Calderón problem

The Calderón problem is modeled by the following elliptic problem with Dirichlet conditions

$$\begin{aligned} L_\gamma u(x) &\equiv \nabla \cdot \gamma(x) \nabla u(x) = 0, && x \in X \\ u(x) &= f(x), && x \in \partial X, \end{aligned} \tag{1}$$

where $X \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂X. In what follows, we assume $n \geq 3$. Here, $\gamma(x)$ is a conductivity coefficient, which we assume is a smooth function, and $f(x)$ is a prescribed Dirichlet data for the elliptic problem.

The Dirichlet-to-Neumann or voltage-to-current map is given by

$$\Lambda_\gamma : \begin{array}{rcl} H^{\frac{1}{2}}(\partial X) & \rightarrow & H^{-\frac{1}{2}}(\partial X) \\ f(x) & \mapsto & \Lambda_\gamma[f](x) = \gamma(x) \dfrac{\partial u}{\partial \nu}(x). \end{array} \tag{2}$$

With $\mathfrak{X} = \mathcal{C}^2(\bar{X})$ and $\mathfrak{Y} = \mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$, we define the measurement operator

$$\mathfrak{M} : \qquad \mathfrak{X} \ni \gamma \mapsto \mathfrak{M}(\gamma) = \Lambda_\gamma \in \mathfrak{Y}. \tag{3}$$

The Calderón problem consists of reconstructing γ from knowledge of the Calderón measurement operator $\mathfrak{M}$. To slightly simplify the derivation of uniqueness, we also make the (unnecessary) assumption that γ and $\nu \cdot \nabla\gamma$ are *known* on ∂X. The main result of this chapter is the following.

Theorem Define the measurement operator $\mathfrak{M}$ as in (3). Then $\mathfrak{M}$ is injective in the sense that $\mathfrak{M}(\gamma) = \mathfrak{M}(\tilde{\gamma})$ implies that $\gamma = \tilde{\gamma}$.

Moreover, we have the following logarithmic stability estimate:

$$\|\gamma(x) - \gamma'(x)\|_{L^\infty(X)} \le C \Big| \log \|\mathfrak{M}(\gamma) - \mathfrak{M}(\tilde{\gamma})\|_{\mathfrak{Y}} \Big|^{-\delta}, \tag{4}$$

for some $\delta > 0$ provided that γ and $\tilde{\gamma}$ are uniformly bounded in $H^s(X)$ for some $s > \frac{n}{2}$.

The proof of the injectivity result is based on two main ingredients. The first ingredient consists of recasting the injectivity result as a statement of whether products of functionals of solutions to elliptic equations such as (2) are dense in the space of, say, continuous functions. The second step is to construct specific sequences of solutions to (2) that positively answer the density question. These specific solutions are Complex Geometric Optics (CGO) solutions.

$$\int_{\partial X}(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})f_1 f_2 d\mu = \int_X (\gamma_1 - \gamma_2)\nabla u_1 \cdot \nabla u_2 dx, \tag{5}$$

where $d\mu$ is the surface measure on ∂X and where u_j is the solution to (1) with γ replaced by γ_j and f replaced by f_j.

The above lemma shows that when $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, the right-hand-side in (1) also vanishes for *any* solutions u_1 and u_2 of (1) with γ given by γ_1 and

γ_2, respectively. We are now thus faced with the question of whether products of the form $\nabla u_1 \cdot \nabla u_2$ are dense in the space of, say, continuous functions. Unfortunately, answering this question affirmatively seems to be a difficult problem. The main difficulty in the analysis of (1) is that the unknown coefficient γ appears in the leading order of the differential operator L_γ. The following Liouville change of variables allows us to treat the unknown coefficient as a *perturbation* to a known operator (with constant coefficients):

$$\gamma^{-\frac{1}{2}} L_\gamma \gamma^{-\frac{1}{2}} = \Delta - q, \qquad q = \frac{\Delta \gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}}. \tag{6}$$

Here Δ is the usual Laplacian operator.

Consider the Schrödinger equation (still calling the solution "u" rather than "v")

$$(\Delta - q)u = 0 \quad \text{in } X, \qquad u = f \quad \text{on } \partial X, \tag{7}$$

with q given by (6). For $f \in H^{\frac{1}{2}}(\partial X)$, we find a unique solution $u \in H^1(X)$ such that $\nu\cdot\nabla u \in H^{-\frac{1}{2}}(\partial X)$. Indeed, the above equation admits a solution since it is equivalent to (1) by the change of variables (6). We then define the Dirichlet-to-Neumann operator

$$\Lambda_q : \begin{array}{rcl} H^{\frac{1}{2}}(\partial X) & \to & H^{-\frac{1}{2}}(\partial X) \\ f(x) & \mapsto & \Lambda_q[f](x) = \dfrac{\partial u}{\partial \nu}(x), \end{array} \tag{8}$$

where u is the solution to (7). We then verify that

$$\Lambda_q f = \gamma^{-\frac{1}{2}} \frac{\partial \gamma}{\partial \nu}\Big|_{\partial X} f + \gamma^{-\frac{1}{2}} \Lambda_\gamma(\gamma^{-\frac{1}{2}}\Big|_{\partial X} f), \qquad f \in H^{\frac{1}{2}}(\partial X). \tag{9}$$

We thus observe that knowledge of Λ_γ, $\gamma|_{\partial X}$ and $\nu \cdot \nabla\gamma|_{\partial X}$ implies knowledge of Λ_q. It turns out that knowledge of Λ_γ implies that of $\gamma|_{\partial X}$ and $\nu \cdot \nabla\gamma|_{\partial X}$, which we assume here.

Our next step is therefore to reconstruct q from knowledge of Λ_q.

Let Λ_{q_j} for $j = 1, 2$ be the two operators associated to q_j and let $f_j \in H^{\frac{1}{2}}(\partial X)$ for $j = 1, 2$ be two Dirichlet conditions. Then we find that

$$\int_{\partial X} (\Lambda_{q_1} - \Lambda_{q_2}) f_1 f_2 d\mu = \int_X (q_1 - q_2) u_1 u_2 dx, \tag{10}$$

where $d\mu$ is the surface measure on ∂X and where u_j is the solution to (7) with q replaced by q_j and f replaced by f_j.

The above lemma shows that when $\Lambda_{q_1} = \Lambda_{q_2}$, then the right-hand-side in (10) also vanishes for *any* solutions u_1 and u_2 of (7) with q replaced by q_1 and q_2, respectively. We are now thus faced with the question of whether products of the form $u_1 u_2$ are *dense* in the space of, say, continuous functions. This is a question that admits an affirmative answer. The main tool in the proof of this density argument is the construction of complex geometric optics solutions. Such solutions are constructed later. The main property that we need at the moment is summarized in the following lemma.

