Inverse transport with isotropic sources and angularly averaged measurements

Guillaume Bal,[*] Ian Langmore,[†] and François Monard[‡]

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Abstract

We consider the reconstruction of a spatially-dependent scattering coefficient in a linear transport equation from diffusion-type measurements. In this setup, the contribution to the measurement is an integral of the scattering kernel against a product of harmonic functions, plus an additional term that is small when absorption and scattering are small. The linearized problem is severely ill-posed. We construct a regularized inverse that allows for reconstruction of the low frequency content of the scattering kernel, up to quadratic error, from the nonlinear map. An iterative scheme is used to improve this error so that it is small when the high frequency content of the scattering kernel is small.

1 Introduction

Linear transport equations are used in many applications such as the propagation of the energy density of waves in heterogeneous media [5, 9, 19, 25], neutrons in nuclear reactors [17], and more recently, near-infra-red photons in tissues and its application in optical tomography, a medical imaging modality [3, 8, 18].

Inverse transport theory consists of reconstructing the constitutive parameters in the transport equation from various measurements. The typical optical parameters one wants to reconstruct are the total absorption coefficient σ and the scattering coefficient k. Several theories have been developed on the reconstruction of such parameters in various settings; see [11, 15, 16, 20, 21]. All rigorous theoretical results of uniqueness and stability [20] are based on full phase-space measurements. What we mean by this is the following. Particle

[*]Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027; gb2030@columbia.edu

[†]Department of Mathematics, University of Washington, Seattle WA, 98195; ilangmor@u.washington.edu

[‡]Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027; fm2234@columbia.edu and Ecole Nationale Supérieure de l'Aéronautique et de l'Espace, 31400 Toulouse, France; fmonard@supaero.fr

densities depend on their position x and their direction v, which in this paper we assume is normalized to $|v| = 1$. Phase space measurements mean that $u(x, v)$ can be arbitrarily chosen and measured at the domain's boundary as a function of its phase-space variables (x, v) in $n + (n - 1)$ dimensions for $n-$dimensional problems. This means having $4(n - 1)$ dimensions of available data to reconstruct the optical parameters. Such data are usually not available in practice [3], although minimization-based reconstructions from smaller data sets still are of reasonable quality [18]. The reconstruction of optical parameters from angularly averaged measurements, but still with phase-space incoming source radiations, was recently addressed in [14]. The purpose of this paper is to consider the reconstruction of optical parameters from isotropic sources and angularly averaged measurements, i.e., from $2(n - 1)$ dimensional data. In dimension $n = 2$, this corresponds to 2-dimensional data sets. The optical parameters therefore realistically need be 2-dimensional as well, i.e., may only depend on the spatial variable x.

The problem thus resembles that of electrical impedance tomography and of the reconstruction of diffusion coefficients from Cauchy data [7, 10, 22]. Diffusion equations may also be used to model solutions of transport equations in highly scattering media [12]. This explains why diffusion models are very popular in optical tomography [3]. In this paper, we are interested in cases where the diffusion approximation does not hold, and yet only diffusion-type measurements are available.

Our results require somewhat non-practical simplifications. We know in the diffusive regime that only one of the optical parameters k and σ may be reconstructed from Cauchy data; see e.g. [4]. In this paper, we are interested in reconstructing the scattering coefficient. The first simplification is thus to assume that $\sigma(x)$ has already been obtained, for instance by angularly dependent measurements of Radon transform type; see [20]. We also assume that the scattering coefficient $k = k(x)$ is independent of the direction of scattering. Our techniques do not allow for the reconstruction of scattering coefficients of the form $f(v, v')k(x)$ (see [20]), even when the phase function $f(v, v')$ is known. The second simplification is that both the known $\sigma(x)$ and the unknown $k(x)$ are sufficiently small. How small they have to be will be made more explicit in the next section. Smallness is required because our inversion formula is based on a linearization of the inverse transport problem, which we can invert explicitly only in the limit $\sigma = 0$. The nonlinear reconstruction is then based on applying standard fixed point arguments.

Surprisingly enough, the solution of the linearized inverse transport problem is performed by using the same complex geometrical optics (CGO) solutions as in the inverse diffusion problem treated in its linearized form in [7] and in its full non-linear form in [22]; see also [24]. As in [7], the main tool used in this inversion is the construction of harmonic CGOs, which allow us to have access to the Fourier transform of $k(x)$ thanks to the density of products of harmonic functions. As a result, as in [7], the linearized inverse generates a severely ill-posed problem with exponential-type stability. These very negative results should be contrasted with the situation where phase space measurements are available, which allow us in many settings to obtain much better behaved Hölder-type stability estimates [14, 20, 26].

The rest of the paper is structured as follows. We state our main hypotheses and main results in section 2. In section 3 we describe the forward problem, in particular the measurements (5) and isotropic sources in section 3.1. In section 3.2, the so-called half-adjoint operator is introduced. This emerges as a result of our measurements, and is the reason we are able to use harmonic solutions. In section 4, we use the half-adjoint operator to solve the linearized inverse problem. In section 5 we define a regularized inverse and use it to obtain our main results. Some conclusions are offered in section 6.

2 Statement of the Main Results

Let $X \subset \mathbb{R}^n, n \geq 2$ be an open bounded strictly convex set with C^2 boundary ∂X. Denote $\Gamma_\pm = \{(x, v) \in \partial X \times \mathbb{S}^{n-1} : \pm\nu_x \cdot v > 0\}$, where ν_x is the outer normal to ∂X at $x \in \partial X$. The stationary linear transport equation for the density $u(x, v)$ we consider in this paper is defined as:

$$\begin{aligned} v \cdot \nabla_x u(x, v) + \sigma(x)u(x, v) - k(x) \int_{\mathbb{S}^{n-1}} u(x, v')dS(v') = 0, \\ u|_{\Gamma_-} = u_-. \end{aligned} \tag{1}$$

Following our discussion in the introduction, we consider the simplified setting of isotropic scattering $k = k(x)$. An existence theory for (1) is recalled in the next section.

We assume that $k, \sigma \in L^\infty(X)$ are bounded functions, extended by 0 outside of X. Smallness assumptions on $k(x)$ are necessary for (1) to admit a solution. Indeed creation of particles need be compensated by absorption and leakage of particles at the domain's boundary in order for (1) to admit a physical solution [12, 17]. Our inversion algorithm requires that we make additional smallness assumptions on $k(x)$ and $\sigma(x)$.

Let us define the operator $L_{\sigma,k}$ as:

$$L_{\sigma,k}h(x) := \int_X \frac{e^{-\int_0^{|x-y|} \sigma(x+s\frac{y-x}{|y-x|})ds}}{|x - y|^{n-1}} k(y)h(y)\, dy. \tag{2}$$

Existence of a solution to (1) is guaranteed provided that the spectral radius $\rho(L_{\sigma,k}) < 1$ [17], which is always satisfied provided that k is sufficiently small. We also define $L_\sigma = L_{\sigma,1}$, where $k \equiv 1$. We then verify that

$$\|L_\sigma\|_p \leq \sup_{x\in X} \int_X \frac{e^{-\int_0^{|x-y|} \sigma(x+sv)ds}}{|x - y|^{n-1}}\, dy, \quad p \in [1, \infty], \tag{3}$$

where $\|\cdot\|_p$ denotes the norm in $\mathcal{L}(L^p, L^p)$. The above bound is straightforward to verify for $p = 1$ and $p = \infty$ and comes for other values of p by the Riesz-Thorin interpolation theorem [6] (see also lemma 3.2 below for a simple proof).

Our first smallness assumption on k is that

$$|k\|_{L^\infty}\|L_\sigma\| < 1, \tag{4}$$

where we denote by $\|L_\sigma\| = \|L_\sigma\|_2$. This assumption, which is less optimal than the condition $\rho(L_{\sigma,k}) < 1$, will allow us to write the transport solution as an infinite series corresponding to increasing orders of scattering and to conveniently estimate the influence of high orders of scattering. This is the only necessary assumption in our first result, theorem 2.1. Additional assumptions will be made explicit in order to prove our non-linear inversion result in theorem 2.2.

