section 4. These results provide a partial answer to the combined reconstruction of the coefficients (a, f). They have the advantage that they fully use the redundancy in the AtRT data to obtain information about the absorption term a. Unfortunately, the lack of unique reconstructions of a for given values of f shows that complete reconstructions can only be expected in favorable situations, although a complete description of such favorable situations remains to be done. Some conclusions and perspectives are offered in section 5.

2. Consistency condition and linearization

2.1. Nonlinear consistency condition. Let us consider (1.1) and boundary measurements at the boundary ∂X of the domain X given by the following Attenuated Radon Transform (AtRT)

$$(2.1) \qquad P_{a,\theta}f(x \cdot \theta^\perp) = \int_{-\infty}^{+\infty} e^{-\int_0^{+\infty} a(x+(t+s)\theta)ds} f(x+t\theta)dt,$$

where a and f are extended by 0 outside of X or X is simply considered as the whole of $\mathbb{R}^2$. The reconstruction of f from (2.1) for a known absorption coefficient a is treated in, e.g., [**2, 3, 4, 9, 10**]. Moreover, the data $P_{a,\theta}f(s)$ satisfy some compatibility conditions [**3, 10**] that may be seen as a generalization of the condition $P_\theta f(s) = P_{-\theta}f(-s)$ when $a \equiv 0$, which essentially states that the integral of a function along a line does not depend on the choice of orientation for the line. In the presence of absorption, the weight depends on the direction along which f is

integrated along the line and so the compatibility condition is significantly more complicated and was first obtained in [**9**]. The reconstruction formula and the compatibility condition take the form

$$(2.2) \qquad f(x) \;=\; \frac{1}{4\pi} \int_{\mathbb{S}^1} \theta^\perp \cdot \nabla_x (T_{a,\theta}(C_{a,\theta} + S_{a,\theta}) e^{\frac{P_\theta a}{2}} P_{a,\theta} f)(x \cdot \theta^\perp) d\theta$$

$$(2.3) \qquad 0 \;=\; \int_{\mathbb{S}^1} (T_{a,\theta}(C_{a,\theta} + S_{a,\theta}) e^{\frac{P_\theta a}{2}} P_{a,\theta} f)(x \cdot \theta^\perp) d\theta,$$

respectively, where $y^\perp = (-y_2, y_1)$ for $y = (y_1, y_2) \in \mathbb{R}^2$, and where we have defined the following operators

$$(2.4) \qquad T_{a,\theta} g(x) \;=\; e^{-D_\theta a(x)} g(x),$$

$$(2.5) \qquad C_{a,\theta} g(x) \;=\; \cos\big(\frac{HP_\theta a(x \cdot \theta^\perp)}{2}\big) H\Big(\cos\big(\frac{HP_\theta a}{2}\big) g \Big)(x \cdot \theta^\perp),$$

$$(2.6) \qquad S_{a,\theta} g(x) \;=\; \sin\big(\frac{HP_\theta a(x \cdot \theta^\perp)}{2}\big) H\Big(\sin\big(\frac{HP_\theta a}{2}\big) g \Big)(x \cdot \theta^\perp),$$

$$(2.7) \qquad D_\theta a(x) \;=\; \frac{1}{2}\Big(\int_0^{+\infty} a(x - t\theta) dt - \int_0^{+\infty} a(x + t\theta) dt \Big),$$

$$(2.8) \qquad P_\theta f(s) \;=\; \int_{-\infty}^{+\infty} f(t\theta + s\theta^\perp) dt,$$

$$(2.9) \qquad Hf(s) \;=\; \frac{1}{\pi} \text{ p.v.} \int_{\mathbb{R}} \frac{f(t)}{s - t} dt,$$

for $(x, \theta, s) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$. Above, we recognize Pf as the two-dimensional Radon transform of f, $D_\theta a(x)$ as the symmetrized beam transform of f and $Hf(s)$ as the Hilbert transform of f where p.v. means that the integral is considered as a principal value.

Note that (2.2) provides a reconstruction formula for $f(x)$ when $a(x)$ is known. The compatibility condition (2.3) provides a constraint for all $x \in \mathbb{R}^2$ that a needs to satisfy for the data $P_{a,\theta} f(s)$ to be in the range of the AtRT. Such a compatibility condition is actually a necessary and sufficient condition for $P_{a,\theta} f(s)$ to be in the range of the AtRT in an appropriate functional setting described in [**10**]. In other words, (2.2) and (2.3) provide a complete description of the information that can be obtained from $P_{a,\theta} f(s)$.

2.2. Linearization of the consistency condition. The above system is a two-by-two system of equations for the two unknown coefficients (f, a). Since f is directly written as a functional of a and the data in (2.2), information about a has to be obtained from (2.3). This nonlinear functional for a is rather complicated and we therefore simplify it by linearizing it in the variable a in the vicinity of $a = 0$. We justify the linearization for $(a, f) \in C_0^\infty(\mathbb{R}^2)^2$ (where $C_0^\infty(\mathbb{R}^2)$ denotes the space of infinitely smooth and compactly supported functions on $\mathbb{R}^2$). Using that $|\sin t - t| \leq \frac{|t|^3}{6}$ and $1 - \cos t \leq \frac{t^2}{2}$ for $t \in \mathbb{R}$, we have that

$$(2.10) \qquad \cos\big(\frac{HP_\theta a(s)}{2}\big) \;=\; 1 + O(\|a\|_{C^\alpha}^2), \text{ in } L^\infty(\mathbb{R}_s),$$

$$(2.11) \qquad \sin\big(\frac{HP_\theta a(s)}{2}\big) \;=\; \frac{HP_\theta a(s)}{2} + O(\|a\|_{C^\alpha}^2), \text{ in } L^\infty(\mathbb{R}_s),$$

$$(2.12) \qquad T_{a,\theta}(x) \;=\; 1 - D_\theta a(x) + O(\|a\|_\infty^2), \text{ in } L^\infty(\mathbb{R}_x^2),$$

where $\|a\|_\infty := \sup_{x \in \mathbb{R}^2} |a(x)|$ and $\|a\|_{C^\alpha} := \|a\|_\infty + \sup_{\substack{(x,y) \in \mathbb{R}^2 \\ x \neq y}} \frac{|a(y)-a(x)|}{|y-x|^\alpha}$ for some $\alpha > 0$. Therefore linearizing (2.3) with respect to a small attenuation a we obtain

(2.13)
$$\int_{\mathbb{S}^1} H\big(e^{\frac{P_\theta a(x \cdot \theta^\perp)}{2}} P_{a,\theta} f\big)(x \cdot \theta^\perp) d\theta - \int_{\mathbb{S}^1} D_\theta a(x) H\big(e^{\frac{P_\theta a(x \cdot \theta^\perp)}{2}} P_{a,\theta} f\big)(x \cdot \theta^\perp) d\theta = O(\|a\|_{C^\alpha}^2),$$

in $L^2_{\mathrm{loc}}(\mathbb{R}^2_x)$ (we recall that the Hilbert transform defines a bounded operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for $1 < p < \infty$). Again using the equality