Lemma. Let $\varrho \in \mathbb{C}^n$ be a complex valued vector such that $\varrho \cdot \varrho = 0$. Let $\|q\|_\infty < \infty$ and $|\varrho|$ be sufficiently large. Then there is a solution u of $(\Delta - q)u = 0$ in X of the form

$$u(x) = e^{\varrho \cdot x}(1 + \varphi(x)), \tag{11}$$

such that

$$|\varrho| \|\varphi\|_{L^2(X)} + \|\varphi\|_{H^1(X)} \leq C. \tag{12}$$

Proof to follow. The principle of such solutions is this. When $q \equiv 0$, then $e^{\varrho \cdot x}$ is a (complex-valued) harmonic function, i.e., a solution of $\Delta u = 0$. The above result shows that q may be treated as a perturbation of Δ. Solutions of $(\Delta - q)u = 0$ are fundamentally not that different from solutions of $\Delta u = 0$.

Now, coming back to the issue of density of product of elliptic solutions.

For u_1 and u_2 solutions of the form (11), we find that

$$u_1 u_2 = e^{(\varrho_1+\varrho_2)\cdot x}(1+\varphi_1+\varphi_2+\varphi_1\varphi_2). \tag{13}$$

If we can choose $\varrho_1+\varrho_2 = ik$ for a fixed k with $|\varrho_1|$ and $|\varrho_2|$ growing to infinity so that $\varphi_1+\varphi_2+\varphi_1\varphi_2$ becomes negligible in the L^2 sense thanks to (12), then we observe that in the limit u_1u_2 equals $e^{ik\cdot x}$. The functions $e^{ik\cdot x}$ for arbitrary $k \in \mathbb{R}^n$ certainly form a dense family of, say, continuous functions.

Let us make a remark on the nature of the CGO solutions and the measurement operator $\mathfrak{M}$. The CGO solutions are complex valued. Since the equations (1) and (7) are linear, we can assume that the boundary conditions $f = f_r + if_i$ are complex valued as a superposition of two real-valued boundary conditions f_r and f_i. Moreover, the results (5) and (10) hold for complex-valued solutions. Our objective is therefore to show that the product of complex-valued solutions to elliptic equations of the form (7) is indeed dense. The construction in dimension $n \geq 3$ goes as follows.

Let $k \in \mathbb{R}^n$ be fixed for $n \geq 3$. We choose $\varrho_{1,2}$ as

$$\varrho_1 = \frac{m}{2} + i\frac{k+l}{2}, \qquad \varrho_2 = -\frac{m}{2} + i\frac{k-l}{2}, \tag{14}$$

where the real-valued vectors l, and m are chosen in $\mathbb{R}^n$ such that

$$m \cdot k = m \cdot l = k \cdot l = 0, \qquad |m|^2 = |k|^2 + |l|^2. \tag{15}$$

We verify that $\varrho_i \cdot \varrho_i = 0$ and that $|\varrho_i|^2 = \frac{1}{2}(|k|^2 + |l|^2)$. In dimension $n \geq 3$, such vectors can always be found. For instance, changing the system of coordinates so that $k = |k|e_1$, we can choose $l = |l|e_2$ with $|l| > 0$ arbitrary and then $m = \sqrt{|k|^2 + |l|^2}e_3$, where (e_1, e_2, e_3) forms a family of orthonormal vectors in $\mathbb{R}^n$. Note that this construction is possible only when $n \geq 3$. It is important to notice that while k is fixed,

$|l|$ can be chosen arbitrarily large so that the norm of ϱ_i can be arbitrarily large while $\varrho_1 + \varrho_2 = k$ is fixed.

Upon combining (10) and (13), we obtain for the choice (14) that $\Lambda_{q_1} = \Lambda_{q_2}$ implies that

$$\Big| \int_X e^{ik\cdot x}(q_1 - q_2)dx \Big| \le \Big| \int_X e^{ik\cdot x}(q_1 - q_2)(\varphi_1 + \varphi_2 + \varphi_1\varphi_2)dx \Big| \le \frac{C}{|l|}$$

thanks to (12) since $|l|(\varphi_1 + \varphi_2 + \varphi_1\varphi_2)$ is bounded in $L^1(X)$ by an application of the Cauchy-Schwarz inequality and $e^{ik\cdot x}(q_1 - q_2)$ is bounded in $L^\infty(X)$. Since the above inequality holds independent of l, we deduce that the Fourier transform of $(q_1 - q_2)$ (extended by 0 outside of X) vanishes, and hence that $q_1 = q_2$. So far we have thus proved that

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \implies \Lambda_{q_1} = \Lambda_{q_2} \implies q_1 = q_2,$$

where q_j and γ_j are related by (6). From (6) still, we deduce that

$$0=\gamma_1^{\frac12}\Delta\gamma_2^{\frac12}-\gamma_2^{\frac12}\Delta\gamma_1^{\frac12}=\nabla\cdot(\gamma_1^{\frac12}\nabla\gamma_2^{\frac12}-\gamma_2^{\frac12}\nabla\gamma_1^{\frac12})=\nabla\cdot(\gamma_1\nabla\Big(\frac{\gamma_2}{\gamma_1}\Big)^{\frac12}). \qquad (16)$$

Since $\gamma_1=\gamma_2$ on ∂X, this is an elliptic equation for $\big(\frac{\gamma_1}{\gamma_2}\big)^{\frac12}$ whose only solution is identically 1. This shows that $\gamma_1=\gamma_2$. This concludes the proof of the uniqueness result

$$\Lambda_{\gamma_1}=\Lambda_{\gamma_2} \implies \gamma_1=\gamma_2. \qquad (17)$$

Let us return to (10) and assume that $\varLambda_{q_1}-\varLambda_{q_2}$ no longer vanishes but is (arbitrarily) small. We first want to assess how errors in $\varLambda_q$ translates into errors in q. For u_j solutions of the form (11), we find that

$$\begin{aligned}
&\Big|\int_X e^{ik\cdot x}(q_1-q_2)dx\Big| \\
&\leq \Big|\int_X e^{ik\cdot x}(q_1-q_2)(\varphi_1+\varphi_2+\varphi_1\varphi_2)dx\Big| + \Big|\int_{\partial X}(\Lambda_{q_1}-\Lambda_{q_2})f_1 f_2 d\mu\Big| \\
&\leq \frac{C}{|l|} + \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\mathfrak{Y}}\|f_1\|_{H^{\frac{1}{2}}(\partial X)}\|f_2\|_{H^{\frac{1}{2}}(\partial X)} \\
&\leq \frac{C}{|l|} + C\|\Lambda_{q_1}-\Lambda_{q_2}\|_{\mathfrak{Y}}\|u_1\|_{H^1(X)}\|u_2\|_{H^1(X)} \\
&\leq \frac{C}{|l|} + C|l|\|\Lambda_{q_1}-\Lambda_{q_2}\|_{\mathfrak{Y}} e^{C|l|}.
\end{aligned}$$

Indeed, $f_j = u_j|_{\partial X}$ and $\|u\|_{H^{\frac{1}{2}}(\partial X)} \leq C\|u\|_{H^1(X)}$ is a standard estimate. This step is where the ill-posedness of the Calderón problem is best displayed.