Our measurements are constructed as follows. For each isotropic source $u_- = f(x)$ at the domain's boundary, we measure the current at point $x \in \partial X$ given by:

$$\int_{\nu_x\cdot v>0} u(x,v)\nu_x \cdot v\, dS(v), \tag{5}$$

where ν is outward the unit normal, and u solves (1). After integrating the current over ∂X with weight function $g(x)$, we obtain the following type of measurements:

$$\mathcal{M}(f,g) := \int_{\partial X} g(x) \int_{\nu_x\cdot v>0} u(x,v)(\nu_x\cdot v)\, dS(v) d\mu(x). \tag{6}$$

The crux of the inversion is to reconstruct $k(x)$ from the contribution in (6) that is linear in k, which corresponds to the single scattering term. When $\sigma = 0$, we show that this contribution to our measurement is equivalent to the integral of k against the product of two harmonic functions. Under suitable regularity conditions, these harmonic functions are arbitrary. Borrowing then techniques from the problem of electrical impedance tomography, we are able to solve this problem with the classical harmonic solutions of Calderón [7]. As in the Calderón problem, however, this inversion is severely ill-posed.

Specifically, the recovery of higher frequencies comes with an error growing exponentially with frequency. We therefore propose a regularized inverse, which attempts to recover $P_\chi k$ with:

$$P_\chi k := \int_{\mathbb{R}^n} \hat{k}(\xi)\chi(\xi)e^{i\xi\cdot x}\, d\xi.$$

Here χ must decay sufficiently fast for large ξ. In particular, we require

$$\chi(\cdot)e^{\frac{|\cdot|}{2}\mathrm{diam}(X)} \in L^1(\mathbb{R}^n). \tag{7}$$

Our error will depend on the bound $\|T^\chi\| := \|T^\chi\|_{L^2((\partial X)^2)\to L^\infty(X)}$, for an operator T^χ that will be introduced in (43) below. If χ is the characteristic function of the ball of radius M, the result (44) gives us

$$\|T^\chi\| \le C_X \int_{|\xi|<M} e^{\frac{|\xi|}{2}\mathrm{diam}(X)}\, d\xi,$$

with C_X depending only on X. In this case, P_χ is the orthogonal projection of an L^2 function onto its low frequency content ($|\xi| < M$). When computing the Fourier transform of k, we extend it to be zero outside of X. Since then the support of k is compact, k always has some high frequency content.

Our main results are the following:

Theorem 2.1 (Recovery of $P_\chi k$). *Suppose that* (4) *holds. Then there exists a constant $C_X > 0$, depending only on X and $\|L_\sigma\|$ such that for all χ satisfying (7), the measurements $\{\mathcal{M}(f,g), (f,g) : f \in L^1(\partial X), g \in L^1(\partial X)\}$ determine $P_\chi k$ up to an error, bounded in L^∞ by*

$$C_X \|T^\chi\| \|k\|_{L^\infty} \left(\|\sigma\|_{L^\infty} + \|k\|_{L^\infty} \right).$$

Using an iterative scheme, and a smallness assumption on k and σ, we are able to improve this result as follows.

Theorem 2.2 (Iterative improvement). *Given χ satisfying (7), $c_1 \in (0,1)$, and σ such that $\|\sigma\|_{L^\infty} < \frac{c_1}{C_X \|T^\chi\|}$, there exists $\varepsilon > 0$ such that for $\|k\|_{L^\infty(X)} < \varepsilon$, the measurements $\{\mathcal{M}(f,g), (f,g) : f \in L^1(\partial X), g \in L^1(\partial X)\}$ determine $P_\chi k$ up to an error bounded in L^∞ by*

$$\frac{c_1}{1-c_1} \|(I - P_\chi)k\|_{L^\infty(X)}.$$

Our iterative method requires a certain constant of contraction in order to converge. Hence the c_1 constant. The constant ε depends on $c_1, \sigma, C_X, \|T^\chi\|$, though when $\|\sigma\|_{L^\infty} \ll \frac{c_1}{C_X \|T^\chi\|}$, we can approximate this constant with the following expression independent of $\|\sigma\|_{L^\infty}$:

$$\varepsilon \approx \frac{c_1(1-c_1)}{2\|L_\sigma\|(C_X \|T^\chi\|)^2 + c_1(1-c_1)}.$$

The general expression of this constant can be established with equations (50) and (52). These constants are not necessarily optimal since, while proving the theorem, we look for sufficient conditions.

3 The Forward Problem

We now return to the forward model (1) and present well-known properties that will be useful in subsequent sections.

We begin with some notation. For $(x,v) \in X \times \mathbb{S}^{n-1}$, let $\tau_\pm(x,v)$ be the distance from x to ∂X traveling in the direction of $\pm v$, and $x_\pm(x,v) = x \pm \tau_\pm(x,v)v$ be the boundary point encountered when we travel from x in the direction of $\pm v$. We also define $\tau = \tau_+ + \tau_-$. We give $\Gamma_\pm$ the measure $d\xi(x,v) = |\nu_x \cdot v| d\mu(x) dS(v)$, where $d\mu, dS$ are the volume forms on $\partial X, \mathbb{S}^{n-1}$ respectively.

Existence of a unique solution to the forward problem (1) is well-established; see e.g. [12, 17]. Let the incoming boundary condition u_- be in $L^1(\Gamma_-, \tau d\xi)$. With $u|_{\Gamma_-} = u_-$, we recast the transport equation as

$$(I - K)u = Ju_-,$$

where we have defined

$$\begin{aligned} Ju_-(y,v) &:= E(x_-(y,v),y)u_-(x_-(y,v),v), && (8)\\ Kf(x,v) &:= \int_0^{\tau_-(x,v)} E(x,x-tv)\int_{\mathbb{S}^{n-1}} k(x-tv)f(x-tv,v')\,dS(v')dt, && (9)\\ E(x,y) &:= \exp\left(-\int_0^{|y-x|}\sigma\Big(x+\frac{y-x}{|y-x|}s\Big)ds\right). && (10)\end{aligned}$$

For future reference, we also define iteratively:

$$E(a_1,\ldots,a_{i+1}) := E(a_1,\ldots,a_i)E(a_i,a_{i+1}). \qquad (11)$$

The operator J is bounded from $L^1(\Gamma_-, \tau d\xi)$ to $L^1(X \times S^{n-1})$ [12]. With the following two smallness assumptions

$$\Big\|\frac{k(x)}{\omega_n\sigma(x)}\Big\|_{L^\infty(X)} \le \beta < 1, \qquad (12)$$

where ω_n is the measure of S^{n-1}, and

$$\|k(x)\omega_n\tau_+(x,v)\|_{L^\infty(X\times S^{n-1})} \le \beta < 1, \qquad (13)$$

we obtain [12, 17] that K has norm less than $\beta < 1$ in $L^1(X\times S^{n-1})$ and $L^1(X \times S^{n-1}, \tau^{-1}dxdS(v))$, respectively. As a consequence, we have by a Neumann series, existence and uniqueness of the solution to the transport equation, and moreover:

$$u = (I-K)^{-1}Ju_- = \sum_{m=0}^{\infty} K^m Ju_-. \qquad (14)$$

Note that the restriction of u on Γ_+ is well defined in $L^1(\Gamma_+, \tau d\xi)$.

3.1 The Surface Distribution Model

Suppose our incoming flux $u_-(x,v) = f(x)$ is independent of the angular variable v. Then, the contribution due to flux at point $y \in \bar{X}$ due to incoming flux coming directly from ∂X is given by

$$Jf(x,v) := E(x_-(x,v),x)f(x_-(x,v)).$$

We will often need to integrate this flux over all directions $v \in \mathbb{S}^{n-1}$ and change variables from $v \in S^{n-1}$ to the boundary point $x_0 \equiv x_-(x,v) \in \partial X$ (at x fixed).