(2.14)
$$P_{a,\theta} f(s) = P_\theta f(s) + O(\|a\|_\infty), \text{ in } L^2(\mathbb{R}_s \times \mathbb{S}^1_\theta),$$

we obtain

(2.15)
$$\int_{\mathbb{S}^1} H P_{a,\theta} f(x \cdot \theta^\perp) d\theta + 2^{-1} \int_{\mathbb{S}^1} H(P_\theta a P_\theta f)(x \cdot \theta^\perp) d\theta \\ - \int_{\mathbb{S}^1} D_\theta a(x) H P_\theta f(x \cdot \theta^\perp) d\theta + O(\|a\|_{C^\alpha}^2) = 0,$$

in $L^2_{\mathrm{loc}}(\mathbb{R}^2_x)$. As the function $(P_\theta a P_\theta f)(s) = (P_{-\theta} a P_{-\theta} f)(-s)$, we obtain by symmetry that $\int_{\mathbb{S}^1} H(P_\theta a P_\theta f)(x \cdot \theta^\perp) d\theta = 0$, which yields

(2.16) $\int_{\mathbb{S}^1} D_\theta a(x) H P_\theta f(x \cdot \theta^\perp) d\theta = \int_{\mathbb{S}^1} H P_{a,\theta} f(x \cdot \theta^\perp) d\theta + O(\|a\|_{C^\alpha}^2), \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^2_x).$

Note that $\int_{\mathbb{S}^1} H P_{a,\theta} f(x \cdot \theta^\perp) d\theta$ is known from the data. The linearization of the compatibility condition (2.3) thus provides an equation for a of the form:

(2.17) $\qquad R_f a(x) := \int_{\mathbb{S}^1} D_\theta a(x) H P_\theta f(x \cdot \theta^\perp) d\theta, \qquad$ known for $x \in \mathbb{R}^2$.

This is a linear integral equation for a whose kernel depends linearly on f.

2.3. Some equivalent formulas. The kernel of R_f strongly depends on the structure of f. Before we consider our non-uniqueness and uniqueness results for (2.17), we recast the equation using several equivalent formulations (assuming that $f \in \mathcal{S}(\mathbb{R}^2)$ and $a \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$, where $\mathcal{S}(\mathbb{R}^2)$ denotes the Schwartz space of infinitely smooth functions f from $\mathbb{R}^2$ to $\mathbb{C}$ such that $\sup_{x \in \mathbb{R}^2} |x|^N |\frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}(x)| < \infty$ for any $N \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, and where $L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$ denotes the space of bounded measurable functions on $\mathbb{R}^2$ that are compactly supported). First, we obtain that

(2.18) $\qquad R_f a(x) = \frac{1}{\pi} \mathrm{p.v.} \int_{\mathbb{R}^4} \frac{a(y) f(z) dy dz}{(x-y)^\perp \cdot (x-z)}.$

This shows that
$$R_f a(x) + R_a f(x) = 0.$$

Upon taking Fourier transforms in $x \to \xi$ above, we obtain that

(2.19) $\qquad \widehat{R_f a}(\xi) = 2\mathrm{p.v.} \int_{\mathbb{R}^2} \frac{\hat{a}(\zeta) \hat{f}(\xi - \zeta)}{\xi^\perp \cdot \zeta} d\zeta,$

where $\hat{g}$ is the Fourier transform of g, $\hat{g}(\xi) = \int_{\mathbb{R}^2} e^{-i\xi \cdot x} g(x) dx$, $\xi \in \mathbb{R}^2$.

Some symmetries in the above expressions can be obtained with a little work. For a vector $x \in \mathbb{R}^2$, we define $\tilde{x} = \frac{x}{|x|}$ its orientation. For $x, y \in \mathbb{R}^2$, $x \neq 0$, we define the vector $\mathrm{sym}(y, \tilde{x})$ by

(2.20) $\qquad \mathrm{sym}(y, \tilde{x}) = (y \cdot \tilde{x})\tilde{x} - (y \cdot \tilde{x}^\perp)\tilde{x}^\perp.$

Then we verify that:

$$(2.21) \quad R_f a(x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}^4} \frac{a(x-y)f(z) - a(x - \text{sym}(y, \tilde{x}))f(\text{sym}(z, \tilde{x}))}{(x-z) \cdot y^{\perp}} dy dz,$$

Note that when $f = f(|z|)$ is a radial source term, then

$$(2.22) \quad R_f a(x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}^4} \left(a(x-y) - a(x - \text{sym}(y, \tilde{x})) \right) \frac{f(|z|)}{(x-z) \cdot y^{\perp}} dy dz.$$

Therefore we obtain that when $f = f(|z|)$ and $a = a(|z|)$, then $R_f a \equiv 0$. This is a first non-uniqueness result that will be generalized in section 3.

In the Fourier domain, the above symmetry condition takes the form

$$(2.23) \quad \widehat{R_f a}(\xi) = \int_{\mathbb{R}^2} \frac{\hat{a}(\xi - \zeta)\hat{f}(\zeta) - \hat{a}(\xi - \text{sym}(\zeta, \tilde{\xi}))\hat{f}(\text{sym}(\zeta, \tilde{\xi}))}{\xi \cdot \zeta^{\perp}} d\zeta.$$

When $f = f(|z|)$ is radial, then we obtain that:

$$(2.24) \quad |\xi| \widehat{R_f a}(\xi) = \int_{\mathbb{R}^2} \frac{\hat{a}(t\tilde{\xi} + s\tilde{\xi}^{\perp}) - \hat{a}(t\tilde{\xi} - s\tilde{\xi}^{\perp})}{s} \hat{f}\left(\sqrt{(|\xi| - t)^2 + s^2} \right) ds dt.$$

The first term in $\hat{a}$ depends on $\tilde{\xi}$ but not on $|\xi|$ while the term in $\hat{f}$ depends on $|\xi|$ but not on $\tilde{\xi}$. We exploit the above formulas to state some uniqueness and non-uniqueness results in the next two sections.

3. Non-uniqueness results

3.1. Non-uniqueness results for the linearized problem for a. We already know that $R_f a$ as a bilinear map of (a, f) has a nontrivial kernel which contains all pairs (a, f) such that both a and f are radial functions. This uses the radial symmetry of $R_f a$ (see (2.22)). We provide new pairs (a, f), where neither a nor f are a priori radial functions, for which $R_f a$ identically vanishes.