Define $\delta q = q_1 - q_2$. So far, we have obtained a control of $\widehat{\delta q}(k)$ uniform in $k \in \mathbb{R}^n$. Upon choosing

$$|l| = \frac{\sigma}{C} \ln \frac{1}{\varepsilon}, \qquad 0 < \sigma < 1,$$

so that $e^{C|l|} = \varepsilon^{-\sigma}$, we find that for $\varepsilon := \min(1, \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathfrak{Y}})$,

$$|\widehat{\delta q}(k)| \leq \eta := C\Big|\ln \varepsilon\Big|^{-1}. \tag{18}$$

Since q is assumed to be bounded and compactly supported, it is square integrable in $\mathbb{R}^n$ so that $\|\delta q\|_{L^2(\mathbb{R}^n)} := E < \infty$. This and the control in (18) allows one to obtain a control of δq in $H^{-s}(\mathbb{R}^n)$ for $s > 0$. Indeed

$$\begin{aligned} \|\delta q\|^2_{H^{-s}(\mathbb{R}^n)} &= \int \langle k \rangle^{-2s} |\widehat{\delta q}|^2 dk \\ &\leq k_0^n \eta^2 + k_0^{-2s} E^2, \end{aligned}$$

by splitting the integration in k into $|k| < k_0$ and $|k| > k_0$ and choosing

$k_0 \geq 1$. We then choose

$$k_0 = \left(\frac{E}{\eta}\right)^{\frac{2}{n+2s}}.$$

This implies

$$\|q_1 - q_2\|_{H^{-s}(\mathbb{R}^n)} \leq C E^{\frac{n}{n+2s}} \big|\ln \varepsilon\big|^{-\frac{2s}{n+2s}}. \tag{19}$$

It remains to convert the estimate on $q_1 - q_2$ into an estimate for $\gamma_1 - \gamma_2$. We find that (16) is replaced by

$$(\gamma_1\gamma_2)^{\frac{1}{2}}(q_1 - q_2) = \nabla\cdot(\gamma_1 \nabla \left(\frac{\gamma_2}{\gamma_1} - 1\right)^{\frac{1}{2}}) \quad \text{in} \quad X, \qquad \left(\frac{\gamma_2}{\gamma_1} - 1\right)^{\frac{1}{2}} = 0 \quad \text{on} \quad \partial X \tag{20}$$

Standard elliptic regularity results and the fact that γ_1 is of class C^2 therefore show that

$$\|\gamma_1 - \gamma_2\|_{H^1(X)} \leq C\|q_1 - q_2\|_{H^{-1}(X)} \leq C\big|\ln \varepsilon\big|^{-\delta}, \tag{21}$$

with $\delta = \frac{2}{2+n}$ if q is bounded in the L^2 sense and $\delta = \frac{2(1+\varsigma)}{n+2(1+\varsigma)}$ if q is bounded. The final result in (4) then follows from interpolating the a priori bound in H^s of $\gamma_1 - \gamma_2$, the above smallness bound in H^1 to obtain a small bound in H^τ for some $\frac{n}{2} < \tau < s$. Then by the Sobolev imbedding of $L^\infty(X)$ into $H^\tau(X)$, we conclude the proof of the Calderón result.

Complex geometrical Optics Solutions

Let us consider the equation $\Delta u = qu$ in X. When $q = 0$, then a rich family of harmonic solutions is formed by the complex exponentials $e^{\varrho \cdot x}$ for $\varrho \in \mathbb{C}^n$ a complex valued vector such that $\varrho \cdot \varrho = 0$. Indeed, we verify that

$$\Delta e^{x \cdot \varrho} = \varrho \cdot \varrho \, e^{x \cdot \varrho} = 0. \tag{22}$$

A vector $\varrho = \varrho_r + i\varrho_i$ is such that $\varrho \cdot \varrho = 0$ if an only if ϱ_r and ϱ_i are orthogonal vectors of the same (Euclidean) length.

When $q \neq 0$, it is tempting to try and write solutions of $\Delta u = qu$ as perturbations of the harmonic solutions $e^{\varrho \cdot x}$, for instance in the form

$$u(x) = e^{\varrho \cdot x}(1 + \varphi(x)).$$

This provides an equation for φ of the form

$$(\Delta + 2\varrho \cdot \nabla)\varphi = q(1 + \varphi). \tag{23}$$

Treating the right-hand side as a source f, the first part of the construction consists of solving the problem

$$(\Delta + 2\varrho \cdot \nabla)\varphi = f, \tag{24}$$

for f a source in X and φ defined on X as well. Surprisingly, the analysis of (24) is the most challenging technical step in the construction of solutions to (23). The construction with $f \in L^2(X)$ is sufficient for the proof of Theorem . In later chapters, we will require more regularity for the solution to (24) and thus prove the following result.
Lemma Let $f \in H^s(X)$ for $s \geq 0$ and let $|\varrho| \geq c > 0$. Then there exists a solution to (24) in $H^{s+1}(X)$ and such that

$$|\varrho| \|\varphi\|_{H^s(X)} + \|\varphi\|_{H^{s+1}(X)} \leq C \|f\|_{H^s(X)}. \tag{25}$$

Proof. We first extend f defined on X to a function still called f defined and compactly supported in $\mathbb{R}^n$ and such that

$$\|f\|_{H^s(\mathbb{R}^n)} \leq C(X)\|f\|_{H^s(X)}.$$

We thus wish to solve the problem

$$(\Delta + 2\varrho \cdot \nabla)\varphi = f, \qquad \text{in} \qquad \mathbb{R}^n. \tag{26}$$

The main difficulty is that the operator $(\Delta + 2\varrho \cdot \nabla)$ has for symbol

$$\mathcal{F}_{x\to\xi}(\Delta + 2\varrho \cdot \nabla)\mathcal{F}^{-1}_{\xi\to x} = -|\xi|^2 + 2i\varrho \cdot \xi.$$

Such a symbol vanishes for $\varrho_r \cdot \xi = 0$ and $2\varrho_i \cdot \xi + |\xi|^2 = 0$. We thus construct a solution that can be seen of the product of a plane wave with a periodic solution with different period. Let us define

$$\varphi = e^{i\varsigma\cdot x}\mathfrak{p}, \qquad f = e^{i\varsigma\cdot x}\mathfrak{f}$$

for some vector $\varsigma \in \mathbb{R}^n$ to be determined. Then we find

$$\Big(\Delta + 2(\varrho + i\varsigma)\cdot\nabla + (2i\varrho\cdot\varsigma - |\varsigma|^2)\Big)\mathfrak{p} = (\nabla + i\varsigma + 2\varrho)\cdot(\nabla + i\varsigma)\mathfrak{p} = \mathfrak{f}. \quad (27)$$

Let us now assume that $\mathfrak{f}$ is supported in a box Q of size $(-L,L)^n$ for L sufficiently large. Then we decompose as Fourier series:

$$\mathfrak{p} = \sum_{k\in\mathbb{Z}^n} \mathfrak{p}_k e^{i\frac{\pi}{L}k\cdot x}, \qquad \mathfrak{f} = \sum_{k\in\mathbb{Z}^n} \mathfrak{f}_k e^{i\frac{\pi}{L}k\cdot x}. \quad (28)$$

