Ray Transforms in Hyperbolic Geometry

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Abstract

We derive explicit inversion formulae for the attenuated geodesic and horocyclic ray transforms of functions and vector fields on two-dimensional manifolds equipped with the hyperbolic metric. The inversion formulae are based on a suitable complexification of the associated vector fields so as to recast the reconstruction as a Riemann Hilbert problem. The inversion formulae have a very similar structure to their counterparts in Euclidean geometry and may therefore be amenable to efficient discretizations and numerical inversions. An important field of application is geophysical imaging when absorption effects are accounted for.

Résumé

Nous présentons des formules d'inversion explicites permettant la reconstruction de fonctions et de champs de vecteurs à partir de leur intégration le long des géodésiques ou des horocycles associés à une métrique hyperbolique en dimension deux d'espace. L'intégration peut contenir un poids tenant compte de phénomènes d'absorption. La méthode de reconstruction se fonde sur une complexification des champs de vecteurs associés à ces intégrales et écrits dans un système de coordonnées adapté, de manière à ce que l'inversion se ramène à résoudre un problème de type Riemann-Hilbert. Les formules d'inversion obtenues ont une structure très proche de celle que l'on connait en géométrie euclidienne, ce qui devrait en permettre une discrétisation numérique aisée. Le principal domaine d'application de cette inversion est l'imagerie en géophysique lorsque l'absorption de l'énergie mesurée est prise en compte.

keywords

Ray transform, hyperbolic geometry, Riemann-Hilbert problem, transport equation.

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1 Introduction

Ray transforms and their inversions are ubiquitous in medical and geophysical imaging. Many successful medical imaging techniques such as CT-scan, PET, and SPECT, are based on inversions of the Radon transform in two space dimensions and more general ray transforms in three space dimensions [23]. To a large extent, most of our understanding of the inner structure of the Earth is also based on the inversion of integral transforms [29]. There, the ray transform may arise in the linearization of a non-linear problem consisting of reconstructing the metric of a Riemannian manifold from travel time measurements [9, 15, 18, 28, 31].

Whereas most medical imaging techniques involve domains equipped with the Euclidean metric, most geophysical reconstructions involve ray transforms in non-Euclidean geometry. This is because the speed of elastic waves generally increases with depth, thus curving the rays back to the Earth surface. Inversions of geodesic ray transforms are rendered more difficult by non vanishing curvatures. In practice, it is often possible to assume as a first approximation that curvature is negative throughout the domain. Such assumptions allow us to show, for instance, that functions are uniquely determined by a sufficient number of ray transforms and that reconstructions are not too unstable [19, 20, 29, 34].

Explicit inversion formulae in non-Euclidean geometry primarily involve manifolds with enough symmetries such as those having constant curvature [6, 12, 13, 17, 30]. These inversions do not account for possible attenuation during propagation along the geodesics. In this paper, we concentrate on the inversion of attenuated ray transforms in two-dimensional hyperbolic geometry. The results are thus also limited to the constant curvature framework although we hope that the methodology used in the inversion may extend to more general metrics with negative curvature. Throughout the paper, we assume that the absorption term is known. The constant curvature model is quite useful in geophysical imaging as it corresponds to the propagation of acoustic waves according to a metric $ds^2 = c^{-2}(z)(dx^2+dz^2)$, where $c(z)$ is proportional to depth z. The constant gradient hypothesis is suitable as a first approximation [29]. Inversion of ray transforms on constant-curvature manifolds have also found an application in electrical impedance tomography, a medical imaging technique that uses electrical current and potential measurements to image properties of human tissues [5, 17].

The method developed in this paper builds on a complexification technique recently used successfully by R.G. Novikov [24, 25] to invert the attenuated ray transform in Euclidean geometry; see also [3, 4, 7, 8, 10, 11, 22] for additional recent references on the problem. This inversion is the mathematical backbone of the medical imaging technique SPECT. The main aspect in the extension of the method to non-Euclidean geometry is a choice of parameterization of the vector fields that allows the extension of a judicious parameter into the complex plane. In the case of the two-dimensional hyperbolic disc D, which is our model of a manifold with constant negative curvature, the vector fields are parameterized by the point $\lambda = e^{i\theta} \in \partial D$ of convergence of the geodesics on the sphere at infinity, and λ is the parameter that is extended into the complex plane. This allows us to recast the inversion of several attenuated ray transforms as a Riemann-Hilbert problem [11], whose solution is provided by the classical Cauchy formula. Throughout the paper we assume that the functions integrated along rays and the absorption parameters are sufficiently smooth so that all integrations make sense. We do not consider range characterizations of attenuated ray transforms, which, to a large extent, may be dealt with by adapting techniques developed in [25, 27].