For X a subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, we denote by χ_X the indicatrix function of X, i.e. the function from $\mathbb{R}^m$ to $\mathbb{R}$ defined by $\chi_X(y) = 1$ if $y \in X$ and $\chi_X(y) = 0$ otherwise. For $x \in \mathbb{R}^2$ and $(r_1, r_2) \in (0, +\infty)^2$, we denote by $D(x, r_1)$ the closed Euclidean disc of $\mathbb{R}^2$ centered at x with radius r_1, and we denote by $C(x, r_1, r_2)$ (resp. $\partial D(x, r_1)$) the Euclidean annulus $\{y \in \mathbb{R}^2 \mid r_1 \leq |y - x| < r_2\}$ (resp. the Euclidean circle $\{y \in \mathbb{R}^2 \mid |y - x_1| = r_1\}$). We have the following non-uniqueness result.

THEOREM 3.1. *The following statements are valid.*

    i. *Let $f_1 \in L^2((0, +\infty)_r, rdr)$, and assume that there exists $r_0 > 0$ such that $f_1(r) = 0$ for a.e. $r \in (0, r_0)$. Set $f(x) = f_1(|x|)$ for $x \in \mathbb{R}^2$. Then for any $a \in L^{\infty}(\mathbb{R}^2)$ such that $\text{supp} a \subseteq D(0, r_0)$ we have $R_f a \equiv 0$.*

    ii. *Let $f = \delta_{\partial D(0,r)}$ for some $r > 0$. Then $R_f a \equiv 0$ for any $a \in L^{\infty}(\mathbb{R}^2)$ such that $\text{supp} a \subseteq D(0, r)$.*

PROOF OF THEOREM 3.1. We recall that

$$(3.1) \quad I(\alpha) := \text{p.v.} \int_0^{2\pi} \frac{1}{\alpha - \cos(\theta)} d\theta = \begin{cases} \frac{\text{sgn}(\alpha) 2\pi}{\sqrt{\alpha^2 - 1}}, & \text{for } |\alpha| > 1, \\ 0, & \text{for } |\alpha| < 1. \end{cases}$$

We prove item i. Let $a \in L^\infty(\mathbb{R}^2)$, $\operatorname{supp} a \subseteq D(0, r_0)$. Note that

$$
\begin{aligned}
HP_\theta f(s) &= \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{f_1(|z|)}{s - z \cdot \theta^\perp} dz \\
&= \frac{1}{\pi} \int_0^{+\infty} f(r) I(sr^{-1}) dr = 2\operatorname{sgn}(s) \int_0^{|s|} \frac{r f_1(r)}{\sqrt{s^2 - r^2}} dr.
\end{aligned}
$$
(3.2)

Thus $HP_\theta f(x \cdot \theta^\perp) = 0$ for $(x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ such that $|x \cdot \theta^\perp| < r_0$. In addition $D_\theta a(x) = 0$ for $(x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ such that $|x \cdot \theta^\perp| > r_0$ since $\operatorname{supp} a \subseteq D(0, r_0)$. Therefore from (2.17) it follows that $R_f a \equiv 0$. Item i is proved.

Now let $f = \delta_{\partial D(0, r_0)}$ for some $r_0 > 0$. We have

$$
HP_\theta \delta_{\partial D(0, r_0)}(s) = -\frac{2r_0}{\sqrt{s^2 - r_0^2}} \operatorname{sgn}(s) \chi_{(r_0, +\infty)}(|s|), \quad (\theta, s) \in \mathbb{S}^1 \times \mathbb{R}.
$$

Thus $HP_\theta \delta_{\partial D(0, r_0)}(s) = 0$ for $s \in (-r_0, r_0)$, which proves item ii. $\square$

Theorem 3.1 applies to source functions of the form $f = \chi_{C(0, r_1, r_2)}$ for $0 < r_1 < r_2$. In that case we have the explicit formula

$$
P_\theta \chi_{D(0, r)}(s) = 2\sqrt{r^2 - s^2} \chi_{(-r, r)}(s),
$$
(3.3)

$$
HP_\theta \chi_{D(0, r)}(s) = 2\big(s - \operatorname{sgn}(s) \chi_{(r, +\infty)}(|s|) \sqrt{s^2 - r^2}\big),
$$
(3.4)

for $s \in \mathbb{R}$ and $r > 0$.

Note that from (2.18) one can deduce the following translation invariance property:

$$
R_f a(x - x_0) = R_{(\tau_{x_0} f)}(\tau_{x_0} a)(x), \quad \text{for } x \in \mathbb{R}^2,
$$
(3.5)

for some $x_0 \in \mathbb{R}^2$ and where $\tau_{x_0} a(x) := a(x - x_0)$ and $\tau_{x_0} f(x) = f(x - x_0)$ for any $x \in \mathbb{R}^2$.

Thus using, in particular, this property and the linearity of the operator $R_f a$ with respect to f, we can construct more involved non-uniqueness examples for the reconstruction of a from $R_f a$.

COROLLARY 3.2. Let $N \in \mathbb{N}$ and $(\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$, $(x_1, \ldots, x_N) \in (\mathbb{R}^2)^N$, and let $(r_{1,j}, \ldots, r_{N,j}) \in (0, +\infty)^N$, $j = 1, 2$ such that $0 < r_{i,1} < r_{i,2}$ for $i = 1 \ldots N$. The following statements are valid.

   i. When $f = \sum_{i=1}^N \alpha_i \chi_{C(x_i, r_{i,1}, r_{i,2})}$ we have $R_f a \equiv 0$ for any $a \in L^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} a \subset \cap_{i=1 \ldots N} D(x_i, r_{i,1})$.
   ii. Similarly when $f = \sum_{i=1}^N \alpha_i \delta_{\partial D(x_i, r_{i,1})}$ we have $R_f a \equiv 0$ for any $a \in L^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} a \subset \cap_{i=1 \ldots N} D(x_i, r_{i,1})$.

Note that in the above result, we obtain that $R_f a$ uniformly vanishes for functions f and a that are not necessarily radial.

3.2. Non-Uniqueness for global problem. Non-uniqueness for the nonlinear problem of the reconstruction of the absorption a from SPECT data $P_{a,\theta} f(s)$ given for all $(\theta, s) \in \mathbb{S}^1 \times \mathbb{R}$ also holds, even when f is assumed to be *known* so that all of the data in $P_{a,\theta} f(s)$ may be used toward the reconstruction of a. For instance if f is a delta function, we can change a so that a subset of its line integrals remains the same and thus get the same data. More precisely, let $f = c\delta_0$, for some

$c \in (0, +\infty)$ and where $\int_{\mathbb{R}^2} \delta_0(x)\phi(x)dx = \phi(0)$ for any $\phi \in C(\mathbb{R}^2, \mathbb{R})$. Then it follows that

$$\int_{\mathbb{S}^1 \times \mathbb{R}} P_{a,\theta} f(s)\phi(\theta, s) d\theta ds = c \int_{\mathbb{S}^1} e^{-\int_0^{+\infty} a(\sigma\theta)d\sigma} \phi(\theta, 0)d\theta,$$

for $\phi \in C(\mathbb{S}^1 \times \mathbb{R})$. Therefore we can recover only $ce^{-\int_0^{+\infty} a(\sigma\theta)d\sigma}$, $\theta \in \mathbb{S}^1$ from the data. Therefore the integrals of a over any half-line originated from the origin are known up to the constant $\ln(c)$ (which is unknown a priori). This is not sufficient to recover a. This was already noticed in [8] where the source f has the form of a finite sum of Dirac measures. In that case, approximation results are given using additional consistency conditions of Helgason-Ludwig type in [8].