The change of variables from the sphere to the convex boundary ∂X is given formally by

$$dS(v) = \frac{|\nu_{x_0} \cdot v|}{|x - x_0|^{n-1}} d\mu(x_0). \tag{15}$$

The above change of variables is justified in the following result, whose proof is postponed to the appendix:

Proposition 3.1 (Change of variables from $\mathbb{S}^{n-1}$ to ∂X)**.** *Let* $\mathbf{S}$ *be a* C^2 *surface in* $\mathbb{R}^n$*.*

1. *Pick any* y *enclosed by* $\mathbf{S}$*. Then for* $f \in L^1(\mathbf{S})$*,*

$$\int_{\mathbf{S}} f(x) \frac{\nu_x \cdot (x - y)}{|x - y|^n}\, d\mathbf{S}(x) = \int_{\mathbb{S}^{n-1}} f(x_-(y, v))\, dS(v). \tag{16}$$

   Here ν_x *is the outward unit normal to* $\mathbf{S}$ *at* x*, and* $d\mathbf{S}$*,* dS *are the volume forms on* $\mathbf{S}$*,* $\mathbb{S}^{n-1}$ *respectively.*

2. *Moreover, if* $\mathbf{S}$ *is the boundary of a strictly convex domain, we have, for any* $y \in \mathbf{S}$ *and* $f \in L^\infty(\mathbf{S})$*,*

$$\int_{\mathbf{S}} f(x) \frac{\nu_x \cdot (x - y)}{|x - y|^n}\, d\mathbf{S}(x) = \int_{\substack{v \cdot \nu_y > 0 \\ v \in \mathbb{S}^{n-1}}} f(x_-(y, v))\, dS(v), \tag{17}$$

   with the same notation as in (16).

Recall that our averaged measurements at point $x \in \partial X$ are given by

$$\int_{\nu_x \cdot v > 0} u(x, v) \nu_x \cdot v\, dS(v),$$

where ν is outward the unit normal, and u solves (1). We take measurements all along ∂X, and compute the weighted average of them with weighting function g. This gives:

$$\int_{\partial X} g(x) \int_{\nu_x \cdot v > 0} u(x, v)(\nu_x \cdot v)\, dS(v) d\mu(x). \tag{18}$$

Suppose now that $u_-(x_0, v_0) = f(x_0)$, i.e., our incoming flux is the same in every direction. We then obtain the following type of measurement:

$$\mathcal{M}(f, g) := \int_{\partial X} g(x) \int_{\nu_x \cdot v > 0} u(x, v)(\nu_x \cdot v)\, dS(v) d\mu(x),$$

where u solves (1) and $u(x, v) = f(x)$ on Γ_-.

The contribution to the outgoing flux density at position x, in direction v, that has not scattered (a.k.a the ballistic contribution) will be

$$Ju_-(x, v) = Jf(x, v) = E(x_-(x, v), x) f(x_-(x, v)).$$

The ballistic contribution to $\mathcal{M}(f,g)$ is therefore,

$$\begin{aligned}
&\int_{\partial X} g(x) \int_{\nu_x\cdot v>0} Jf(x,v)|\nu_x\cdot v|\, dS(v)d\mu(x)\\
&= \int_{\partial X} g(x) \int_{\nu_x\cdot v>0} E(x_-(x,v),x)f(x_-(x,v))|\nu_x\cdot v|\, dS(v)d\mu(x)\\
&= \int_{\partial X} g(x) \int_{\partial X} f(x_0)\left(\frac{E(x_0,x)|\nu_x\cdot v||\nu_{x_0}\cdot v|}{|x_0-x|^{n-1}}\right) d\mu(x_0)d\mu(x)\Big|_{v=\frac{x-x_0}{|x-x_0|}}\\
&= \langle T_0, f\otimes g\rangle_{L^2((\partial X)^2)}, \qquad T_0(x_0,x) := \frac{E(x_0,x)|\nu_x\cdot v||\nu_{x_0}\cdot v|}{|x_0-x|^{n-1}},
\end{aligned} \tag{19}$$

where T_0 is known since σ is. T_0 is integrable due to the fact that [13, Lemma 3.15]

$$|(x-y)\cdot\nu_y| \le c|x-y|^2, \tag{20}$$

so that the ballistic contribution to $\mathcal{M}(f,g)$ is defined. The second equality is due to the change of variables (15) or equation (17) in proposition 3.1.

The contribution to the outgoing flux density at position x, in direction v, that has scattered once will be $KJf(x,v)$. The single scattering contribution to (5) at point x is then:

$$\begin{aligned}
&\int_{\nu_x\cdot v_1>0} KJf(x,v_1)(\nu_x\cdot v_1)\, dS(v_1)\\
&= \int_{\nu_x\cdot v_1>0}\int_0^{\tau_-(x,v_1)} E(x,x-t_1v_1)\int_{\mathbb{S}^{n-1}} k(x-t_1v_1)Jf(x-t_1v_1,v_0)\, dS(v_0)dt(\nu_x\cdot v_1)\, dS(v_1)\\
&= \int_{\nu_x\cdot v_1>0}\int_0^{\tau_-(x,v_1)} \int_{\mathbb{S}^{n-1}} k(x-t_1v_1)E(x,x-t_1v_1,x-t_1v_1-t_0v_0)\\
&\qquad\qquad\qquad \times f(x-t_1v_1-t_0v_0)dS(v_0)dt(\nu_x\cdot v)\, dS(v_1)\\
&= \int_X\int_{\mathbb{S}^{n-1}} k(x_1)f(x_-(x_1,v_0))E(x_-(x_1,v_0),x_1,x)\frac{|\nu_x\cdot v_1|}{|x_1-x|^{n-1}}\, dS(v_0)dx_1\\
&= \int_X\int_{\partial X} f(x_0)E(x_0,x_1,x)k(x_1)\frac{|\nu_{x_0}\cdot v_0||\nu_x\cdot v_1|}{|x_0-x_1|^{n-1}|x_1-x|^{n-1}}\, dx_1d\mu(x_0).
\end{aligned} \tag{21}$$

The third equality comes from the change of variables $x_1 = x - t_1v_1$, $dx_1 = t_1^{n-1}dt_1dS(v_1)$. The last equality is due to equation (16) of proposition 3.1.

The total single scattering contribution to $\mathcal{M}(f,g)$ is therefore

$$\begin{aligned}
&\int_{\partial X} g(x)\int_{\partial X} f(x_0)\int_X E(x_0,x_1,x)k(x_1)(t_0t_1)^{1-n}(\nu_{x_0}\cdot v_0)(\nu_x\cdot v_1)\, dx_1d\mu(x_0)d\mu(x)\\
&= \int_X k(x_1)\left(\int_{\partial X} f(x_0)\frac{E(x_0,x_1)|\nu_{x_0}\cdot v_0|}{|x_0-x_1|^{n-1}}\, d\mu(x_0)\right)\left(\int_{\partial X} g(x)\frac{E(x_1,x)|\nu_x\cdot v_1|}{|x_1-x|^{n-1}}\, d\mu(x)\right) dx_1,
\end{aligned} \tag{22}$$

where $v_0 = (x_1-x_0)|x_1-x_0|^{-1}$, and $v_1 = (x-x_1)|x-x_1|^{-1}$.

An inductive argument shows that for $m \geq 2$,

$$K^m h(x,v) = \int_0^{\tau_-(x,v)} \int_{X^{m-1}} \int_{\mathbb{S}^{n-1}} E(x, x - t_m v, x_{m-1}, x_{m-2}, \dots, x_1) k(x - t_m v) \\ \times k(x_{m-1}) \dots k(x_1) (t_1, \dots, t_{m-1})^{1-n} h(x_1, v_0) \, dS(v_0) dx_1 \dots dx_{m-1} dt_m.$$

Following a procedure similar to (21), we are able to represent the contribution to our measurements due to m scattering events in a compact form. We have the contribution to $\mathcal{M}(f,g)$ due to m scattering events equal to

$$\langle T_m(k), f \otimes g \rangle_{L^2((\partial X)^2)},$$

where we define T_m as follows.

Definition 3.1 (m^{th} scattering kernel)**.**

$$T_m(k)(x_0, x) := \int_{X^m} k(x_1) \cdots k(x_m) \frac{E(x_0, \dots, x_m, x) |\nu_{x_0} \cdot v_0| |\nu_x \cdot v_m|}{|x_0 - x_1|^{n-1} \cdots |x_m - x|^{n-1}} dx_1 \dots dx_m.$$

This leads to

$$\mathcal{M}(f,g) = \langle T_0, f \otimes g \rangle_{L^2((\partial X)^2)} + \sum_{m=1}^{\infty} \langle T_m(k), f \otimes g \rangle_{L^2((\partial X)^2)}. \tag{23}$$

Note that T_0 and $T_m(k)$, taken at points x and x_0, are the measurements given source $f = \delta_{\{x_0\}}$, and weight $g = \delta_{\{x\}}$.