We then find that (27) is equivalent in the Fourier domain to

$$\mathfrak{p}_k = \frac{1}{-|\frac{\pi}{L}k + \varsigma|^2 + 2i\varrho\cdot(\varsigma + \frac{\pi}{L}k)}\mathfrak{f}_k \quad (29)$$

The imaginary part of the denominator is given by $2\varrho_r\cdot(\frac{\pi}{L}k + \varsigma)$. It remains to choose

$$\varsigma = \frac{1}{2}\frac{\pi}{L}\frac{\varrho_r}{|\varrho_r|},$$

to obtain that the above denominator never vanishes since $k \in \mathbb{Z}^n$. Moreover, for such a choice, we deduce that

$$\Big| - \Big|\frac{\pi}{L}k + \varsigma\Big|^2 + 2i\varrho \cdot \Big(\varsigma + \frac{\pi}{L}k\Big)\Big| \geq C|\varrho|,$$

for some constant C independent of ϱ. This shows that

$$|\mathfrak{p}_k| \leq C|\varrho|^{-1}|\mathfrak{f}_k|.$$

Since $\mathfrak{f} \in H^s(Q)$, we deduce that $\|\mathfrak{f}\|^2_{H^s(Q)} = \sum_{k\in\mathbb{Z}^n} |k|^{2s}|\mathfrak{f}_k|^2 < \infty$, from which we deduce that

$$\|\mathfrak{p}\|_{H^s(Q)} \leq C|\varrho|^{-1}\|\mathfrak{f}\|_{H^s(Q)}.$$

It remains to restrict the constructed solution to X (and realize that $e^{i\varsigma\cdot x}$ is smooth) to obtain that $|\varrho|\|\varphi\|_{H^s(X)} \leq C\|f\|_{H^s(X)}$ and the first step in (25).

The result on $\|\varphi\|_{H^{s+1}(X)}$ requires that we obtain bounds for $|k|\mathfrak{p}_k$. For

$|k|$ small, say $|k| \leq \frac{8L}{\pi}|\varrho|$, then we use the same result as above to obtain

$$|k||\mathfrak{p}_k| \leq C|\mathfrak{f}_k|, \qquad |k| \leq \frac{8L}{\pi}|\varrho|.$$

For the other values of $|k|$, we realize that the denominator in (29) causes no problem and that

$$|k||\mathfrak{p}_k| \leq C|k|^{-1}|\mathfrak{f}_k|, \qquad |k| > \frac{8L}{\pi}|\varrho|.$$

This shows that $|k||\mathfrak{p}_k| \leq C|\mathfrak{f}_k|$ for some constant C independent of k and $|\varrho|$. The proof that $\|\varphi\|_{H^{s+1}(X)} \leq C\|f\|_{H^s(X)}$ then proceeds as above. This concludes the proof of the fundamental lemma of CGO solutions to Schrödinger equations. $\square$

We now come back to the perturbed problem (23). We assume that q is a complex-valued potential in $H^s(X)$ for some $s \geq 0$. We say that

$q \in L^\infty(X)$ has regularity s provided that for all $\varphi \in H^s(X)$, we have

$$\|q\varphi\|_{H^s(X)} \leq q_s \|\varphi\|_{H^s(X)}, \tag{30}$$

for some constant q_s. For instance, when $s = 0$, when $q_s = \|q\|_{L^\infty(X)}$. Then we have the following result.

Theorem Let us assume that $q \in H^s(X)$ is sufficiently smooth so that $q_s < \infty$. Then for $|\varrho|$ sufficiently large, there exists a solution φ to (23) that satisfies

$$|\varrho| \|\varphi\|_{H^s(X)} + \|\varphi\|_{H^{s+1}(X)} \leq C\|q\|_{H^s(X)}. \tag{31}$$

Moreover, we have that

$$u(x) = e^{\varrho \cdot x}(1 + \varphi(x)) \tag{32}$$

is a Complex Geometrical Optics solution in $H^{s+1}(X)$ to the equation

$$\Delta u = qu \qquad \text{in } X.$$

Proof. Let T be the operator which to $f \in H^s(X)$ associates $\varphi \in H^s(X)$ the solution of (26) constructed in the proof of Lemma . Then (23) may be recast as

$$(I - Tq)\varphi = Tq.$$

We know that $\|T\|_{\mathcal{L}(H^s(X))} \leq C_s|\varrho|^{-1}$. Choosing $|\varrho|$ sufficiently large so that $|\varrho| > C_s q_s$, we deduce that $(I - Tq)^{-1} = \sum_{m=0}^{\infty}(Tq)^m$ exists and is a bounded operator in $\mathcal{L}(H^s(X))$. We have therefore constructed a solution so that $q(1+\varphi) \in H^s(X)$. The estimate (25) yields (31) and concludes the proof of the theorem. □

Let us now consider the elliptic equation (1). The change of variables in (6) shows that $u = \gamma^{-\frac{1}{2}}v$ with v a solution of $\Delta v = qv$, is a solution of (1). We therefore have the

Corollary. Let γ be sufficiently smooth so that $q = \gamma^{-\frac{1}{2}}\Delta\gamma^{\frac{1}{2}}$ verifies the hypotheses of Theorem . Then for $|\varrho|$ sufficiently large, we can find a solution u of $\nabla \cdot \gamma \nabla u = 0$ on X such that

$$u(x) = \frac{1}{\gamma^{\frac{1}{2}}(x)} e^{\varrho \cdot x}(1 + \varphi(x)), \tag{33}$$

and such that (31) holds. For instance, for $s \in \mathbb{N}$, we verify that (30) holds provided that γ is of class $C^{s+2}(X)$. The case $s = 0$ with γ of class $C^2(X)$ is the setting of Theorem .

Unique Continuation and Cauchy problem

The proofs of the (weak) unique continuation principle (UCP) stating that a solution vanishing on an open set vanishes everywhere, of the Cauchy data problem, stating that a solution vanishing on an open set of the boundary with vanishing Neumann conditions on that open set has to vanish everywhere, are based on Carleman estimates.
We show that the Laplacian controls lower-order derivatives locally by means of Carleman estimates.
Let $\psi(x)$ be a function such that $\psi = \exp(\phi)$ with $\phi \geq 0$ on a domain of interest. We are looking at the operator

$$P_{\tau\psi} = e^{\tau\psi}(-\Delta)e^{-\tau\psi} = -e^{\tau\psi}\nabla e^{-\tau\psi} \cdot e^{\tau\psi}\nabla e^{-\tau\psi} = -(\nabla - \tau\nabla\psi)\cdot(\nabla - \tau\nabla\psi)$$

and want to show that for an appropriate function ψ defined on a bounded domain X and for $u \in C_c^\infty(X)$, we have

$$\|P_{\tau\psi}u\| \geq C\Big(\tau^{\frac{1}{2}}\|\nabla u\| + \tau^{\frac{3}{2}}\|u\|\Big)$$

for a positive constant C independent of τ sufficiently large.