Note that when f is not assumed to be known and is expected to be reconstructed from (2.2), then the remaining information for a has to be found in the compatibility condition (2.3), which contains less information than the full $P_{a,\theta} f(s)$. There are therefore clear obstructions to the reconstruction of (f, a) even in the non-linear setting.

4. Uniqueness results

4.1. Reconstructions with nonlocal sources. We give examples of source functions f such that $R_f a$ uniquely determines the absorption $a \in L^\infty_{\text{comp}}(\mathbb{R}^2)$. In these examples, the source function f is not compactly supported and in that sense f is not local. We will denote by e_1 (resp. e_2) the unit vector $(1, 0)$ (resp. $(0, 1)$) and by $\mathcal{S}(\mathbb{R})$ the Schwartz space of infinitely smooth functions f from $\mathbb{R}$ to $\mathbb{C}$ such that $\sup_{x \in \mathbb{R}} |x|^k \left|\frac{d^j f}{dx^j}(x)\right| < \infty$ for any $(j, k) \in \mathbb{N}^2$. We will also denote by ˆ the one-dimensional Fourier transform. We have the following result:

PROPOSITION 4.1. Let $f_1 \in \mathcal{S}(\mathbb{R})$ and let $f(x) = f_1(x_1)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Then the following formulas are valid

$$(4.1) \qquad R_f a(x) = -8\pi^2 D_{e_2} a(x) D_{e_1} f(x),$$

$$(4.2) \qquad e_2 \cdot \nabla_x R_f a(x) = -8\pi^2 a(x) D_{e_1} f(x),$$

where D_{e_1} and D_{e_2} are defined by (2.7). Moreover R_f is one-to-one when the support of $D_{e_1} f$ is $\mathbb{R}^2$.

PROOF OF PROPOSITION 4.1. First we have $\hat{f}(\zeta) = 2\pi \hat{f}_1(\zeta_1)\delta(\zeta_2)$, and using (2.19), it follows that $R_f a$ is given by

$$(4.3) \qquad \widehat{R_f a}(\xi) = 4\pi \text{p.v.}\left(\int_{\mathbb{R}} \frac{\hat{a}(\xi_1 - \zeta_1, \xi_2)\hat{f}_1(\zeta_1)}{\xi_2 \zeta_1} d\zeta_1\right).$$

We recall that the inverse one-dimensional Fourier transform of the principal value distribution is given $-\frac{\text{sgn}(s)}{2i}$. Hence

$$(4.4) \qquad \hat{g}(s)\text{p.v.}\frac{1}{s} = -\frac{1}{2i}\widehat{g * \text{sgn}},$$

for $g \in L^\infty_{\text{comp}}(\mathbb{R}) \cup \mathcal{S}(\mathbb{R})$, where $*$ denotes the convolution product. Therefore applying an inverse Fourier transform in the ξ_2 variable (denoted by $\mathcal{F}^{-1}_{\xi_2 \to x_2}$) in

both sides of (4.3), we obtain

$$\mathcal{F}_{x_1 \to \xi_1} R_f a(\xi_1, x_2) = -\frac{2\pi}{i} \mathcal{F}_{x_1 \to \xi_1} \left(\int_{\mathbb{R}} a(., x_2 - s) \text{sgn}(s) ds \right) *_{\xi_1} \left(\hat{f}_1(\zeta_1) \text{p.v.} \frac{1}{\zeta_1} \right)$$

$$(4.5) \qquad = -\frac{4\pi}{i} \mathcal{F}_{x_1 \to \xi_1} D_{e_2} a(., x_2) *_{\xi_1} \left(\hat{f}_1(\zeta_1) \text{p.v.} \frac{1}{\zeta_1} \right)$$

where $\mathcal{F}_{x_1 \to \xi_1}$ denotes the one-dimensional Fourier transform in the ξ_1 variable, and where $*_{\xi_1}$ denotes the convolution product with respect to ξ_1. Then by an inverse Fourier transform in ξ_1 and using (4.4) we obtain

$$(4.6) \qquad R_f a(x) = -4\pi^2 D_{e_2} a(x) \int_{\mathbb{R}} f_1(x_1 - s) \text{sgn}(s) ds,$$

which proves (4.1) (we use the following property of the one-dimensional Fourier transform "$\hat{g}_1 * \hat{g}_2 = 2\pi \widehat{g_1 g_2}$"). Then using (4.1) and using the equality

$$(4.7) \qquad \theta \cdot \nabla_x D_\theta a(x) = a(x),$$

we obtain (4.2) (we also used the equality $e_2 \cdot \nabla_x D_{e_1} f(x) = 0$). $\qquad \square$

When $f(x_1, x_2) = \delta(x_1)$, we obtain

$$(4.8) \qquad R_f a(x) = -4\pi^2 \text{sgn}(x_1) D_{e_2} a(x), x \in \mathbb{R}^2,$$

and

$$(4.9) \qquad e_2 \cdot \nabla_x R_f a(x) = -4\pi^2 \text{sgn}(x_1) a(x), x \in \mathbb{R}^2,$$

which proves the injectivity of R_f for $f(x_1, x_2) = \delta(x_1)$.

We can generalize formula (4.1) as follows. Let $f(x) = f_1(x_1) + f_2(x_2)$, where $(f_1, f_2) \in \mathcal{S}(\mathbb{R})^2$. Then

$$(4.10) \qquad R_f a(x) = -8\pi^2 D_{e_2} a(x) D_{e_1} \tilde{f}_1(x) + 8\pi^2 D_{e_1} a(x) D_{e_2} \tilde{f}_2(x),$$

where $\tilde{f}_1(x) = f_1(x_1)$ and $\tilde{f}_2(x) = f_2(x_2)$. This provides new examples of sources f such that R_f is one-to-one.

4.2. Constant source on the support of a. We now give examples where $R_f a$ uniquely determines a up to the radial part of a. In these examples, f is equal to a non-vanishing constant on the support of a. The notation $\chi_{D(0,r)}$ is introduced in section 3.1.

THEOREM 4.2. *When $f = \chi_{D(0,r)}$ for some $r > 0$, and when $a \in L^\infty(\mathbb{R}^2)$, $\text{supp} a \subseteq D(0,r)$, then $R_f a$ uniquely determines a up to its radial part and we have the following formula*

$$(4.11) \qquad \Delta R_f a(x) = -4\pi x^\perp \cdot \nabla_x a(x),$$

where the equality holds in the distributional sense.