The measurement viewpoint (23) will play the dominant role from now on. We will attempt to find suitable boundary functions f and g to extract the necessary information on the unknown parameter $k(x)$.

3.2 The Half-Adjoint Operator

In this section we introduce the "half-adjoint" operator along with some of its basic properties. We first recall that the Newton potential, the fundamental solution of $\Delta_y N(x,y) = \delta_x(y)$ is given by

$$N(x,y) := \frac{1}{c_n |x-y|^{n-2}} \;\; (n > 2); \qquad \frac{1}{c_2} \log|x-y| \;\; (n = 2), \tag{24}$$

where

$$c_n := (2-n)\omega_n \;\; (n \geq 3); \qquad c_2 := 2\pi \;\; (n = 2), \qquad \omega_n = \mathrm{Vol}(\mathbb{S}^{n-1}).$$

Given $x \in \partial X$, $y \in \mathbb{R}^n$, one can check that

$$\partial_{\nu_x} N(x,y) = \frac{\nu_x \cdot (x-y)}{\omega_n |x-y|^n}.$$

Since the above kernel is central to the following calculations, we recall that:

$$\int_{\partial X} \partial_\nu N(x,y)\, dx = \begin{cases} 0, & y \in \mathbb{R}^n - \bar{X} \\ 1, & y \in X \\ \frac{1}{2}, & y \in \partial X \end{cases}, \tag{25}$$

so that care must be taken with boundary values; see the sections on "double layer potentials" in e.g. [2, 13].

We are now ready to define the following.

Definition 3.2 (Half-Adjoint Operator)**.** For $y \in X$,

$$Af(y) := \omega_n \int_{\partial X} f(x) E(x,y) \partial_\nu N(x,y)\, d\mu(x). \tag{26}$$

When $E = 1$, $\omega_n^{-1} A$ is a harmonic function called the "double layer potential." $\omega_n^{-1} Af$ is a "moment" of the "double layer potential." For $y \notin \partial X$, $\omega_n^{-1} Af(y)$ is equal to the potential at y due to a distribution f of dipoles on ∂X.

We now note that:

$$T_1(k)(x_0, x) := \int_X k(x_1) \frac{E(x_0, x_1, x)|\nu_{x_0} \cdot v_0||\nu_x \cdot v_1|}{|x_0 - x_1|^{n-1}|x_1 - x|^{n-1}}\, dx_1. \tag{27}$$

With our definitions of A, T_1 we may re-write our single-scattering measurement (22) as

$$\langle T_1(k), f \otimes g \rangle_{L^2((\partial X)^2)} = \langle k, Af\, Ag \rangle_{L^2(X)}. \tag{28}$$

We also have that the resultant contribution to $\mathcal{M}(f,g)$ due to two scattering events is:

$$\begin{aligned} &\langle T_2(k), f \otimes g \rangle_{L^2((\partial X)^2)} \\ &= \int_X \int_X k(x_1) k(x_2) \frac{E(x_1, x_2)}{|x_1 - x_2|^{n-1}} Af(x_1) Ag(x_2)\, dx_1 dx_2, \\ &= \int_X k(x) Af(x) L_{\sigma,k} Ag(x) dx. \end{aligned} \tag{29}$$

and from m scattering events,

$$\begin{aligned} &\langle T_m(k), f \otimes g \rangle_{L^2((\partial X)^2)} \\ &= \int_X \cdots \int_X \frac{k(x_1) \cdots k(x_m) E(x_1, \ldots, x_m)}{|x_1 - x_2|^{n-1} \cdots |x_{m-1} - x_m|^{n-1}} Af(x_1) Ag(x_m)\, dx_1 \cdots dx_m \\ &= \int_X k(x) Af(x) L_{\sigma,k}^{m-1} Ag(x) dx. \end{aligned} \tag{30}$$

The relation (28) suggests an inversion based on finding boundary values f and g such that their products $AfAg$ are dense. As it turns out, this is possible when $\sigma = 0$ because the product of harmonic functions is dense. Proving this requires some facts about double-layer potentials.

Let A_0 be the half-adjoint operator A defined in (26) when $E \equiv 1$. Now, when $f \in L^1(\partial X)$, the defining property of the Newton potential shows that $A_0 f \in C(X)$ is harmonic. The main question now is whether any harmonic function in X may be prescribed as $A_0 f$ for a suitable boundary term f. That this is possible is based on the following classical result on the jump conditions of the double layer potential:

$$\lim_{X \ni x' \to y \in \partial X} \omega_n^{-1} A_0 f(x') = \frac{1}{2} f(y) + \omega_n^{-1} \mathscr{A}_0 f(y), \tag{31}$$

where

$$\mathscr{A}_0 f(y) = \omega_n \int_{\partial X} f(x) \partial_{\nu_x} N(x,y)\, d\mu(x), \tag{32}$$

where the integral is defined in the usual sense; see [13], (20), and the proof of Lemma 3.3 below. When X has boundary ∂X of class $\mathcal{C}^1$, the operator $\mathscr{A}_0$ is compact on $L^2(\partial X)$ and does not admit $-\frac{1}{2}$ as an eigenvalue [2, §2.2]. This shows that

$$f \mapsto \frac{1}{2} f + \omega_n^{-1} \mathscr{A}_0 f, \qquad \text{is an isomorphism} \quad L^2(\partial X) \to L^2(\partial X). \tag{33}$$

We have thus obtained that the $L^2(\partial X)$-valued trace of any harmonic function in, say $H^{\frac{1}{2}}(X)$, may be written as $\lim_{X \ni x' \to y \in \partial X} A_0 f(x')$ for some $f \in L^2(\partial \Omega)$. More formally, we have:

Lemma 3.1 (Pseudo-inverse for the A_0 operator)**.** *The operator defined by*

$$A_0^\dagger : \left\{ \begin{array}{ccc} H^{\frac{1}{2}}(X) & \to & L^2(\partial X) \\ u & \mapsto & A_0^\dagger u := (\frac{1}{2} I + \omega_n^{-1} \mathscr{A}_0)^{-1} (\omega_n^{-1} u|_{\partial X}) \end{array} \right. ,$$

is continuous and such that $A_0 A_0^\dagger u = u|_X$ *for all harmonic function* $u \in H^{\frac{1}{2}}(X)$.

Proof. The above isomorphism (33) shows that $A_0^\dagger$ is a well-posed operator. Now for u harmonic, for $f = A_0^\dagger u$, we have $A_0 f$ harmonic and $A_0 f$ and u having the same trace on ∂X. Thus $A_0 f = u$. $\square$

When $\partial X = \mathbb{S}^1$, we can find an explicit expression for $A_0^\dagger$. First, the Poisson kernel on $\{x \in \mathbb{R}^n : |x| \le 1\}$ is given by

$$P(x,y) = \frac{1 - |y|^2}{\omega_n |x-y|^n}, \quad x \in \mathbb{S}^{n-1}, \, y \in \text{unit ball}. \tag{34}$$

If $n = 2$, we can show (using the fact $|x| = 1$) that

$$P(x,y) + \frac{1}{2\pi} = 2 \partial_{\nu_x} N(x,y),$$

which leads to

$$\begin{aligned} A_0 f(y) &= \frac{2\pi}{2}\int_{\mathbb{S}^1} P(x,y) f(x)\, dx + \frac{1}{2}\int_{\mathbb{S}^1} f(x)\, dx \\ &= \pi(\tilde{f}(y) + \tilde{f}(0)). \end{aligned} \tag{35}$$

Where $\tilde{f}$ satisfies $\Delta \tilde{f} = 0$ in X, and $\tilde{f}|_{\partial X} = f$. Here we have used the defining property of the Poisson kernel, and the mean value theorem for harmonic functions. This analysis tells us that in this special geometry,

$$A_0^{\dagger} u(x) = \frac{2u(x) - u(0)}{2\pi}, \quad x \in \mathbb{S}^1.$$

We close this section with some properties on the operators A and T_i that will be useful in later sections. First we state and prove the following lemma.