Let φ be a smooth function bounded below and such that $|\nabla\varphi| \geq c_0 > 0$ on $\bar{X}$. Then for λ sufficiently large and $\psi = \exp(\lambda\varphi)$, there is a positive constant C independent of τ and $u \in H_0^2(X)$ such that for all τ sufficiently large, we have

$$\|e^{\tau\psi}Lu\| \geq C\Big(\tau^{\frac{1}{2}}\|e^{\tau\psi}\nabla u\| + \tau^{\frac{3}{2}}\|e^{\tau\psi}u\|\Big)$$

for any operator of the form $L = a\Delta + b\cdot\nabla + c$ for $a(x)$ scalar bounded below by a positive constant and a, b, c bounded above (component by component) or in divergence form $L = \nabla\cdot a\nabla + b\cdot\nabla + c$ provided a is Lipschitz in the latter case. Lipschitz continuity of a is then optimal to obtain unique continuation results as such results are known not to hold for Hölder continuous coefficient $a(x)$.

Similar estimates hold for a tensor valued as well.

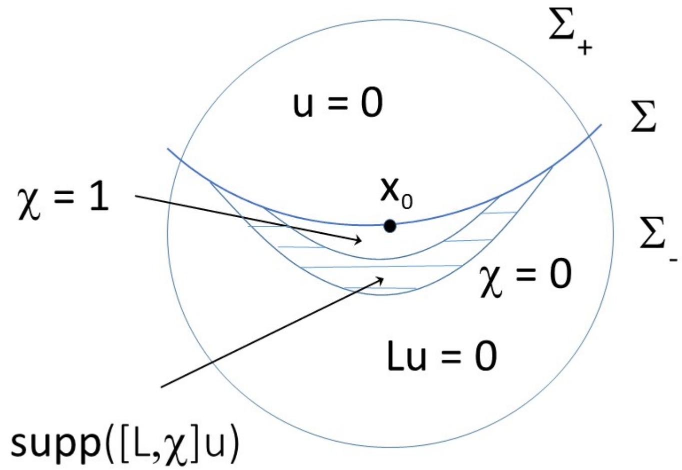
We now want to use the above Carleman estimate to obtain unique continuation principles.

Let $\phi(x)$ be a smooth real-valued function with non-vanishing $\nabla\phi$ in the vicinity V of a point $x_0 \in \mathbb{R}^n$. Set $\Sigma = \phi^{-1}(0)$ in V and $\Sigma_\pm$ the parts of V where $\pm\phi > 0$. We assume that $u = 0$ on Σ_+ and that $Lu = 0$ on V. We want to show that $u = 0$ in an open neighborhood of x_0. This is done as follows. We define

$$\varphi(x) = \phi(x) + \delta^3 - 3\delta|x - x_0|^2$$

for some $\delta > 0$ sufficiently small. We then verify that $\nabla\varphi \neq 0$ in V and that φ is bounded from below. Note that $\varphi(x_0) = \delta^3 > 0$. The level sets $\varphi = 0$ and $\varphi = -\delta^3$ are hypersurfaces intersecting Σ_- within a distance δ of x_0; see above figure.

Let us define $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on $\{\varphi > 0\} \cap B(0,\delta)$, equal to 0 on $\{\varphi < -\delta^3\}$ and hence with support of $\nabla\chi$ in Σ_- given by $\{-\delta^3 < \varphi < 0\}$. This is also the domain where $u\nabla\chi$ is supported and hence where $[L,\chi]u$ is supported. Define $\psi = e^{\lambda\varphi} - 1$ as above with λ sufficiently large, so that φ and ψ share the same 0$-$level set.



Now we have

$$\|u_{|\chi=1}\| \leq \|e^{\tau\psi}u\chi\| \leq \tau^{-\frac{3}{2}}\|e^{\tau\psi}L(\chi u)\| = \tau^{-\frac{3}{2}}\|e^{\tau\psi}[L,\chi]u\| \leq \tau^{-\frac{3}{2}}\|[L,\chi]u\|$$

since $e^{\tau\psi} \geq 1$ where $\chi = 1$ and $u \neq 0$ and $e^{\tau\psi} \leq 1$ where $[L,\chi]u$ is supported.

Finally, for u sufficiently smooth, $\|[L,\chi]u\|$ is bounded so that, sending

$\tau \to \infty$, we observe that $u_{|\chi=1} = 0$. This domain includes an open set including the point x_0.

This proves unique continuation across a surface locally. When $u = 0$ on one side of the surface and $Lu = 0$ in the vicinity of the surface, then $u = 0$ in a (possibly smaller) vicinity of the surface simply by displacing x_0 along the surface.

From this, we deduce the weak unique continuation principle for second-order elliptic equations (with scalar coefficients in the above proof although the method extends to the general elliptic (scalar!) case). This goes as follows.

Theorem. Let X be a bounded connected open domain and $u \in H^2(X)$ with $Lu = 0$ on X and L with sufficiently smooth coefficient that the above Carleman estimates hold. Let X_0 be an open subdomain with $\bar{X}_0 \subset X$ and assume that $u = 0$ on X. Then $u = 0$ on X.

Proof. Let us first assume that for open balls $B_0 \subset B_1 \subset X$, we have $u = 0$ on B_1 when $u = 0$ on B_0. We define B_t a continuous family of balls such that $B_0 \subset B_s \subset B_t \subset B_1$ for $s < t$. From the above construction, we observe that if $u = 0$ on B_s, then $u = 0$ on a slightly larger ball $B_{s'}$ for $s' > s$. This is obtained by constructing functions ϕ such that the level set of ϕ is the boundary of B_s locally in the vicinity of $x_0 \in \partial B_s$, then φ positive in the vicinity of x_0, and finally ψ such that the Carleman estimate holds. This shows that u vanishes in the vicinity of x_0. Now rotating x_0 along ∂B_s yields the result. We also observe from the construction that the vicinity is independent of s since the coefficients are uniformly sufficiently smooth on X. This shows that $u = 0$ on each ball B_s including B_1.

Now let x be any point in X and V a bounded connected (since X is connected) open domain including x and X_0 such that $\bar{V} \subset X$. By compactness, we can cover $\bar{V}$ with a finite number of open balls supported in X. Let us add to the collection a ball in X_0 where u vanishes. Assume that u vanishes on a ball B_k and that $B_k \cap B_{k+1} \neq \emptyset$. Then there is a ball in the intersection where u vanishes so that u vanishes on B_{k+1} as well. This shows that u vanishes on all balls and hence at x, which was arbitrary. Therefore, u vanishes on X. $\square$

We also have the local UCP result for Cauchy data

Theorem. Let X be a bounded connected open domain with smooth boundary and $u \in H^2(X)$ with $Lu = 0$ on X and L with sufficiently smooth coefficient that the above Carleman estimates hold. Let Γ be a non-empty open subset of ∂X and assume that u and ∇u vanish on Γ. Then $u = 0$ in X.