The paper is structured as follows. Section 2 recalls important results on the geodesics on D equipped with the hyperbolic metric. The main results of this paper on the inversion of ray transforms are presented in section 3. We consider successively the inversion of the geodesic ray transform, the inversion of the attenuated geodesic ray transform, and the inversion of the attenuated horocyclic ray transform. The derivation of the results is postponed to subsequent sections. In section 4, the (non-attenuated) ray transform is inverted. The main three steps of the derivation are the complexification of the vector fields, the derivation of the Riemann Hilbert problem, and finally the solution to that problem. The inversion of the attenuated geodesic ray transform is considered in section 5 and that of the dual transform along horocycles in section 6. Section 7 then extends the inversion of the attenuated geodesic transform to the vectorial case. Whereas it is known that only the solenoidal part of a vector field may be reconstructed from its ray transform in the absence of absorption, we show that the full vector field can be reconstructed in the presence of known absorption. This generalizes similar results in Euclidean geometry. We briefly comment on the ray inversion on the half-plane equipped with the hyperbolic metric in section 8 and offer some conclusions in section 9.

2 Preliminaries

This section reviews important properties of the unit disc $D = \{z \in \mathbb{C}, |z| < 1\}$ equipped with the Riemannian metric

$$ds_g^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} = \frac{dx^2 + dy^2}{(1 - |z|^2)^2}. \tag{1}$$

The curvature is constant and equal to -4 in this model. We define

$$\langle X, Y \rangle = g(X, Y), \qquad \|X\| = \langle X, X \rangle^{1/2}, \tag{2}$$

the inner product and norm associated to the Riemannian structure. The (volume) measure on (D, g) is denoted by $dm_g(z)$. We verify that

$$dm_g(z) = \frac{dxdy}{(1 - |z|^2)^2}. \tag{3}$$

The form of the geodesics on (D, g) is well-understood. We will mostly follow the presentation in [12]; see also [16, 17].

The boundary ∂D of D, the "sphere at infinity", is the unit circle parameterized by the complex numbers $e^{i\theta}$ for $0 \le \theta < 2\pi$. Let us introduce the map

$$z : \left\{ \begin{array}{ccl} \mathbb{R}^2 & \to & D \\ (t, s) & \mapsto & z(t, s) = \dfrac{\sinh t - ise^{-t}}{\cosh t - ise^{-t}}. \end{array} \right. \tag{4}$$

We verify that z is a C^∞-diffeomorphism. Moreover, the maps $t \to z(t,s)$ at fixed $s \in \mathbb{R}$ provide all the geodesics of (D,g) that converge to $e^{i0} \in \partial D$ (as $t \to +\infty$); see [12]. In the (t,s) coordinates, the geodesic field is $\frac{\partial}{\partial t}$. All the other geodesics of (D,g) can be obtained by rotation around the origin in D. We thus define the set of geodesics parameterized by $\theta \in [0, 2\pi)$ and $s \in \mathbb{R}$ as

$$\xi(s,\theta) = \{e^{i\theta} z(t,s), t \in \mathbb{R}\}. \tag{5}$$

These are all the geodesics converging to $e^{i\theta} \in \partial D$ as $t \to +\infty$.

We identify $z = x + iy$, where (x,y) are the Cartesian coordinates on D. As is customary in complex analysis, we change coordinates and introduce

$$z = x + iy, \quad \bar{z} = x - iy, \tag{6}$$

for $|z| < 1$ and view z and $\bar{z}$ as independent variables. A basis for the tangent space $T_z D$ is then given by

$$\partial \equiv \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{7}$$

In the variables $(z, \bar{z})$, the diffeomorphism in (4) may be recast as

$$z : (t,s) \mapsto z(t,s) = \left(\frac{\sinh t - ise^{-t}}{\cosh t - ise^{-t}}, \frac{\sinh t + ise^{-t}}{\cosh t + ise^{-t}}\right). \tag{8}$$