Lemma 3.2. *Suppose that for every $y \in Y$,*

$$\int_X |k(x,y)|\, dx < C_1, \quad \textit{and for every } x \in X, \int_Y |k(x,y)|\, dy < C_2.$$

Then for $1/p + 1/q = 1$, $p \in [1, \infty]$,

$$\left\| \int_X k(x,\cdot) f(x)\, dx \right\|_{L^p(Y)} \leq C_1^{1/p} C_2^{1/q} \|f\|_{L^p(X)}.$$

Proof. The proof is classical and may be found e.g. in [23] when $X = Y$. For completeness, we recall it here. First consider the case $p \in (1, \infty)$. The identity $ab \leq a^p/p + b^q/q$ applied to $f(x)g(y)$ shows that

$$\Big| \int_Y \int_X k(x,y) f(x) g(y)\, dx dy \Big| \leq C_1 \frac{\|f\|_{L^p}^p}{p} + C_2 \frac{\|g\|_{L^q}^q}{q}.$$

Now the same computation with f replaced by tf, and g replaced by $t^{-1}g$, $t > 0$, and simple calculus minimization show that

$$\Big| \int_Y \int_X k(x,y) f(x) g(y)\, dx dy \Big| \leq C_1^{1/p} C_2^{1/q} \|f\|_{L^p(X)} \|g\|_{L^q(Y)}.$$

This proves the lemma for the case $p \in (1, \infty)$. The special cases $p = 1$ and $p = \infty$ are easily checked. □

This allows us to prove the following

Lemma 3.3 (L^p mapping property of half-adjoint operator)**.** *There exists C_X depending on X such that for $p \in [1, \infty]$,*

$$\|Af\|_{L^p(X)} \leq C_X \|f\|_{L^p(\partial X)}.$$

Proof. Recall that the kernel of the integral operator A is (up to some constant) $\partial_{\nu_y} N(x,y)$, with $y \in \partial X$, $x \in X$. Since ∂X is of class $\mathcal{C}^2$, lemma 3.20 from [13] and (20) show that there exists C such that

$$\int_{\partial X} |\partial_{\nu_y} N(x,y)|\, dy < C,\ \forall x \in X, \text{ and } \int_X |\partial_{\nu_y} N(x,y)|\, dx < C,\ y \in \partial X.$$

The result therefore is a direct application of lemma 3.2. □

Lemma 3.4 (Mapping property of T_m)**.** *There exists C_X, depending only on X such that for $m \geq 1$, we have*

$$\|T_m(k)\|_{L^2((\partial X)^2)} \leq C_X \|k\|_{L^\infty} \|L_{\sigma,k}\|_2^{m-1} \leq C_X \|k\|_{L^\infty}^m \|L_\sigma\|_2^{m-1}. \tag{36}$$

Proof. When $m = 1$, the proof follows from (28) and lemma 3.3. For $m \geq 2$, we find that

$$\begin{aligned}
&|\langle T_m(k), f \otimes g\rangle_{L^2((\partial X)^2)}| \\
&\leq \|k\|_{L^\infty} \int_X \cdots \int_X k(x_1)\cdots k(x_{m-1}) \frac{Af(x_1)E(x_1,\ldots,x_m)Ag(x_i)}{|x_1-x_2|^{n-1}\cdots|x_{m-1}-x_m|^{n-1}}\, dx_i \cdots dx_1 \\
&= \|k\|_{L^\infty} \int_X Af(x_1) L_{\sigma,k}^{m-1} Ag(x_1)\, dx_1 \\
&\leq \|k\|_{L^\infty} \|L_{\sigma,k}\|_2^{m-1} \|Af\|_{L^2} \|Ag\|_{L^2}.
\end{aligned}$$

The proof then follows from lemma 3.3 and the obvious bound $\|L_{\sigma,k}\|_2 \leq \|k\|_{L^\infty} \|L_\sigma\|_2$. □

4 The Linearized Inverse Problem

Lemma 3.1 and (28) motivate an attempt to invert the operator T_1 by finding dense products of harmonic functions. For the disk, one such choice would be (in polar coordinates) $r^k e^{ik\theta}$. This choice, along with its stability has been explored in [1]. Here we opt for the more general, and familiar complex geometrical optics (CGO) solutions of Calderón [7]. So let $\mathbb{C}^n \ni \rho = 1/2(\xi + i\eta)$, where $\xi, \eta \in \mathbb{R}^n$, $\xi \cdot \eta = \sum_{i=1}^n \xi_i \eta_i = 0$, and $|\xi| = |\eta|$. Then the functions $e^{i\rho\cdot x}$, and $e^{i\bar\rho\cdot x}$ are harmonic, and $e^{i\rho\cdot x} e^{i\bar\rho\cdot x} = e^{i\xi\cdot x}$.

Definition 4.1 (Oscillatory boundary values)**.** With ρ as above, we define

$$f_\xi(x) := A_0^\dagger e^{i\rho\cdot x}, \text{ and } g_\xi(x) := A_0^\dagger e^{i\bar\rho\cdot x}, \quad x \in \partial X.$$

Both functions are in $L^2(\partial X)$. Assuming that a coordinate system is chosen such that $|e^{\eta\cdot x}| \leq e^{|\eta|\mathrm{diam}(X)/2} = e^{|\xi|\mathrm{diam}(X)/2}$, from the construction of $A_0^\dagger$, these functions satisfy the following estimate:

$$\|f_\xi\|_{L^2(\partial X)}, \|g_\xi\|_{L^2(\partial X)} \leq \frac{\alpha_0}{\omega_n} |\partial X|^{\frac{1}{2}} e^{|\xi| \frac{\mathrm{diam}(X)}{4}}, \tag{37}$$

where $\alpha_0 = \|(\frac{1}{2}I + \omega_n^{-1}\mathscr{A}_0)^{-1}\|$ and $|\partial X|$ is the (Lebesgue) measure of ∂X.

We have a pseudo-inverse for A_0. However for $\sigma \neq 0$, $\langle T_1(k), f \otimes g\rangle \neq \langle k, A_0 f A_0 g\rangle$. We therefore introduce the following notation to deal with non vanishing absorption, which we treat as a perturbation.

Let us define:

$$\begin{aligned}
[T_1^0(k)](x_0,x) &= \int_X k(x_1) \frac{|\nu_{x_0}\cdot v_0||\nu_x \cdot v_1|}{|x_0-x_1|^{n-1}|x_1-x|^{n-1}}\, dx_1,\\
[T_1^\sigma(k)](x_0,x) &= \int_X k(x_1)\,[E(x_0,x_1,x)-1]\,\frac{|\nu_{x_0}\cdot v_0||\nu_x \cdot v_1|}{|x_0-x_1|^{n-1}|x_1-x|^{n-1}}\, dx_1,
\end{aligned}$$

so that $T_1(k) = T_1^0(k) + T_1^\sigma(k)$. We can prove a mapping property of T_1^σ, similar to lemma 3.4.