Proof. Note that the conditions are $u = 0$ and $\nu \cdot \nabla u = 0$ on Γ with $\nu(x)$ the outward unit normal to X at $x \in \partial X$. Let $x_0 \in \Gamma$ and let B be a sufficiently small ball centered at x_0 so that the intersection of ∂X and

B_0 is in Γ. Let us extend u by 0 in the open part Y_1 of B that is not in X. We then extend the coefficients in L to smooth coefficients in B that preserve ellipticity. Then $Lu = 0$ in Y_1 and in Y_2, the intersection of B with X. Moreover, the compatibility conditions at $\partial Y_1 \cap \partial Y_2$ ensure that $u \in H^2(B)$. We have therefore constructed an extension u such that $Lu = 0$ on B and $u = 0$ in Y_1. From the preceding theorem, this means that $u = 0$ in the whole of B, and hence in Y_2 and finally all of X. $\square$

Runge Approximation

Can one control solutions in a subdomain B from solutions in X? It should be clear that such a control cannot be exact. Solutions in B need not be smooth on ∂B. When the coefficients in the vicinity of ∂B are smooth, then solutions on X have to be smooth there. So control has to be approximate at best. What makes the approximate control possible is the unique continuation principle (UCP) we already used in the analysis of the Cauchy problem for elliptic equations.

We denote $L = \nabla \cdot a\nabla + b \cdot \nabla + c$ and recall that UCP holds for such an operator when a is elliptic and of class C^1 while all other coefficients are bounded. We proved such a result in the case where a is scalar. Such results extend to the anisotropic setting as well. We assume L invertible on X when augmented with Dirichlet conditions.

Theorem[Runge approximation]Let X_0 be an open subset with closure in X and let u be a solution in $H^1(X_0)$ of $Lu = 0$ on X_0. Then there is a sequence of function $u_\varepsilon \in H^1(X)$ solutions of $Lu_\varepsilon = 0$ in X such that $v_\varepsilon \to u$ in $L^2(X_0)$, where $v_\varepsilon = u_{\varepsilon|X_0}$ is the restriction of u_ε to X_0.

Proof. The proof goes by contradiction as an application of a geometric version of the Hahn-Banach theorem. Let

$$F = \{v_{|X_0};\ v \in H^1(X), Lv = 0 \text{ on } X\}$$

and

$$G = \{u \in H^1(X_0), Lu = 0 \text{ on } X_0\}.$$

Both are subspaces of $L^2(X_0)$. Let $\bar{F}$ and $\bar{G}$ be the closures of F and G for the $L^2(X_0)$ topology. The Runge approximation states that $\bar{F} = \bar{G}$. Assume otherwise. We verify that $\bar{F}$ is a closed convex subset of $L^2(X_0)$ since F is a linear space. Assume the existence of $u \in \bar{G}$ with $u \notin \bar{F}$.

Then $\{u\}$ is a compact subset of $L^2(X_0)$ and Hahn-Banach states that $\bar{F}$ and $\{u\}$ are separated, in other words, there is an element $f \in L^2(X_0)$ identified with its dual such that $(f, v) < \alpha < (f, u)$ for some $\alpha > 0$, say. Since v may be replaced by λv for any $\lambda \in \mathbb{R}$, this implies $(f, v) = 0$ for each $v \in \bar{F}$ while $(f, u) > 0$. Let us now prove that this contradicts the unique continuation principle (UCP).

Let us extend f by 0 in X outside X_0 and still call f the extension. Let us solve

$$L^* w = f, \quad \text{in} \quad X, \qquad w = 0, \quad \text{on } \partial X.$$

This equation admits a unique solution by assumption on L (and the Fredhiolm alternative). Then, for $v \in F$, we have

$$0 = (Lv, w) - (v, L^* w) = -\int_{\partial X} (an \cdot \nabla w) v d\sigma.$$

This holds for arbitrary trace $v_{|\partial X}$ in $H^{\frac{1}{2}}(\partial X)$, from which we deduce that $an \cdot \nabla w = 0$ on ∂X. However, we also have $w = 0$ on ∂X so that all Cauchy data associated to w vanish. From the UCP, we deduce that $w \equiv 0$, and hence that $f \equiv 0$, which is incompatible with the existence of $u \notin \bar{F}$. This proves the result. $\square$

The approximation we obtain is in the $L^2(X_0)$ sense. This is not sufficient to obtain point-wise linear independence of Hessians of v_j. However, such an estimate clearly comes from elliptic interior regularity. Indeed, we have $u_\varepsilon - u$ small in X_0 and $L(u_\varepsilon - u_0) = 0$ in X_0. Elliptic regularity with coefficients in $C^{p,\alpha}$ shows that $u_\varepsilon - u$ is also small in $C^{p,\alpha}(X_1)$ for any X_1 open with closure in X_0. This concludes our control of internal derivatives of elliptic solutions from the boundary. The boundary controls are obviously the sequence of traces $f_\varepsilon = u_{\varepsilon|\partial X}$. We choose ε small enough so that the Hessian of u_ε and that of u are sufficiently small. This proves the linear independent of Hessians and gradients of the functions $u_{j,\varepsilon}$ approximating u_j. By continuity of elliptic solutions with respect to perturbations in the boundary conditions, we therefore obtain the existence of an open set of boundary conditions f_j such that the resulting solutions u_j satisfy the independence conditions on the domain X_1. This may be repeated for a covering of X by domains of the form X_1 (including by domains that cover the boundary ∂X using a slightly different regularity theory leading to the same results).

Sub-elliptic stability in PAT

Here are mathematical assumptions on the coefficients and a definition of *well-chosen* illuminations.

(H1). We denote by $\mathfrak{X}$ the set of coefficients (γ, σ, Γ) that are of class $W^{1,\infty}(X)$, are bounded above and below by fixed positive constants, and such that the traces $(\gamma, \sigma, \Gamma)_{|\partial X}$ on the boundary ∂X are fixed (known) functions.

(H2). The illuminations f_j are positive functions on ∂X that are the restrictions on ∂X of functions of class $C^3(\bar{X})$.

(H3). We say that $\mathfrak{f}_2 = (f_1, f_2)$ is a pair of *well-chosen* illuminations with corresponding functionals $(H_1, H_2) = (\mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_1, \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_2)$ provided that **(H2)** is satisfied and the vector field

$$\beta := H_1 \nabla H_2 - H_2 \nabla H_1 = H_1^2 \nabla \frac{H_2}{H_1} = H_1^2 \nabla \frac{u_2}{u_1} = -H_2^2 \nabla \frac{H_1}{H_2} \qquad (34)$$

is a vector field in $W^{1,\infty}(X)$ such that

$$|\beta|(x) \geq \alpha_0 > 0, \qquad \text{a.e. } x \in X. \tag{35}$$

(H3'). We say that $\mathfrak{f}_2 = (f_1, f_2)$ is a pair of *weakly well-chosen* illuminations with corresponding functionals $(H_1, H_2) = (\mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_1, \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_2)$ provided that **(H2)** is satisfied and the vector field β defined in (34) is in $W^{1,\infty}(X)$ and $\beta \neq 0$ a.e. in X.