We denote by z_* the push-forward defined by $z_* f = f \circ z^{-1}$. Defining the functions

$$s(z) = z_* s(z), \quad t(z) = z_* t(z), \quad P(z) = z_* e^{2t}(z),$$

we find that the inverse of the map z is characterized by

$$e^{2t}(z) = P(z) = \frac{1 - |z|^2}{|1 - z|^2}, \quad s(z) = \frac{\bar{z} - z}{2i|1 - z|^2} = \frac{1}{2i}\Big(\frac{1}{1 - \bar{z}} - \frac{1}{1 - z}\Big), \quad t(z) = \frac{1}{2}\ln P(z).$$

Geodesic vector field. The geodesic vector field whose integral curves are the geodesics of (D,g) converging to e^{i0} as $t \to +\infty$ is given by

$$X(e^{i0}) = z_* \frac{\partial}{\partial t} = z_*\Big(\frac{\partial z}{\partial t}\Big)\partial + z_*\Big(\frac{\partial \bar{z}}{\partial t}\Big)\bar{\partial}. \tag{9}$$

Some algebra shows that

$$\frac{\partial z}{\partial t} = (1 - |z|^2)\frac{1 - z}{1 - \bar{z}}, \qquad \frac{\partial \bar{z}}{\partial t} = (1 - |z|^2)\frac{1 - \bar{z}}{1 - z}.$$

We also deduce that

$$\frac{\partial z}{\partial s} = \frac{-ie^{-2t}}{(\cosh t - ise^{-t})^2} = -i(1 - z)^2, \qquad \frac{\partial \bar{z}}{\partial s} = \frac{ie^{-2t}}{(\cosh t + ise^{-t})^2} = i(1 - \bar{z})^2.$$

The Jacobian of the transformation $(t,s)\mapsto(z,\bar z)$ is thus given by

$$z_*\partial z(t,s) = z_*\begin{vmatrix} \dfrac{\partial z}{\partial t} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial \bar z}{\partial t} & \dfrac{\partial \bar z}{\partial s} \end{vmatrix} = 2i(1-|z|^2)|1-z|^2. \tag{10}$$

The Jacobian of the transformation $(t,s)\mapsto(x,y)$ for $z=x+iy$ is therefore $(1-|z|^2)|1-z|^2$, which does not vanish on D. Moreover, we find that

$$dm_g(z) = \frac{1}{(1-|z|^2)^2}dxdy = \frac{|1-z|^2}{1-|z|^2}dtds = e^{-2t}dtds. \tag{11}$$

The above calculations provide the following expression for the geodesic field:

$$X(e^{i0}) = (1-|z|^2)\Big(\frac{1-z}{1-\bar z}\partial + \frac{1-\bar z}{1-z}\bar\partial\Big). \tag{12}$$

The geodesic fields with integral curves converging to other points on ∂D may be obtained by rotation. Let us define in the "$z,\bar z$" variables the function

$$e^{i\theta} : (z,\bar z) \mapsto (e^{i\theta}z, e^{-i\theta}\bar z), \tag{13}$$

which maps a point $z\in D$ to $e^{i\theta}z\in D$. The geodesic vector field $X(e^{i\theta})$ on D corresponding to geodesics converging to $e^{i\theta}$ as $t\to+\infty$ is thus given by

$$X(e^{i\theta}) = e^{i\theta}_* X(e^{i0}) = (e^{i\theta}\circ z)_*\frac{\partial}{\partial t}. \tag{14}$$

More explicitly, we deduce from (12) that

$$X(e^{i\theta}) = (1-|z|^2)\Big(\frac{1-e^{-i\theta}z}{1-e^{i\theta}\bar z}e^{i\theta}\partial + \frac{1-e^{i\theta}\bar z}{1-e^{-i\theta}z}e^{-i\theta}\bar\partial\Big). \tag{15}$$

The analysis and complexification of the above geodesic vector field is the main ingredient in the derivation of the reconstruction formulae for the integral transforms considered in this paper.