Lemma 4.1. *There exists C_X, depending only on X such that*

$$\|T_1^\sigma(k)\|_{L^2((\partial X)^2)} \leq C_X \|\sigma\|_{L^\infty}\|k\|_{L^\infty}. \tag{38}$$

Proof. The proof is identical to that of lemma 3.4, except that we use the relation $|e^{-a}-1| \leq a$ (valid for $a \geq 0$) to show that $|E(x_0,x_1,x)-1| \leq 2\mathrm{diam}(X)\|\sigma\|_{L^\infty}$. □

This yields a refined version of (28):

$$\begin{aligned}
\langle T_1(k), f\otimes g\rangle_{L^2((\partial X)^2)} &= \langle T_1^0(k), f\otimes g\rangle_{L^2((\partial X)^2)} + \langle T_1^\sigma(k), f\otimes g\rangle_{L^2((\partial X)^2)}\\
&= \langle k, A_0 f A_0 g\rangle_{L^2(X)} + \langle T_1^\sigma(k), f\otimes g\rangle_{L^2(X)}.
\end{aligned} \tag{39}$$

Now setting boundary values equal to f_ξ and g_ξ, (28) gives

$$\langle T_1^0(k), f_\xi \otimes g_\xi\rangle_{L^2((\partial X)^2)} = \langle k, A_0 f_\xi\, A_0 g_\xi\rangle_{L^2(X)} = \langle k, e^{i(\xi,\cdot)}\rangle_{L^2(X)} = \hat{k}(\xi).$$

Here we are using (ξ, y) to denote the dot product. This leads to:

$$k(x) = \int_{\mathbb{R}^n} \hat{k}(\xi) e^{i\xi\cdot x}\, d\xi = \int_{\mathbb{R}^n} \langle T_1^0(k), f_\xi\otimes g_\xi\rangle e^{i\xi\cdot x}\, d\xi.$$

We may also define the following operator, whose domain contains the range of T_1^0:

$$\begin{aligned}
(T_1^0)^{-1}h(x) &:= \int_{\mathbb{R}^n} \widehat{(T_1^0)^{-1}h}(\xi) e^{i\xi\cdot x}\, d\xi\\
&= \int_{\mathbb{R}^n} \langle h, f_\xi \otimes g_\xi\rangle_{L^2((\partial X)^2)} e^{i\xi\cdot x}\, d\xi.
\end{aligned} \tag{40}$$

This formal inverse operator to T_1^0 will be useful in the analysis of the nonlinear inversion problem.

5 The Nonlinear Problem

5.1 The Regularized Inverse

Inverting T_1^0 is a severely ill-posed problem. To show this, put

$$L_D^2(X) := \{h \in L^2(X) : \mathrm{dist}\,(\mathrm{supp}(h), \partial X) \geq D > 0\}.$$

It is easy enough to see that $T_1^0 : L_D^2(X) \to H^s((\partial X)^2)$ is bounded for every s. Therefore $T_1^0 : L_D^2 \to H^s$ is compact for every s. We may then construct a sequence of unit vectors $h_n \in L^2(X)$ such that $T_1^0 h_n \to 0$ in H^s.

We therefore look for some regularized version of $(T_1^0)^{-1}$ that is a bounded operator from $L^2((\partial X)^2) \to L^\infty(X)$. To this end, using the bound (37), we notice that

$$\begin{aligned} |\langle h, f_\xi \otimes g_\xi \rangle| &\leq \|h\|_{L^2((\partial X)^2)} \|f_\xi\|_{L^2(\partial X)} \|g_\xi\|_{L^2(\partial X)} \\ &\leq \|h\|_{L^2((\partial X)^2)} \frac{\alpha_0^2}{\omega_n^2} |\partial X| e^{\frac{|\xi|}{2}\mathrm{diam}(X)}. \end{aligned}$$

Our measurements and knowledge of σ give us access to:

$$\begin{aligned} \mathcal{M}(f_\xi, g_\xi) - \langle T_0, f_\xi \otimes g_\xi \rangle = \sum_{m=1}^{\infty} \langle T_m(k), f_\xi \otimes g_\xi \rangle \\ = \quad \langle T_1^0(k), f_\xi \otimes g_\xi \rangle + \langle T_1^\sigma k, f_\xi \otimes g_\xi \rangle + \sum_{m=2}^{\infty} \langle T_m(k), f_\xi \otimes g_\xi \rangle \\ = \quad \hat{k}(\xi) + R(\xi), \end{aligned} \tag{41}$$

where

$$|R(\xi)| \leq C_X \alpha_0^2 |\partial X| \left(\|\sigma\|_{L^\infty} \|k\|_{L^\infty} + \|k\|_{L^\infty}^2 \right) e^{\frac{|\xi|}{2}\mathrm{diam}(X)}. \tag{42}$$

This bound was obtained by using lemmas 3.4 and 4.1. To deal with error that will grow exponentially with frequency, we introduce a cutoff χ and define the following regularized version of T_0^{-1} in (40):

Definition 5.1 (Regularized Inverse)**.** We define:

$$T^\chi h(x) := \int_{|\xi|<M} \langle h, f_\xi \otimes g_\xi \rangle_{L^2((\partial X)^2)} \chi(\xi) e^{i\xi \cdot x}\, d\xi, \tag{43}$$

where we require that:

$$\chi(\xi) e^{\frac{|\xi|}{2}\mathrm{diam}(X)} \in L^1(\mathbb{R}^n).$$

The cut-off $\chi(\xi)$ should be seen as a function equal to 1 for small values of ξ corresponding to frequencies that we wish to invert accurately, and equal to 0 for large values of ξ corresponding to frequencies that cannot be reconstructed because of noise in the data. The simplest example of such a function is the indicatrix function χ_M equal to 1 for $|\xi| < M$ and equal to 0 elsewhere.

The following results are immediate:

$$\begin{aligned} T^\chi T_1^0(k) = P_\chi k, \quad \text{where} \quad P_\chi h(x) := \int_{\mathbb{R}^n} \hat{h}(\xi)\chi(\xi)e^{ix\cdot\xi}\, d\xi, \\ \|T^\chi h\|_{L^\infty(X)} \leq \alpha_0^2 |\partial X| \|\chi(\cdot) e^{\frac{|\cdot|}{2}\operatorname{diam}(X)}\|_{L^1(\mathbb{R}^n)} \|h\|_{L^2((\partial X)^2)}. \end{aligned} \tag{44}$$

We now illustrate a step-by-step procedure for implementing the regularized inverse. First, set the boundary values to $f = f_\xi$ and $g = g_\xi$, with $\xi \in \mathbb{R}^n$. Our measurement is then

$$\mathcal{M}(f_\xi, g_\xi) = \left\langle T_0 + \sum_{m=1}^{\infty} T_m(k), f_\xi \otimes g_\xi \right\rangle. \tag{45}$$

Subtracting the known contribution $\langle T_0, f_\xi \otimes g_\xi \rangle$, and multiplying by $\chi(\xi)$ we have thereby computed

$$\left(T^\chi \sum_{m=1}^{\infty} T_m(k) \right)^\wedge (\xi).$$

We do the same thing for all $\xi \in \operatorname{supp}(\chi)$. Then after multiplying by $e^{ix\cdot\xi}$, and integrating over $\operatorname{supp}(\chi)$ we have computed

$$\begin{gathered} T^\chi \left(\sum_{m=1}^{\infty} T_m(k) \right) = k_\chi + T^\chi \left(T_1^\sigma(k) + \sum_{m=2}^{\infty} T_m(k) \right), \\ \text{where} \\ k_\chi := P_\chi k. \end{gathered} \tag{46}$$

We thus obtain from our measurements a reconstruction of $P_\chi k$, the low-frequency part of k, up to an error that is formally quadratic in (k, σ). In the next section, we make this statement more precise and prove our main theorems 2.1 and 2.2.

5.2 Proofs of Main Results

Using this framework, we may prove theorem 2.1.

Proof of theorem 2.1. (46) shows that our measurements determine k_χ up to the error term $T^\chi(T_1^\sigma k + \sum_{m=2}^\infty T_m(k))$. Using lemmas 3.4, 4.1, and (44) we have

$$\left\| T^\chi T_1^\sigma(k) + T^\chi \sum_{m=2}^{\infty} T_m(k) \right\|_{L^\infty(X)} \leq C_X \|T^\chi\| \left(\|\sigma\|_{L^\infty} \|k\|_{L^\infty} + \sum_{m=2}^{\infty} \|L_\sigma\|^{m-1} \|k\|_{L^\infty}^m \right).$$

Our smallness assumption on k that $\|k\|_\infty \|L_\sigma\| < 1$ ensures that the series converges, and the theorem is proved. □

Our iterative scheme is motivated by (46), and allows us to improve our estimate of k_χ.