PROOF OF THEOREM 4.2. From (3.4), it follows that

$$(4.12) \qquad HPf(\theta, x \cdot \theta^\perp) = 2x \cdot \theta^\perp, \quad \text{for } x \in D(0,r)$$

and

$$(4.13) \qquad R_f a(x) = 2 \int_0^{2\pi} D_\theta a(x)(x \cdot \theta^\perp) d\theta, \quad \text{when } \text{supp} a \subseteq D(0,r).$$

Therefore we obtain

$$(4.14) \quad R_f a(x) = -2 \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \cdot y^\perp a(y) dy = 4\pi \int_{\mathbb{R}^2} \nabla_y G(x-y) \cdot y^\perp a(y) dy,$$

where $G(y) = (2\pi)^{-1} \ln(|y|)$ is the Green function for the Laplacian in $\mathbb{R}^2$, $\Delta G = \delta_0$. Hence for $\phi \in C_0^\infty(\mathbb{R}^2)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} R_f a(x) \Delta \phi(x) dx &= 4\pi \int_{\mathbb{R}^2} a(y) y^\perp \cdot \nabla_y \Big(\int_{\mathbb{R}^2} G(x) \Delta \phi(x+y) dx \Big) dy \\ &= 4\pi \int_{\mathbb{R}^2} a(y) y^\perp \cdot \nabla_y \phi(y) dy, \end{aligned}$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Combining Theorems 3.1 and 4.2 we obtain examples of sources f of the form of the sum of the indicatrix function of an Euclidean Disc D_0 centered at 0 and a superposition of indicatrix functions of Euclidean annuli or a superposition of delta functions of circles such that any absorption $a \in L^\infty(\mathbb{R}^2)$ supported inside D_0 (with the additional constraints on the support of a with respect to Theorem 3.1) is reconstructed from $R_f a$ up to its radial part by the formula (4.11).

For such a source and absorption (f, a), a can be completely reconstructed from $R_f a$ provided that $P_{\theta_0} a(s)$ is also known for some $\theta_0 \in \mathbb{S}^1$ and for any $s \in \mathbb{R}$. In X-ray tomography, these integrals $P_{\theta_0} a(s)$ are known for all $s \in \mathbb{R}$ when a full transversal scan in the fixed direction θ_0 is performed on the object of interest. Such results show that combined with very limited tomographic projections of a, unique reconstructions of both a and f may be feasible.

4.3. Formulas for radial sources f. Using (2.18) (resp. (2.19)), we first give a general formula that relates the Fourier decomposition of $R_f a$ (resp. $\widehat{R_f a}$) to the Fourier decomposition of a (resp. $\hat{a}$) when f is a radial function. Then we provide an example of a smooth (and Gaussian) radial source f such that $R_f a$ uniquely determines a up to its radial part. However, the stability of the reconstruction is very poor as we shall see.

Let $f(x) = f_1(|x|) \in L^2(\mathbb{R}^2)$. Performing the changes of variables $y = s\tilde{x} + t\tilde{x}^\perp$ and $z = r(\cos(\omega), \sin(\omega))$ ($dy = ds\, dt$, $dz = r\, dr\, d\omega$), in equation (2.22) and using (3.1), it follows that

$$\begin{aligned} R_f a(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\big(a(x - s\tilde{x} - t\tilde{x}^\perp) - a(x - s\tilde{x} + t\tilde{x}^\perp)\big)}{\sqrt{s^2 + t^2}} \\ &\quad \times \int_0^{+\infty} f_1(r) I(r^{-1} x \cdot \widetilde{y}^\perp) d\omega dr \\ &= -2 \int_{(s,t) \in \mathbb{R} \times (0, +\infty)} \frac{a(s\tilde{x} - t\tilde{x}^\perp) - a(s\tilde{x} + t\tilde{x}^\perp)}{\sqrt{(|x| - s)^2 + t^2}} \\ (4.15) &\quad \times g\Big(\frac{t|x|}{\sqrt{(|x| - s)^2 + t^2}} \Big) ds dt, \end{aligned}$$

where

$$(4.16) \qquad\qquad g(r) = \int_0^r \frac{f_1(s)s}{\sqrt{r^2 - s^2}} ds, \text{ for } r > 0.$$

Let us introduce the Fourier decomposition of $R_f a$ and a given by $R_f a(r\omega) = \sum_{m\in\mathbb{Z}}(R_f a)_m(r)e^{im\theta}$ and $a(r\omega) = \sum_{m\in\mathbb{Z}} a_m(r)e^{im\theta}$, where $\omega = (\cos(\theta), \sin(\theta))$. Set $x = r\omega$ in (4.15) for $(r,\theta) \in (0,+\infty) \times (0,2\pi)$. Then performing the change of variables $(s,t) = \sigma(\cos(\phi), \sin(\phi))$ in (4.15) (where $(\sigma,\phi) \in (0,+\infty) \times (0,\pi)$), we have

$$(4.17) \qquad (R_f a)_m(r) = 4i \int_0^{+\infty} \sigma a_m(\sigma) \int_0^\pi \frac{\sin(m\phi) g\left(\frac{r\sigma \sin(\phi)}{\sqrt{r^2+\sigma^2-2r\sigma\cos(\phi)}}\right)}{\sqrt{r^2 + \sigma^2 - 2r\sigma\cos(\phi)}} d\phi d\sigma.$$

A formula similar to (4.17) holds for the Fourier transform of $R_f a$ and a: from (2.24), it follows that
$$(4.18)$$
$$r\widehat{R_f a}_m(r\omega) = 4i \int_0^{+\infty} \hat{a}_m(\sigma)\left(\int_0^\pi g_1\left(\sqrt{\sigma^2 + r^2 - 2\sigma r \cos(\phi)}\right)\frac{\sin(m\phi)}{\sin(\phi)}d\phi\right)d\sigma,$$

for $(r,\theta) \in (0,+\infty) \times (0,2\pi)$. Here, we have defined $\omega = (\cos(\theta),\sin(\theta))$, $\widehat{R_f a}(r\omega) = \sum_{m\in\mathbb{Z}} \widehat{R_f a}_m(r)e^{im\theta}$ and $\hat{a}(r\omega) = \sum_{m\in\mathbb{Z}} \hat{a}_m(r)e^{im\theta}$, as well as $g_1(\sigma) := \hat{f}((\sigma,0))$ for $\sigma \in (0,+\infty)$.

We have seen in section 3 examples of sources f such that no reconstruction of a_m is possible for any $m \in \mathbb{Z}$ from knowledge of $(R_f a)_m$. We also have seen in section 4.2 examples of sources f such that a_m can be reconstructed from $(R_f a)_m$ for $m \neq 0$ provided that a is compactly supported inside the support of f. In this latter example, f was also constant on the support of a. We now provide a example of a source f such that $R_f a$ determines a up to its radial part and such that f restricted to any nonempty open subset of $\mathbb{R}^2$ is not constant.