Horocyclic vector field. A vector field orthogonal to $X(e^{i0})$ at all points in D is given by

$$X^\perp(e^{i0}) = z_*\frac{\partial}{\partial s} = i\Big(-(1-z)^2\partial + (1-\bar z)^2\bar\partial\Big). \tag{16}$$

We also define $X^\perp(e^{i\theta}) = e^{i\theta}_* X^\perp(e^{i0})$. The integral curves of $X^\perp(e^{i\theta})$ are called the horocycles; see (38) below. We easily verify that $\|X(e^{i\theta})\| = 1$ for the norm defined in (2) (recall that $dz\wedge d\bar z = 2idxdy$), which is consistent with the fact that $\frac{\partial}{\partial t}$ is the geodesic vector field in the (t,s) coordinates. However, $X^\perp(e^{i0})$ is not normalized and we introduce the normalized vector field

$$\check X^\perp(e^{i0}) = P(z)X^\perp(e^{i0}) = i(1-|z|^2)\Big(-\frac{1-z}{1-\bar z}\partial + \frac{1-\bar z}{1-z}\bar\partial\Big). \tag{17}$$

More generally, we define

$$\check X^\perp(e^{i\theta}) = P(e^{-i\theta}z)X^\perp(e^{i\theta}) = i(1-|z|^2)\Big(-\frac{1-e^{-i\theta}z}{1-e^{i\theta}\bar z}e^{i\theta}\partial + \frac{1-e^{i\theta}\bar z}{1-e^{-i\theta}z}e^{-i\theta}\bar\partial\Big). \tag{18}$$

We verify that $\|\check X^\perp(e^{i\theta})\| = 1$ and that $\check X^\perp(e^{i\theta})$ is the rotation of $X(e^{i\theta})$ by an angle $-\pi/2$.

A note on the constant in the curvature. Let g_α be the Riemannian structure defined by

$$ds^2_{g_\alpha} = \frac{1}{\alpha^2} ds^2_g = \frac{1}{\alpha^2} \frac{dx^2 + dy^2}{(1 - |z|^2)^2}. \tag{19}$$

We verify that the geodesics of (D, g_α) are obviously still given by $\xi(s, \theta)$, but that the geodesic coordinate t need be replaced by $\alpha^{-1}t$ so that the geodesic vector field becomes $X_\alpha(e^{i\theta}) = \alpha X(e^{i\theta})$. Moreover the constant curvature K_α of the metric g_α is given by

$$K_\alpha = -4\alpha^2. \tag{20}$$

We thus see that a constant curvature of $K_\alpha = -1$ is obtained for $\alpha = 1/2$, and that the corresponding geodesic vector fields are given by $X_{1/2}(e^{i\theta}) = \frac{1}{2} X(e^{i\theta})$.

3 Integral transforms and inversion formulae

This section introduces three integral transformations and presents explicit inversion formulae for each of them. The ray transform along geodesics for (D, g) is introduced in (21) and inverted in (29). The result is next generalized to the attenuated ray transform defined in (32) and inverted in (36). Finally we consider the attenuated ray transform along horocycles for (D, g) in (40) and invert it in (48). The derivation of the inversion formulae is postponed to subsequent sections.

The ray transform. For a sufficiently smooth function $f(z)$ on D, we define the geodesic ray transform (or Radon transform) on $\mathbb{R} \times [0, 2\pi)$ as

$$Rf(s, \theta) \equiv \hat{f}(s, \theta) = \int_{\xi(s,\theta)} f(z) dm_g(z) = \int_{\mathbb{R}} f(e^{i\theta} z(t, s)) dt. \tag{21}$$

We also introduce the notation $R_\theta f(s) = Rf(s, \theta)$. Several inversion formulae have been obtained for the Radon transform in two dimensional hyperbolic geometry; see for instance [6, 12, 13, 17, 30]. In this paper, we consider another inversion formula based on the analysis of the complexification of the vector fields introduced in the preceding section. We thus see the above integral as a ray transform rather than a Radon transform, although both transforms are equivalent in two space dimensions. The method builds on earlier works done in Euclidean geometry [3, 11, 24, 25] and will be presented in detail in section 4. The central ingredient in the method is the (geodesic) transport equation

$$X(e^{i\theta}) u(z, e^{i\theta}) = f(z), \tag{22}$$

with the boundary condition that $u(z, e^{i\theta})$ vanishes at "$-\infty$" on the geodesic $\xi(s, \theta)$, i.e.,

$$\lim_{t \to -\infty} u(e^{i\theta} z(t, s), e^{i\theta}) = 0.$$

Here $u(z, e^{i\theta})$ is defined on $D \times S^1$. The ray transform in (21) is then simply given by

$$Rf(s, \theta) = \lim_{t \to +\infty} u(e^{i\theta} z(t, s), e^{i\theta}),$$

where $u(z, e^{i\theta})$ is the unique solution to (22).