Remark. Note that **(H3')** is satisfied as soon as $\frac{f_1}{f_2} \neq C$ is not a constant. Indeed, if $\beta = 0$ on a set of positive measure, then $\nabla \frac{u_2}{u_1} = 0$ on that same set. Yet, $\frac{u_2}{u_1}$ solves the elliptic equation (37) below. It is known that under sufficient smoothness conditions on the coefficients, the critical points of solutions to such elliptic equations are of measure zero unless these solutions are constant.

Hypothesis **(H3)** will be useful in the analysis of the stability of the reconstructions. For the uniqueness result, the weaker hypothesis **(H3')** is sufficient. Note that almost all illumination pairs $\mathfrak{f}_2$ satisfy **(H3')**, which is a mere regularity statement. Beyond the regularity assumptions on (γ, σ, Γ), the domain X, and the boundary conditions f_j, the only real assumption we impose is thus (35). In general, there is no guaranty that the gradient of $\frac{u_2}{u_1}$ does not vanish. Not all pairs of illuminations $\mathfrak{f}_2 = (f_1, f_2)$ are *well-chosen* although most are *weakly well-chosen*. That the vector field β does not vanish is a *sufficient* condition for the stability estimates presented below to be satisfied. It is not *necessary*. As we shall see, guaranteeing (35) is relatively straightforward in dimension $n = 2$. It is much complicated in dimension $n \geq 3$. The only available methodology to ensure that (35) holds for a large class of conductivities is based on the same method of complex geometric optics (CGO) solutions already used to solve the Calderón problem in Chapter 7.

Under these hypotheses, we obtain the following result:

Theorem. Let $\mathfrak{X}$ be defined as in **(H1)** and let $\mathfrak{f}_2$ be *well chosen illuminations* as indicated in **(H2)** and **(H3')**. Let $I \in \mathbb{N}^*$ and $\mathfrak{f} = (f_1, \ldots, f_I)$ be a set of (arbitrary) illuminations satisfying **(H2)**. Then we have the following:

(i). The measurement operator $\mathfrak{M}_{\mathfrak{f}_2}$ uniquely determines $\mathfrak{M}_{\mathfrak{f}}$ (meant in the sense that $\mathfrak{M}_{\mathfrak{f}_2}(\gamma, \sigma, \Gamma) = \mathfrak{M}_{\mathfrak{f}_2}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma})$ implies that $\mathfrak{M}_{\mathfrak{f}}(\gamma, \sigma, \Gamma) = \mathfrak{M}_{\mathfrak{f}}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma})$).

(ii). The measurement operator $\mathfrak{M}_{\mathfrak{f}_2}$ uniquely determines the two following functionals of (γ, σ, Γ) (meant in the same sense as above):

$$\chi(x) := \frac{\sqrt{\gamma}}{\Gamma\sigma}(x), \qquad q(x) := \Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} + \frac{\sigma}{\gamma}\Big)(x). \tag{36}$$

(iii). Knowledge of the two functionals χ and q uniquely determines (in the same sense as above) $\mathfrak{M}_{\mathfrak{f}_2} = (H_1, H_2)$. In other words, the reconstruction of (γ, σ, Γ) is unique up to any transformation that leaves (χ, q) invariant.

Proof. Let us start with (i). Upon multiplying the equation for u_1 by u_2, the equation for u_2 by u_1, and subtracting both relations, we obtain

$$\begin{aligned} -\nabla\cdot(\gamma u_1^2)\nabla\frac{H_2}{H_1} &= 0, && \text{in } X \\ \gamma u_1^2 &= \gamma_{|\partial X} f_1^2, && \text{on } \partial X. \end{aligned} \tag{37}$$

This is a transport equation in conservative form for γu_1^2. More precisely, this is a transport equation $\nabla\cdot\rho\tilde{\beta}=0$ for ρ with $\rho_{|\partial X}=1$ and

$$\tilde{\beta}=\chi^2\beta=(\gamma u_1^2)\nabla\frac{H_2}{H_1}.$$

Since $\tilde{\beta}\in W^{1,\infty}(X)$ and is divergence free, the above equation for ρ admits the unique solution $\rho\equiv 1$ since (35) holds. Indeed, we find that $\nabla\cdot(\rho-1)^2\tilde{\beta}=0$ by application of the chain rule with $\rho_{|\partial X}-1=0$ on ∂X.

Upon multiplying the equation by $\frac{H_2}{H_1}$ and integrating by parts, we find

$$\int_X (\rho-1)^2\chi^2 H_1^2 \Big|\nabla \frac{H_2}{H_1}\Big|^2 dx = 0.$$

Using **(H3')** and the above remark, we deduce that $\rho = 1$ on X by continuity. This proves that γu_1^2 is uniquely determined. Dividing by $H_1^2 = (\Gamma\sigma)^2 u_1^2$, this implies that $\chi > 0$ defined in (36) is uniquely determined. Note that we do not need the full $W^{1,\infty}(X)$ regularity of β in order to obtain the above result. However, we still need to be able to apply the chain rule to obtain an equation for $(\rho-1)^2$ and conclude that the solution to the transport equation is unique.

Let now f be an arbitrary boundary condition.Replacing H_2 above by H

yields

$$
\begin{aligned}
-\nabla \cdot \chi^2 H_1^2 \nabla \frac{H}{H_1} &= 0, && \text{in } X \\
H &= \Gamma_{|\partial X} \sigma_{|\partial X} f, && \text{on } \partial X.
\end{aligned}
\tag{38}
$$

This is a well-defined elliptic equation with a unique solution $H \in H^1(X)$ for $f \in H^{\frac{1}{2}}(\partial X)$. This proves that $H = \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f$ is uniquely determined by (H_1, H_2) and concludes the proof of (i).

Let us next prove (ii). We have already seen that χ was determined by $\mathfrak{M}_{\mathfrak{f}_2} = (H_1, H_2)$. Define now $v = \sqrt{\gamma} u_1$, which is also uniquely determined based on the results in (i). Define

$$
q = \frac{\Delta v}{v} = \frac{\Delta(\sqrt{\gamma} u_1)}{\sqrt{D} u_1},
$$

which is the Liouville change of variables used to solve the Calderón problem. Since u_1 is bounded from below and is sufficiently smooth, the following calculations show that q is given by (36). Indeed, we find that

$$\nabla\cdot\gamma\nabla u_1 = \nabla\cdot(\sqrt{\gamma}\nabla v) - \nabla\cdot((\nabla\sqrt{\gamma})v) = \sqrt{\gamma}\Delta v - (\Delta\sqrt{\gamma})v = \sigma u_1 = \frac{\sigma}{\sqrt{\gamma}}v. \tag{39}$$

Finally, we prove (iii). Since q is now known, we can solve

$$(\Delta - q)v_j = 0, \quad X, \qquad v_j = \sqrt{\gamma_{|\partial X}}g_j \quad \partial X, \qquad j = 1, 2.$$