Proof of theorem 2.2. In this proof, C_X denotes the maximum of the constants from lemma 3.4, and lemma 4.1. Each depends only on X. Defining $D := T^\chi(\sum_{m=1}^\infty T_m(k))$, we arrive at our iterative scheme.

$$\begin{aligned} k_\chi^0 &= D, \\ k_\chi^{\nu+1} &= T^\chi\left(\sum_{m=1}^\infty T_m(k)\right) - T^\chi\left(T_1^\sigma(k_\chi^\nu) + \sum_{m=2}^\infty T_m(k_\chi^\nu)\right) \\ &= D - F(k_\chi^\nu) \;\; \nu = 0,1,2,\dots \end{aligned} \tag{47}$$

The idea is that $\sum_{m=1}^\infty T_m(k)$ is our measured data, and F is a mapping we are able to compute since σ is known. To show convergence of the scheme, we will use the contraction mapping principle. To this end, we define the operator

$$G(k_\chi) := D - F(k_\chi),$$

and we will show that, under certain conditions on $\|k\|_{L^\infty}$ and $\|\sigma\|_{L^\infty}$, we can find a closed set $B \subset L^\infty(X)$, such that G is a contraction mapping on B and $G(B) \subset B$. Moreover, if $\|k\|_{L^\infty}$ is sufficiently small so that $D \in B$, then the iterated scheme (47) will converge.

Step 1: Condition for G to be a contraction. We first prove the following estimate:

Lemma 5.1. *For all $k_\chi, \tilde{k}_\chi, \sigma \in L^\infty(X)$, the following inequality holds:*

$$\|F(k_\chi) - F(\tilde{k}_\chi)\|_{L^\infty} \le \|T^\chi\| C_X \|k_\chi - \tilde{k}_\chi\|_{L^\infty}\left(\|\sigma\|_{L^\infty} + \frac{M\|L_\sigma\|(2 - M\|L_\sigma\|)}{(1 - M\|L_\sigma\|)^2}\right), \tag{48}$$

where $M = \max(\|k_\chi\|_{L^\infty}, \|\tilde{k}_\chi\|_{L^\infty})$.

Proof. We start with the inequality:

$$\|F(k_\chi) - F(\tilde{k}_\chi)\|_{L^\infty} \le \|T^\chi\|\left(\|T_1^\sigma(k_\chi - \tilde{k}_\chi)\|_{L^2} + \left\|\sum_{m=2}^\infty T_m(k_\chi) - T_m(\tilde{k}_\chi)\right\|_{L^2}\right). \tag{49}$$

Repeated use of the relation $ab - \tilde{a}\tilde{b} = (a - \tilde{a})b + \tilde{a}(b - \tilde{b})$ shows that the term $T_m(k)_\chi - T_m(\tilde{k}_\chi)$ looks like $T_m(k)$, except that instead of $k(x_1)\cdots k(x_m)$ it has the term

$$\begin{aligned} &k_\chi(x_1)\cdots k_\chi(x_m) - \tilde{k}_\chi(x_1)\cdots\tilde{k}_\chi(x_m) \\ &\qquad = \sum_{j=1}^m \tilde{k}_\chi(x_1)\cdots\tilde{k}_\chi(x_{j-1})[k_\chi(x_j) - \tilde{k}_\chi(x_j)]k_\chi(x_{j+1})\cdots k_\chi(x_m), \end{aligned}$$

so we may use lemma 3.4 to see that

$$\|T_m(k_\chi) - T_m(\tilde{k}_\chi)\|_{L^2} \le mC_X\|L_\sigma\|^{m-1}M^{m-1}\|k_\chi - \tilde{k}_\chi\|_{L^\infty}.$$

Now since

$$\sum_{m=2}^{\infty} m x^m = \frac{x(2-x)}{(1-x)^2}, \quad |x| < 1,$$

we can sum the last equation for $m = 2 \ldots \infty$ provided that $M\|L_\sigma\| < 1$. The result is the following inequality:

$$\sum_{m=2}^{\infty} \|T_m(k_\chi) - T_m(\tilde{k}_\chi)\|_{L^2} \leq C_X \|k_\chi - \tilde{k}_\chi\|_{L^\infty} \frac{M\|L_\sigma\|(2 - M\|L_\sigma\|)}{(1 - M\|L_\sigma\|)^2}.$$

We then use lemma 4.1 and sum the terms in (49) to obtain the desired result. □

Since D is constant, we will have exactly the same estimate if we replace F by G. Let us fix now $c_1 \in (0,1)$. G will be a c_1-contraction as soon as

$$\|\sigma\|_{L^\infty} + \frac{M\|L_\sigma\|(2 - M\|L_\sigma\|)}{(1 - M\|L_\sigma\|)^2} \leq \frac{c_1}{T^\chi C_X}.$$

Now we fix $\|\sigma\|_{L^\infty} < c_1(\|T^\chi\| C_X)^{-1}$, and the previous condition becomes

$$M \leq \rho_{c_1}, \qquad \text{where} \quad \rho_{c_1} := \frac{1}{\|L_\sigma\|}\left(1 - \left(1 + \frac{c_1}{\|T^\chi\| C_X} - \|\sigma\|_{L^\infty}\right)^{-\frac{1}{2}}\right). \tag{50}$$

In other words, if we call B the $\|\cdot\|_{L^\infty(X)}$-ball of radius ρ_{c_1} and center 0, then lemma 5.1 under the condition (50) ensures that G is a c_1-contraction on B. Let us find now the condition under which $G(B) \subset B$.

Step 2: condition for B to be G-stable. We look for a bound on $\|D\|_{L^\infty}$ such that B is G-stable. For any $k_\chi \in B$, provided that F and G are c_1-contractions on B and $F(0) = 0$, we have

$$\|G(k_\chi)\|_{L^\infty} \leq \|D\|_{L^\infty} + \|F(k_\chi)\|_{L^\infty} \leq \|D\|_{L^\infty} + c_1 \rho_{c_1}.$$

Thus in order to get $\|G(k_\chi)\|_{L^\infty} \leq \rho_{c_1}$, we need that

$$\|D\|_{L^\infty} \leq (1 - c_1)\rho_{c_1}. \tag{51}$$

Using lemma 3.4 and the fact that $\|k\|_{L^\infty}\|L_\sigma\| < 1$, we have the following estimate on D

$$\|D\|_{L^\infty} \leq C_X \|T^\chi\| \frac{\|k\|_{L^\infty}}{1 - \|L_\sigma\|\|k\|_{L^\infty}}.$$

Thus (51) will hold if k satisfies

$$\|k\|_{L^\infty} \leq \frac{(1 - c_1)\rho_{c_1}}{C_X\|T^\chi\| + (1 - c_1)\rho_{c_1}\|L_\sigma\|}. \tag{52}$$

As a result of the first two steps, lemma 5.1, and equation (50), we see that the hypothesis (52) ensures G is a contraction mapping on B, and that $D \in B$.

Thus in virtue of the contraction mapping principle, the iterated scheme (47) will converge to an element $k_\chi^* \in B$ such that

$$k_\chi^* = D - F(k_\chi^*). \tag{53}$$

Step 3: Error estimates We now show that the difference between k_χ^* and k_χ is small if $k_\chi^\perp := k - k_\chi$ is small. If we define $D_0 = T^\chi \sum_{i=1}^\infty T_i(k_\chi)$, some straightforward calculations show that

$$k_\chi = D_0 - F(k_\chi), \tag{54}$$

which relies on the fact that

$$T^\chi T_1^0 k_\chi = P_\chi k_\chi = k_\chi.$$

Subtracting (53) from (54) we have

$$\|k_\chi - k_\chi^*\|_{L^\infty} \le \|D_0 - D\|_{L^\infty} + c_1 \|k_\chi - k_\chi^*\|_{L^\infty}.$$

Since $c_1 < 1$, we may absorb the second term on the right hand side into the left hand side, yielding

$$\begin{aligned}
&\|k_\chi - k_\chi^*\|_{L^\infty} \le \frac{1}{1-c_1}\|D_0 - D\|_{L^\infty} = \frac{1}{1-c_1}\left\| T^\chi \sum_{m=1}^\infty (T_m(k_\chi) - T_m(k_\chi + k_\chi^\perp))\right\|_{L^\infty} \\
&= \frac{1}{1-c_1}\left\| T^\chi \left(T_1(k_\chi) - T_1(k_\chi + k_\chi^\perp) + \sum_{m=2}^\infty (T_m(k_\chi) - T_m(k_\chi + k_\chi^\perp))\right)\right\|_{L^\infty} \\
&= \frac{1}{1-c_1}\left\| T^\chi \left(T_1^\sigma(k_\chi) - T_1^\sigma(k_\chi + k_\chi^\perp) + \sum_{m=2}^\infty (T_m(k_\chi) - T_m(k_\chi + k_\chi^\perp))\right)\right\|_{L^\infty} \\
&= \frac{1}{1-c_1}\|F(k_\chi) - F(k_\chi + k_\chi^\perp)\|_{L^\infty} \le \frac{c_1}{1-c_1}\|k_\chi^\perp\|_{L^\infty}.
\end{aligned}$$