The geodesics in hyperbolic geometry may also be parameterized by the point at infinity $e^{i\theta}$ as $t \to \infty$ and a point $z \in D$ on that geodesic. We denote by $\xi(z; e^{i\theta})$ such a geodesic, slightly abusing notation by using the same symbol ξ as in the definition (5). We define $\xi_+(z; e^{i\theta})$ as the part of the geodesic joining z to $e^{i\theta}$ and $\xi_-(z; e^{i\theta})$ as the complement of $\xi_+(z; e^{i\theta})$ in $\xi(z; e^{i\theta})$. Recall that $e^{i\theta}_* s(z) = (e^{i\theta} \circ z)_* s = s(e^{-i\theta} z)$. We verify that $\xi_+(z; e^{i\theta}) = \big\{ e^{i\theta} z(t, s(e^{-i\theta} z)), t > t(e^{-i\theta} z) \big\}$. Following the definition in Euclidean geometry [3], we define the symmetrized beam transform as

$$D_\theta f(z) = \frac{1}{2}\Big(\int_{\xi_-(z;e^{i\theta})} f(z) dm_g(z) - \int_{\xi_+(z;e^{i\theta})} f(z) dm_g(z) \Big). \tag{23}$$

We verify that

$$D_\theta f(z) = \frac{1}{2} \int_{\mathbb{R}} f\big(e^{i\theta} z(t, s(e^{-i\theta} z))\big) \operatorname{sign}\big(t(e^{-i\theta} z) - t\big) dt, \tag{24}$$

and that the symmetrized beam transform inverts the transport equation:

$$X(e^{i\theta}) D_\theta f(z) = f(z). \tag{25}$$

We also define the classical Hilbert transform

$$Hf(t) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t-s} ds. \tag{26}$$

Finally we introduce the adjoint operator to the ray transform

$$R^* g(z) = \int_0^{2\pi} \big(e^{i\theta} \circ z\big)_* \big(g(s,\theta) e^{2t}\big) d\theta = \int_0^{2\pi} g(s(e^{-i\theta} z), \theta) P(e^{-i\theta} z) d\theta. \tag{27}$$

That R and R^* are in duality for the inner product in $L^2(\mathbb{R} \times (0, 2\pi))$ and $L^2(D; dm_g(z))$ is obtained as follows:

$$\begin{aligned} &\int Rf(s,\theta) g(s,\theta) ds d\theta = \int f(e^{i\theta} z(t,s)) g(s,\theta) e^{2t} (e^{-2t} dt ds) d\theta \\ &= \int_D \Big(\int_0^{2\pi} \big(e^{i\theta} \circ z\big)_* \big(g(s,\theta) e^{2t}\big) d\theta \Big) f(\zeta) dm_g(\zeta) = \int_D (R^* g)(\zeta) f(\zeta) dm_g(\zeta), \end{aligned}$$

where we have used that $(e^{i\theta})^* dm_g(z) = dm_g(z)$ by rotational invariance. We also use the notation $R^*_\theta g(z) = g(s(e^{-i\theta} z), \theta) P(e^{-i\theta} z)$. The same derivation shows that R^*_θ is the adjoint operator to R_θ for the inner products in $L^2(\mathbb{R})$ and $L^2(D; dm_g(z))$.

Reconstruction formula for the ray transform. We are now in a position to state the following reconstruction formula. First, the data need to satisfy the following compatibility condition

$$R^* H \hat{f}(z) = 0, \qquad \text{for all } z \in D. \tag{28}$$

Then the function $f(z)$ is uniquely determined by $\hat{f}(s, \theta)$ and is given by

$$f(z) = \frac{1}{4\pi} \int_0^{2\pi} P^2(e^{-i\theta} z) \Big(H \frac{\partial}{\partial s} \hat{f} \Big) \big(s(e^{-i\theta} z), \theta\big) d\theta. \tag{29}$$

The above reconstruction formula can also be restated as the following decomposition of identity

$$I = \frac{1}{4\pi} R^* P H \frac{\partial}{\partial s} R = \frac{1}{4\pi} R^* H P \frac{\partial}{\partial s} R. \tag{30}$$

Here, we have denoted by P the operator of multiplication by $P(e^{-i\theta}z)$. Since the multiplicative factor is nothing but $(e^{i\theta}z)_* e^{2t}$, we verify that $PH = HP$ since the Hilbert transform acts on the s-variable. Note that the presence of the multiplicative operator P is here simply because $\frac{\partial}{\partial s}$ is not a unit vector field, whereas $e^{2t}\frac{\partial}{\partial s}$ is. The formulae (29) and (30) thus take the form of the classical filtered back-projections available in Euclidean geometry [21].