PROPOSITION 4.3. *Let $f(x) = e^{-|x|^2}$ for $x \in \mathbb{R}^2$ and let $a \in L^\infty_{\text{comp}}(\mathbb{R}^2)$. Then $R_f a$ uniquely determines a up to its radial part.*

PROOF OF PROPOSITION 4.3. Let $a \in L^\infty_{\text{comp}}(\mathbb{R}^2)$. First we have $\hat{f}(\xi) = \pi e^{-\frac{|\xi|^2}{4}}$. Therefore using (4.18) ("$g_1(s) = \pi e^{-\frac{s^2}{4}}$", for $s \in (0,+\infty)$) we obtain

$$(4.19) \qquad F_m(r) \quad := \quad -\frac{ire^{\frac{r^2}{4}}}{4\pi}\widehat{R_f a}_m(r\omega)$$

$$(4.20) \qquad\qquad = \quad \int_0^{+\infty} \hat{a}_m(\sigma)e^{-\frac{\sigma^2}{4}}\left(\int_0^\pi e^{\frac{\sigma r \cos(\phi)}{2}}\frac{\sin(m\phi)}{\sin(\phi)}d\phi\right)d\sigma,$$

for $r \in (0,+\infty)$ and for $m \in \mathbb{Z}$. The functions F_m are entire functions on $\mathbb{C}$ and thus they are determined by their derivatives at $r = 0$,

$$(4.21) \qquad\qquad 2^n \frac{d^n F_m}{dr^n}(0) = \int_0^{+\infty} \hat{a}_m(\sigma)e^{-\frac{\sigma^2}{4}}\sigma^n d\sigma I_{n,m},$$

where

$$(4.22) \qquad\qquad I_{n,m} = \int_0^\pi (\cos(\phi))^n \frac{\sin(m\phi)}{\sin(\phi)}d\phi.$$

We use the following Lemma.

LEMMA 4.4. *Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, $m \neq 0$. Then when $n + m$ is odd, $\pm m > 0$, we have $\pm I_{n,m} > 0$.*

The proof of Lemma 4.4 is given at the end of section 4.3.

Let $m \neq 0$. Then, using (4.21) and Lemma 4.4, we obtain

$$(4.23) \qquad \int_0^{+\infty} h_m(\sigma)\sigma^n d\sigma = \begin{cases} \dfrac{2^{2n}}{I_{2n,m}} \dfrac{d^{2n}F_m}{dr^{2n}}(0), & \text{when } m \text{ is odd,} \\ \dfrac{2^{2n+1}}{I_{2n+1,m}} \dfrac{d^{2n+1}F_m}{dr^{2n+1}}(0), & \text{when } m \text{ is even,} \end{cases}$$

where

$$(4.24) \qquad h_m(\sigma) = \begin{cases} \dfrac{\hat{a}_m(\sqrt{\sigma})}{2\sqrt{\sigma}} e^{-\frac{\sigma}{4}}, & \text{when } m \text{ is odd,} \\ \dfrac{\hat{a}_m(\sqrt{\sigma})e^{-\frac{\sigma}{4}}}{2}, & \text{when } m \text{ is even.} \end{cases}$$

Here, we performed the change of variables "σ"$= \sigma^2$ on the integral on the right-hand side of (4.21). Then the Laplace transform $\mathcal{L}h_m(\lambda) := \int_0^{+\infty} e^{-\lambda\sigma} h_m(\sigma)d\sigma$ for $m \neq 0$ is analytic on the strip $\{\lambda \in \mathbb{C} \mid \Re\lambda > -1\}$ and is given by the formulas

$$(4.25) \qquad \mathcal{L}h_m(\lambda) = \begin{cases} \displaystyle\sum_{n=0}^{+\infty} \dfrac{(-1)^n 2^{2n}}{n! I_{2n,m}} \dfrac{d^{2n}F_m}{dr^{2n}}(0)\lambda^n, & \text{when } m \text{ is odd,} \\ \displaystyle\sum_{n=0}^{+\infty} \dfrac{(-1)^n 2^{2n+1}}{n! I_{2n+1,m}} \dfrac{d^{2n+1}F_m}{dr^{2n+1}}(0)\lambda^n, & \text{when } m \text{ is even,} \end{cases}$$

in a neighborhood of 0. Inverting a Laplace transform, we recover h_m and thus $\hat{a}_m$ from $\widehat{R_f a}_m$ for $m \neq 0$. This proves that $R_f a$ uniquely determine $\hat{a}$ up to its radial part. Hence $R_f a$ uniquely determines a up to its radial part. $\qquad\square$

The reconstruction procedure we just presented seems to be highly ill-posed as it involves the inversion of a Laplace transform. Such a result should not be surprising. The above reconstruction works for arbitrary Gaussian sources of the form $e^{-|\frac{x}{\eta}|^2}$ for all $\eta > 0$. When $\eta \to 0$ and after proper rescaling, this corresponds to a source term that approximates the delta distribution with support at $x = 0$. We have seen that reconstructions of a were not possible in this limit and thus cannot expect reconstructions from peaked Gaussian source terms to be stable.

In the Appendix, we give an alternative proof of Proposition 4.3 using a different reconstruction procedure that also involves inverting a Laplace transform.

PROOF OF LEMMA 4.4. We prove by induction that $I_{n,m} > 0$ when $n + m$ is odd, $(n,m) \in \mathbb{N}^2$, $m > 0$. First when $m = 1$ and n is even we have $I_{n,1} = \int_0^\pi \cos(\phi)^n d\phi > 0$ (since $\cos(\phi)^n > 0$ for $\phi \in (0,\pi)$, $\phi \neq \frac{\pi}{2}$). When $n = 0$ and m is odd, $m > 0$, then

$$I_{0,m} = \Re\left(\int_0^{2\pi} \frac{(e^{i\phi})^m - (e^{-i\phi})^m}{e^{i\phi} - e^{-i\phi}} d\phi\right)$$

$$(4.26) = \sum_{j=0}^{m-1} \Re\left(\int_0^\pi e^{i(2j-(m-1))\phi}d\phi\right) = \sum_{j=0}^{m-1} \int_0^\pi \cos((2j-(m-1))\phi)d\phi = \pi > 0,$$

where $\Re z$ denotes the real part of a complex number z. Then the proof of the statement for $m > 0$ follows by induction from the identity

$$(4.27) \qquad\qquad\qquad 2I_{n,m} = I_{n-1,m-1} + I_{n-1,m+1},$$

and from $I_{0,m} > 0$ for odd $m > 0$, $I_{n,1} > 0$ for even $n \geq 0$, and $I_{n,0} = 0$ for $n \in \mathbb{N}$. (Identity (4.27) follows from the identity $2\sin(m\phi)\cos(\phi) = \sin((m+1)\phi) + \sin((m-1)\phi)$ for $\phi \in (0, \pi)$.) Then, we note that $I_{n,-m} = -I_{n,m}$ for $(n, m) \in \mathbb{N} \times \mathbb{Z}$ and thus the statement is also proved for $m < 0$. □

4.4. Formulas for arbitrary sources f. We conclude this section by providing a general formula that relates the Fourier series decomposition of $R_f a$ with the Fourier series decomposition of arbitrary smooth and sufficiently decaying source term f and of arbitrary absorption function $a \in L^\infty_{\text{comp}}$.