Because q is of the specific form (36) as a prescribed functional of (γ, σ, Γ), it is known that $(\Delta - q)$ does not admit 0 as a (Dirichlet) eigenvalue, for otherwise, 0 would also be a (Dirichlet) eigenvalue of the elliptic operator

$$(-\nabla\cdot\gamma\nabla + \sigma)\cdot = (-\sqrt{\gamma}(\Delta - q)\sqrt{\gamma})\cdot. \tag{40}$$

The latter calculation follows from (39). Thus v_j is uniquely determined for $j = 1, 2$. Now,

$$H_j = \Gamma \sigma u_j = \frac{\Gamma \sigma}{\sqrt{\gamma}} v_j = \frac{v_j}{\chi}, \qquad j = 1, 2,$$

and is therefore uniquely determined by (χ, q). This concludes the proof that (χ, q) uniquely determines $\mathfrak{M}_{\mathfrak{f}_2}$. $\square$

Theorem. Let $\mathfrak{M}_{\mathfrak{f}_2}(\gamma, \sigma, \Gamma) = (H_1, H_2)$ be the measurements corresponding to the coefficients (γ, σ, Γ) such that **(H1), (H2), (H3)** hold. Let $\mathfrak{M}_{\mathfrak{f}_2}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma}) = (\tilde{H}_1, \tilde{H}_2)$ be the measurements corresponding to the same illuminations $\mathfrak{f}_2 = (f_1, f_2)$ with another set of coefficients $(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma})$ such that **(H1), (H2)** hold. Define $\delta\mathfrak{M}_{\mathfrak{f}_2} = \mathfrak{M}_{\mathfrak{f}_2}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma}) - \mathfrak{M}_{\mathfrak{f}_2}(\gamma, \sigma, \Gamma)$.Then we find that

$$\|\chi - \tilde{\chi}\|_{L^p(X)} \le C \|\delta\mathfrak{M}_{\mathfrak{f}_2}\|^{\frac{1}{2}}_{(W^{1,\frac{p}{2}}(X))^2}, \qquad \text{for all } 2 \le p < \infty. \tag{41}$$

Let us assume, moreover, that $\gamma(x)$ is of class $C^3(\bar{X})$. Then we have the estimate

$$\|\chi-\tilde{\chi}\|_{L^p(X)} \leq C\|\delta\mathfrak{M}_{\mathfrak{f}_2}\|^{\frac{1}{3}}_{(L^{\frac{p}{2}}(X))^2}, \qquad \text{for all } 2 \leq p < \infty. \tag{42}$$

By interpolation, the latter result implies that

$$\|\chi-\tilde{\chi}\|_{L^\infty(X)} \leq C\|\delta\mathfrak{M}_{\mathfrak{f}_2}\|^{\frac{p}{3(d+p)}}_{(L^{\frac{p}{2}}(X))^2}, \qquad \text{for all } 2 \leq p < \infty. \tag{43}$$

Proof. Define $\nu = \chi^2$ and $\tilde{\nu} = \tilde{\chi}^2$ with χ defined in (36) and β and $\tilde{\beta}$ as in (34). Then we find that

$$\nabla \cdot \frac{\nu - \tilde{\nu}}{\nu}(\nu\beta) + \nabla \cdot \tilde{\nu}(\beta - \tilde{\beta}) = 0.$$

Note that $\nu\beta = \chi^2 H_1^2 \nabla \frac{H_2}{H_1}$ is a divergence-free field. Let φ be a twice differentiable, non-negative, function from $\mathbb{R}$ to $\mathbb{R}$ with $\varphi(0) = \varphi'(0) = 0$. Then we find that

$$\nabla \cdot \varphi\Big(\frac{\nu - \tilde{\nu}}{\nu}\Big)(\nu\beta) + \varphi'\Big(\frac{\nu - \tilde{\nu}}{\nu}\Big)\nabla \cdot \tilde{\nu}(\beta - \tilde{\beta}) = 0.$$

Let us multiply this equation by a test function $\zeta \in H^1(X)$ and integrate by parts. Since $\nu = \nu'$ on ∂X, we find

$$\int_X \varphi\Big(\frac{\nu-\tilde\nu}{\nu}\Big)\nu\beta\cdot\nabla\zeta dx + \int_X \tilde\nu(\beta-\tilde\beta)\nabla\cdot\Big[\zeta\varphi'\Big(\frac{\nu-\tilde\nu}{\nu}\Big)\Big]dx = 0.$$

Upon choosing $\zeta = \frac{H_2}{H_1}$, we find

$$\int_X \varphi\nu H_1^2\Big|\nabla\frac{H_2}{H_1}\Big|^2 dx + \int_X \tilde\nu(\beta-\tilde\beta)\cdot\nabla\frac{H_2}{H_1}\varphi' dx + \int_X \tilde\nu(\beta-\tilde\beta)\cdot\nabla\frac{\nu-\tilde\nu}{\nu}\frac{H_2}{H_1}\varphi'' dx = 0.$$

Above, φ stands for $\varphi(\frac{\nu-\tilde\nu}{\nu})$ in all integrals. By assumption on the coefficients, $\nabla\frac{\nu-\tilde\nu}{\nu}$ is bounded a.e.. This is one of our main motivations for assuming that the optical coefficients are Lipschitz. The middle term is seen to be smaller than the third term and so we focus on the latter one. Upon taking $\varphi(x) = |x|^p$ for $p \geq 2$ and using assumption **(H3)**, we find that

$$\|\nu-\tilde\nu\|^p_{L^p(X)} \leq C\int_X |\beta-\tilde\beta||\nu-\tilde\nu|^{p-2}dx.$$

By an application of the Hölder inequality, we deduce that

$$\|\nu - \tilde{\nu}\|_{L^p(X)} \leq C\|\beta - \tilde{\beta}\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(X)}.$$

We next write $\beta - \tilde{\beta} = (H_1 - \tilde{H}_1)\nabla H_2 + \tilde{H}_1(\nabla(H_2 - \tilde{H}_2) - \ldots$ and use the fact that the elliptic solutions and the coefficients are in $W^{1,\infty}(X)$ to conclude that (41) holds.

The other results are obtained by regularity theory and interpolation. Indeed from standard regularity results with coefficients in $W^{1,\infty}(X)$, we find that the solutions to the diffusion equation are of class $W^{3,q}(X)$ for all $1 \leq q < \infty$. Since the coefficient γ is of class $C^3(\bar{X})$, then the measurements H_j are of class $W^{3,q}(X)$ for all $1 \leq q < \infty$. Standard Sobolev estimates show that

$$\|H_j - \tilde{H}_j\|_{W^{1,q}(X)} \leq C\|H_j - \tilde{H}_j\|^{\frac{2}{3}}_{L^q(X)}\|H_j - \tilde{H}_j\|^{\frac{1}{3}}_{W^{3,q}(X)}.$$

The last term is bounded by a constant, which gives (42) for $q = \frac{p}{2}$. Another interpolation result states that

$$\|\varphi\|_\infty \leq \|\nabla\varphi\|_\infty^\theta \|\varphi\|_p^{1-\theta}, \qquad \theta = \frac{d}{d+p}.$$

This provides the stability result in the uniform norm (43). $\square$