Inversion of the attenuated ray transform. In many practical applications, the signal emitted by the source term $f(z)$ may be attenuated before it can reach the detectors. We model the signal by using the following geodesic transport equation with absorption term

$$X(e^{i\theta})u(z, e^{i\theta}) + a(z)u(z, e^{i\theta}) = f(z), \tag{31}$$

and vanishing incoming conditions

$$\lim_{t\to-\infty} u(e^{i\theta}z(t,s), e^{i\theta}) = 0.$$

We assume that $a(z)$ is known, smooth, non-negative, and has compact support to simplify. We recall that

$$X(e^{i\theta})D_\theta a(z, e^{i\theta}) = a(z),$$

whence

$$X(e^{i\theta})(e^{D_\theta a}u)(z, e^{i\theta}) = e^{D_\theta a}(z, e^{i\theta})f(z).$$

We call the attenuated ray transform the quantity

$$R_a f(s,\theta) \equiv R_{a,\theta} f(s) = \int_{\xi(s,\theta)} e^{D_\theta a}(z, e^{i\theta})f(z)dm_g(z). \tag{32}$$

Note that this is the limit of $e^{D_\theta a}u$ as $t \to +\infty$ along the geodesic $\xi(s,\theta)$. Since we assume that $a(z)$ is known here, the attenuated ray transform is obtained in practice by measuring $u(z, e^{i\theta})$ at infinity on the geodesic $\xi(s,\theta)$.

In Euclidean geometry, the attenuated ray transform may be inverted by using the Novikov formula [3, 7, 10, 22, 24, 25]. A similar formula holds in hyperbolic geometry. Let us define the intermediate quantity

$$\varphi(z, e^{i\theta}) = iP^{-1}(e^{-i\theta}z)R^*_{-a,\theta}H_a R_{a,\theta} f(z), \tag{33}$$

where, as a generalization of operators defined in Euclidean geometry [3, 14], we have

$$\begin{aligned} &(R^*_{a,\theta}g)(z) = P(e^{-i\theta}z)e^{D_\theta a(z)}g(s(e^{-i\theta}z)), \qquad H_a = C_cHC_c + C_sHC_s \\ &C_c g(s,\theta) = g(s,\theta)\cos\Big(\frac{H\hat{a}(s,\theta)}{2}\Big), \quad C_s g(s,\theta) = g(s,\theta)\sin\Big(\frac{H\hat{a}(s,\theta)}{2}\Big). \end{aligned} \tag{34}$$

The operators $R_{a,\theta}$ and $R^*_{a,\theta}$ are in duality for the $L^2(\mathbb{R})$ and $L^2(D; dm_g(z))$ inner products. The derivation is the same as for R^* in (27). Note that $R_{a,\theta} f(s,\theta)$ is the measured data so that $\varphi(z, e^{i\theta})$ in (33) may directly be estimated from the measurements. We now come to the inversion formula. First, the data need to satisfy the following compatibility condition

$$\frac{1}{2\pi}\int_0^{2\pi} P(e^{-i\theta}z)\varphi(z, e^{i\theta})d\theta = 0, \qquad \text{for all } z \in D. \tag{35}$$

This condition generalizes (28). Moreover, the source term $f(z)$ is uniquely determined by its attenuated ray transform and is given by

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} \check{X}^\perp(e^{i\theta})\big(R^*_{-a,\theta}H_a[R_{a,\theta}f]\big)(z, e^{i\theta})d\theta. \tag{36}$$

The vector field $\check{X}^\perp(e^{i\theta})$ is defined in (18). We thus obtain a decomposition of identity very similar to that in the Euclidean case:

$$I = \frac{1}{4\pi}\int_0^{2\pi} \check{X}^\perp(e^{i\theta})R^*_{-a,\theta}H_aR_{a,\theta}d\theta. \tag{37}$$

In the presence of non-constant absorption $a(z)$, the operators $\check{X}^\perp(e^{i\theta})$ and $R^*_{-a,\theta}$ do not commute. When $a \equiv 0$, these two operators do commute, $H_a \equiv H$, and $R_{a,\theta} \equiv R_\theta$ so that (37) generalizes (30) to the case of non-constant absorption.