Let us introduce the Fourier decomposition of $R_f a$, a and f, $R_f a(r\omega) = \sum_{m \in \mathbb{Z}} (R_f a)_m(r) e^{im\theta}$, $a(r\omega) = \sum_{m \in \mathbb{Z}} a_m(r) e^{im\theta}$ and $f(r\omega) = \sum_{m \in \mathbb{Z}} f_m(r) e^{im\theta}$ where $\omega = (\cos(\theta), \sin(\theta))$ and $r \in (0, +\infty)$.

Then using (2.18), we can prove that
$$(4.28)$$
$$(R_f a)_m(r) = -\frac{i}{\pi} \sum_{j \in \mathbb{Z}} \int_{(0,+\infty)^2 \times (0,2\pi)^2} \frac{s\sigma a_j(s) f_{m-j}(\sigma) \sin(j\omega + (m-j)\varphi) ds d\sigma d\varphi d\omega}{r(-\sigma\sin(\varphi) + s\sin(\omega)) + s\sigma\sin(\varphi - \omega)}.$$

The analysis of this decomposition is left open.

5. Conclusions and perspectives

The above uniqueness and non-uniqueness results offer a partial answer to the use of the compatibility condition to solve for a for a given f. Ideally, (2.3) or its linearization (2.17) should be coupled with (2.2) to obtain a system of equations for (f, a). Unfortunately, the non-uniqueness results prevent us from stating a positive reconstruction result for the vector (f, a). It is in fact clear that (2.2)-(2.3) is not uniquely solvable in general. For which class of sources f do (2.2)-(2.3) or its linearization (2.2)-(2.17) admit unique solutions is, however, not clear at present. The above study provides partial answers. The main conclusion is that the class of (f, a) for which some non-uniqueness arises is quite large.

From the practical viewpoint, a is typically reconstructed first using a standard CT-scan and then f is reconstructed by using, e.g., (2.2). Note, however, that each CT-scan results in a small does of radiation being absorbed by the patient. The reconstruction of (f, a) from knowledge of $P_a f(\theta, s)$ with minimal additional information about the line integrals of a would therefore have practical value.

We repeat that the coupled system (2.2)-(2.3) provides a full description of the range of the AtRT operator and as such is a mathematically sound starting point for studies of simultaneous reconstructions of f and a. The above linearization about $a = 0$ can be generalized to linearizations about other values of a. The resulting expressions are, however, considerably more complicated than the simple expression obtained in (2.16) leading to the definition of (2.17). These expressions do not seem to be as simple to analyze as the operator in (2.17) and are left open for future studies.

We conclude this paper by the following remark on the nonlinear problem with constant attenuation μ in the disc $D(0, 1)$. When the source f is a radial smooth function that is compactly supported inside $D(0, 1)$, we cannot reconstruct (μ, f) from the data $P_{a,\theta} f(s)$, $(s, \theta) \in \mathbb{R} \times \mathbb{S}^1$, where $a = \mu\chi_{D(0,1)}$. Indeed, when $f \in C_0^\infty(\mathbb{R}^2)$ is radial and compactly supported inside the disc $D(0, 1)$, then $g(\theta, s) := P_{a,\theta} f(s) = e^{-\mu\sqrt{1-s^2}} \int_{-\infty}^{+\infty} e^{\mu t} f(t\theta + s\theta^\perp) dt$ belongs to $C_0^\infty(\mathbb{S}^1 \times \mathbb{R})$ and is radial and compactly supported inside $\mathbb{S}^1 \times (-1, 1)$. Therefore, using the range characterization

of the ERT [**1, 7**] (and a support theorem for the ERT, see [**11**]), one obtains that for any $\mu' \in \mathbb{R}$ there exists a function $f_{\mu'} \in C_0^\infty(\mathbb{R}^2)$ which is radial and compactly supported inside $D(0,1)$ such that $e^{\mu'\sqrt{1-s^2}}g(\theta,s) = \int_{-\infty}^{+\infty} e^{\mu't}f_{\mu'}(t\theta + s\theta^\perp)dt$, i.e. $P_{a,\theta}f = P_{a',\theta}f_{\mu'}$, $\theta \in \mathbb{S}^1$ where $a' = \mu'\chi_{D(0,1)}$. Hence the obstruction for the identification problem for the ERT [**5, 13**] still holds for the similar problem for the AtRT in the disc $D(0,1)$.

Acknowledgment

Appendix A. Alternative proof of Proposition 4.3

First we have $\hat{f}(\xi) = \pi e^{-\frac{|\xi|^2}{4}}$ and from (2.19) it follows that

$$(A.1) \qquad \widehat{R_f a}(\xi) = 2\pi e^{-\frac{|\xi|^2}{4}}\,\mathrm{p.v.} \int_{\mathbb{R}^2} \frac{\hat{a}(s\tilde{\xi} + t\tilde{\xi}^\perp)e^{-\frac{s^2+t^2}{4}}e^{\frac{|\xi|s}{2}}}{t}dtds,$$

for $\xi \in \mathbb{R}^2$, $\xi \neq 0$. Thus at fixed $\omega \in \mathbb{S}^1$, $e^{\lambda^2}\widehat{R_f a}(2\lambda\omega) = \mathcal{B}h_\omega(-\lambda)$ where $h_\omega(s) = 2\pi\,\mathrm{p.v.}\int_{-\infty}^{+\infty}\frac{\hat{a}(s\omega + t\omega^\perp)e^{-\frac{s^2+t^2}{4}}}{t}dt$, and where $\mathcal{B}$ denotes the two-sided Laplace transform, $\mathcal{B}h_1(\lambda) := \int_{-\infty}^{+\infty} e^{-\lambda s}h_1(s)ds$ for $\lambda \in (-\delta,\delta)$ and $h_1 \in L^1(\mathbb{R}, e^{\delta|s|}ds)$ for some $\delta > 0$. Thus inverting the Laplace transform, we obtain

$$(2\pi)^{-1}\mathcal{B}^{-1}\Big[e^{\lambda^2}\widehat{R_f a}(2\lambda\omega)\Big](s) = \mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{\hat{b}(s\omega + t\omega^\perp)}{t}dt.$$