Here, the third equality comes from the decomposition $T_1 = T_1^0 + T_1^\sigma$ and the property $T^\chi T_1^0 k = P_\chi k$ (see 44). □

6 Conclusions

We have shown that the reconstruction of the smooth part of the scattering coefficient in a transport equation could be obtained when arbitrary isotropic sources are used and the corresponding angularly averaged outgoing currents are measured, i.e., in the setting of "diffusion-type" measurements. This corresponds to the practical setting in many applications of inverse transport [3, 18, 27]. The accuracy of the reconstruction is proportional to the size of the non-smooth part of the scattering coefficient. However, we have assumed that the total absorption coefficient σ could be reconstructed by other means and was sufficiently small. These two hypotheses are very constraining from a practical viewpoint. Nonetheless, the results we have presented give a realistic theoretical backbone to practical reconstructions of optical parameters from diffusion-type measurements.

The measurements are similar to what is available in the reconstruction of diffusion coefficients from boundary measurements, as in the application to electrical impedance tomography. [7, 24]. They are thus of the same type as the measurements available in the diffusion approximation to the transport equation, which arises in the limit of vanishing mean free path and is overwhelmingly used in optical tomography [3, 12]. Our reconstructions, however, work for small values of k and σ, i.e., in a transport regime where the diffusion approximation is not valid. It is therefore somewhat surprising that the same complex geometric optics solutions may be used in both our context and that of the reconstruction of diffusion coefficients.

We would like to stress that the reconstruction of optical parameters from diffusion-type measurements is a severely ill-posed problem. More precisely, whenever σ is smooth, the forward map $k \mapsto \sum_{i=0}^{\infty} T_i k$ takes k supported inside X to a C^∞ function on $(\partial X)^2$. This explains why the stability estimates we have obtained are of exponential type, as in Calderón's problem [7]. These results are in sharp contrast to the results obtained when either the source or the measurements (or both) are allowed to depend on the angular variable. In such instances, better stability estimates of Hölder type, which render the reconstruction a mildly ill-posed problem, are available in many settings [14, 21, 26].

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A Appendix

Proof of Proposition 3.1. Proof of equation (16) :
First, assume $f \in C^1(\mathbf{S})$, and extend it to $\tilde{f} \in C^1(\mathbb{R}^2 \setminus \{y\})$, where $\tilde{f}$ is constant along rays originating at y. We can think of $\mathbb{S}^{n-1}$ as a spherical surface $\mathbb{S}_y^{n-1} \subset \mathbb{R}^n$, centered at y. We then show

$$\int_{\mathbf{S}} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} \, d\mathbf{S}(x) = \int_{\mathbb{S}_y^{n-1}} \tilde{f}(x) \, dS(x).$$

The proposition then follows for C^1 f since $\tilde{f}|_{\mathbf{S}} = f$, and $\tilde{f}(x) = f(x_-(y,v))$ when $v = (x-y)|x-y|^{-1}$. We can extend the result to $f \in L^1$ by density since for fixed y, $x \mapsto \nu_x \cdot (x-y)|x-y|^{-1}$ is bounded.

Proceeding, we note that

$$\nabla \tilde{f}(x) \cdot (x-y) = 0. \tag{55}$$

We first prove that for any C^1 hypersurfaces S_1 and S_2 such that S_1 encloses y and S_2 encloses S_1, the following equality holds:

$$\int_{S_1} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} d\boldsymbol{S}(x) = \int_{S_2} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} d\boldsymbol{S}(x). \tag{56}$$

Indeed, if V denotes the volume between S_1 and S_2, the divergence theorem applied to the function $\tilde{f}(x)\nabla N(x,y)$ in the volume V yields

$$\begin{aligned}\int_V \nabla \cdot (\tilde{f}(x)\nabla N(x,y)) \; dx &= \int_{\partial V} (\tilde{f}(x)\nabla N(x,y)) \cdot \nu_x \; d\boldsymbol{S}(x) \\ &= \int_{S_2} \tilde{f}(x)\partial_{\nu_x} N(x,y) \; d\boldsymbol{S}(x) - \int_{S_1} \tilde{f}(x)\partial_{\nu_x} N(x,y) \; d\boldsymbol{S}(x).\end{aligned}$$

We now show that the left-hand side of the previous equation is zero: After writing

$$\nabla \cdot (\tilde{f}(x)\nabla N(x,y)) = \nabla \tilde{f}(x) \cdot \nabla N(x,y) + \tilde{f}(x)\Delta N(x,y),$$

(all the operators apply on the x variable), we first notice that $\Delta N(x,y) = 0$ for all $x \neq y$, in particular on V. Second, as $N(x,y)$ is a radial function of x with respect to y then its gradient is collinear to the vector $x-y$. Using (55), we see that the scalar product $\nabla \tilde{f}(x) \cdot \nabla N(x,y)$ is zero on V. Finally, since $\omega_n \partial_{\nu_x} N(x,y) = \frac{\nu_x \cdot (x-y)}{|x-y|^n}$, the equality (56) holds.

If S is either enclosing or enclosed in $\mathbb{S}_y^{n-1}$, then the proof is done. Otherwise, pick any hypersurface S' which encloses both S and $\mathbb{S}_y^{n-1}$ then applying the first part of this proof to S and S', then to S' and $\mathbb{S}_y^{n-1}$, yields by transitivity

$$\int_S \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} d\boldsymbol{S}(x) = \int_{\mathbb{S}_y^{n-1}} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} d\boldsymbol{S}(x) = \int_{\mathbb{S}_y^{n-1}} \tilde{f}(x) d\boldsymbol{S}(x),$$

hence the result.

Proof of equation (17) : As in the proof of equation (16), we extend $f \in C^1$ to $\tilde{f}$, which is constant along rays originating at y, and zero on all rays $\vec{r}$ such that $\vec{r} \cdot \nu_y > 0$. The strict convexity of $\mathbf{S}$ allows us to do that. We then show that

$$\int_{\mathbf{S}} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} \, d\mathbf{S}(x) = \int_{\mathbb{S}_y^{n-1}} \tilde{f}(x) \, dS(x).$$

The corollary then follows for $f \in C^1(\mathbf{S})$ by the same reasoning as in the proof of equation (16). We may then extend the result for L^∞ f since for $y \in \partial X$, $x \mapsto \nu_x \cdot (x-y)|x-y|^{-1} \in L^1(\partial X)$ as can be seen using (20).

Let B_ε be the ball of radius ε centered at y, and $\mathbf{S}_\varepsilon = \mathbf{S} - B_\varepsilon$. $\mathbf{S}_\varepsilon$ is not a closed surface, so we cannot directly apply proposition 3.1 to it. Form a new closed C^2 surface $\mathbf{S}'_\varepsilon$ closing $\mathbf{S}_\varepsilon$ in such a way that all but a small part (whose volume is $O(\varepsilon^n)$) of $\mathbf{S} \setminus \mathbf{S}_\varepsilon$ lies on the side of the tangent plane to $\mathbf{S}$ at y on which $\tilde{f}$ is identically zero. This is possible since the surface is C^2. Call this small part P_ε. Using the proof of equation (16)

$$\begin{aligned}
\int_{\mathbb{S}_y^{n-1}} \tilde{f}(x)\, dS(x) &= \int_{\mathbf{S}'_\varepsilon} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n}\, d\mathbf{S}(x) \\
&= \int_{\mathbf{S}_\varepsilon} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n}\, d\mathbf{S}(x) + \int_{P_\varepsilon} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n}\, d\mathbf{S}(x) \\
&= \int_{\mathbf{S}_\varepsilon} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n}\, d\mathbf{S}(x) + O(\varepsilon).
\end{aligned}$$

Using (20), we see that the integral over $\mathbf{S}_\varepsilon$ becomes the integral over the entire boundary $\mathbf{S}$ as $\varepsilon \to 0$. □

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