where b is the smooth function defined by $\hat{b}(\xi) = \hat{a}(\xi)e^{-\frac{|\xi|^2}{4}}$, $\xi \in \mathbb{R}^2$, or equivalently by $b = \pi^{-1}a*f$ where $*$ denotes the convolution product. Then using that $\widehat{\mathrm{p.v.}(\frac{1}{t})} = -i\pi\mathrm{sgn}(t)$, we obtain

$$(A.2) \quad \mathrm{p.v.}\int_{-\infty}^{+\infty}\frac{\hat{b}(s\omega + t\omega^\perp)}{t}dt = -\pi i \int_{-\infty}^{+\infty}\mathrm{sgn}(\sigma)\int_{\mathbb{R}} e^{-isr}b(r\omega + \sigma\omega^\perp)drd\sigma,$$

for any $s \in \mathbb{R}$. Then applying an inverse Fourier transform in the s variable (denoted by $\mathcal{F}_{s\to k}^{-1}$) to the left and right hand sides of (A.2) we obtain

$$\begin{aligned} \mathcal{F}_{s\to k}^{-1}\Big\{(2\pi)^{-1}\mathcal{B}^{-1}\Big[e^{\lambda^2}\widehat{R_f a}(2\lambda\omega)\Big](.)\Big\}(k) &= -\pi i \int_{-\infty}^{+\infty}\mathrm{sgn}(\sigma)b(k\omega + \sigma\omega^\perp)d\sigma \\ (A.3) \qquad &= 2\pi i D_{\omega^\perp}b(k\omega), \end{aligned}$$

for $k \in \mathbb{R}$, where $D_{\omega^\perp}$ is defined by (2.7). The question is therefore now whether we can reconstruct $b(x)$ from knowledge of the transform $D_{\omega^\perp}b(k\omega)$, which is an interesting integral geometry problem in itself that does not seem to have been addressed in the literature.

We decompose $b(x)$ in Fourier series: $b(r\omega) = \sum_{m\in\mathbb{Z}}b_m(r)e^{im\theta}$, $(r,\theta) \in (0,+\infty)\times(0,2\pi)$, $\omega = (\cos(\theta),\sin(\theta))$. We will prove that b_m for $m \neq 0$ is uniquely determined by $R_f a$ through (A.3) and through $D_{\omega^\perp}b(k\omega)$ given for any $\omega \in \mathbb{S}^1$ and

$k \in \mathbb{R}$. First using a change of variables similar to the one used for the derivation of (4.15) we obtain

$$(A.4) \qquad D_{\omega^\perp} b(t\omega) = -i \sum_m \operatorname{sgn}(m) e^{im\theta} \int_t^{+\infty} b_m(r) U_{|m-1|}\left(\frac{t}{r}\right) dr,$$

for $t > 0$, $\omega = (\cos(\theta), \sin(\theta))$, $\theta \in (0, 2\pi)$, where U_m denotes the m-th Tchebyschev polynomial function of the second kind: $U_m(t) = \frac{\sin((m+1)\arccos(t))}{\sqrt{1-t^2}}$, $t \in [-1,1]$, $m \in \mathbb{N}$. In addition at fixed $m \in \mathbb{N}$, $m \geq 1$, for $\delta > 0$ and for any function $h \in L^1((0, +\infty)_r, e^{\delta r^2} dr)$ we have

$$(A.5) \qquad \int_t^{+\infty} h(r) U_{m-1}\left(\frac{t}{r}\right) dr \text{ given for all } t > 0 \text{ uniquely determines } h.$$

Therefore from (A.4) and (A.3) it follows that $R_f a$ uniquely determines b_m for $m \neq 0$. Hence $R_f a$ uniquely determines b up to its radial part. Using the equality $\hat{b}(\xi) = \hat{a}(\xi) e^{-\frac{|\xi|^2}{4}}$, we obtain that $R_f a$ uniquely determines a up to its radial part.

It remains to prove (A.5). Let $m \in \mathbb{N}$, $m \geq 1$ and let $h \in L^1((0, +\infty)_r, e^{\delta r^2} dr)$ for some $\delta > 0$. Let $h_m(\lambda) := \int_0^{+\infty} e^{\lambda t} \int_t^{+\infty} h(r) U_{m-1}(\frac{t}{r}) dr dt$. We have $h_m(\lambda) = \int_0^{+\infty} r h(r) \int_0^1 e^{\lambda r t} U_{m-1}(t) dt dr$. The function h_m is an analytic function for λ in $\mathbb{C}$. Then h_m is uniquely determined by all its derivative at $\lambda = 0$, and we have

$$(A.6) \qquad \frac{d^n h_m}{d\lambda^n}(0) = \int_0^{+\infty} r^{n+1} h(r) dr \int_0^1 t^n U_{m-1}(t) dt, \text{ for } n \in \mathbb{N}.$$

We prove at the end of this section that

$$(A.7) \qquad I_{n,m} = \int_0^1 t^n U_{m-1}(t) dt > 0,$$

for each n, m, $m > 0$ and $n + m$ odd. Thus we have

$$(A.8) \qquad \int_0^{+\infty} e^{-\lambda r} \frac{h(\sqrt{r})}{2} dr = \int_0^{+\infty} e^{-\lambda r^2} r h(r) dr = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! I_{2n,m}} \frac{d^{2n} h_m}{d\lambda^{2n}}(0) \lambda^n,$$

in a neighborhood of 0 when m is odd, and
$$(A.9)$$
$$\int_0^{+\infty} e^{-\lambda r} \frac{\sqrt{r} h(\sqrt{r})}{2} dr = \int_0^{+\infty} e^{-\lambda r^2} r^2 h(r) dr = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! I_{2n+1,m}} \frac{d^{2n+1} h_m}{d\lambda^{2n}}(0) \lambda^n,$$

in a neighborhood of 0 when m is even, $m \neq 0$. Inverting again a Laplace transform, we recover h from $\int_t^{+\infty} h(r) U_{m-1}\left(\frac{t}{r}\right) dr$ given for all $t > 0$.

We prove (A.7). Note that $I_{n,m} = \int_0^{\frac{\pi}{2}} \cos(\theta)^n \sin(m\theta) d\theta$. For $m = 0$ we have $I_{n,m} = 0$ for $n \in \mathbb{N}$. Then we prove (A.7) by induction in n. For $n = 0$ and $m > 0$, m odd, we have $I_{0,m} = \frac{1}{m} > 0$, and for $m = 1$, $n \in \mathbb{N}$ and n even we have $I_{n,1} = (n+1)^{-1} > 0$. Now assume that we prove that $I_{n,m} > 0$ for some $n \geq 0$ and for any $m > 0$, $n + m$ odd. Then for $m > 0$ such that $n + m + 1$ is odd, we have $2I_{n+1,m} = I_{n,m-1} + I_{n,m+1}$. By assumption $I_{n,m+1} > 0$ and we have $I_{n,m-1} \geq 0$. Therefore $I_{n+1,m} > 0$. This proves (A.7).   □

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