Inversion of the dual transforms. So far we have considered the reconstruction of functions integrated along geodesics. It is known that functions may be uniquely determined by their integrals along quite general families of curves; see for instance [19, 20, 26, 31, 32]. Here we consider the (much more specific) reconstruction of functions from their integrals over all possible horocycles. The horocycles are the circles in the unit disc D that are tangent to its boundary ∂D. More precisely, they may be parameterized by $t \in \mathbb{R}$ and $\theta \in [0, 2\pi)$ and defined as

$$\hbar(t, \theta) = \big\{e^{i\theta}z(t, s), s \in \mathbb{R}\big\}. \tag{38}$$

The horocycles are orthogonal (both for the metric g and the Euclidean metric) to the geodesics and are integral curves of the vector fields $X^\perp(e^{i\theta})$ and its normalized alter-ego $\check{X}^\perp(e^{i\theta})$. The (dual) transport equation modeling integration along the horocycles is thus given by

$$\check{X}^\perp(e^{i\theta})u^\perp(z, e^{i\theta}) + a(z)u^\perp(z, e^{i\theta}) = f(z), \tag{39}$$

with the same vanishing incoming conditions as before in the limit $s \to -\infty$.

In the absence of absorption ($a \equiv 0$), we define the horocycle transformation of the source term $f(z)$ as the limit as $s \to +\infty$ of the above transport equation (with $a \equiv 0$) namely,

$$R^\perp_\theta f(t, \theta) = \check{f}(t, \theta) = \int_{\hbar(t,\theta)} f(z)dm_g(z) = e^{-2t}\int_{\mathbb{R}} f\big(e^{i\theta}z(t, s)\big)ds. \tag{40}$$

The last equality stems from the relations $z_*(ds)(X^\perp) = 1 = P(z)ds_g(X^\perp) = ds_g(\check{X}^\perp)$, where ds_g is the length measure associated to the metric g on the horocycle $\hbar(t, \theta)$.

We now define the symmetrized horocycle transform

$$D_\theta^\perp f(z) = \frac{P^{-1}(e^{-i\theta}z)}{2}\int_{\mathbb{R}} f\big(e^{i\theta}z(t(e^{-i\theta}z),s)\big)\,\mathrm{sign}\big(s(e^{-i\theta}z)-s\big)ds, \tag{41}$$

and verify that

$$\check{X}^\perp(e^{i\theta})D_\theta^\perp a(z) = a(z). \tag{42}$$

This allows us to observe that

$$\check{X}^\perp(e^{i\theta})(e^{D_\theta^\perp a}u)(z,e^{i\theta}) = e^{D_\theta^\perp a}(z,e^{i\theta})f(z).$$

We thus call the horocycle transform the following integral

$$R_a^\perp f(t,\theta) \equiv R_{a,\theta}^\perp f(t) = \int_{\hbar(t,\theta)} e^{D_\theta^\perp a}(z,e^{i\theta})f(z)dm_g(z). \tag{43}$$

This integral may also be seen as the limit as $s\to+\infty$ of the solution $e^{D_\theta^\perp a}u(z,e^{i\theta})$ of the transport equation (39) by construction.

Let us introduce the weighted Hilbert transform

$$H^\perp f(s) = \frac{2}{\pi}\mathrm{p.v.}\int_{\mathbb{R}}\frac{f(t)}{e^{2(s-t)}-1}dt, \tag{44}$$

and the auxiliary quantity

$$i\varphi^\perp(z,e^{i\theta}) = R_{-a,\theta}^{\perp *}H_a^\perp R_{a,\theta}^\perp f(z), \tag{45}$$

where we have defined the following operators

$$\begin{aligned}&(R_{a,\theta}^{\perp *}g)(z) = e^{D_\theta^\perp a(z)}g(t(e^{-i\theta}z)), \qquad H_a^\perp = C_c^\perp H^\perp C_c^\perp + C_s^\perp H^\perp C_s^\perp\\ &C_c^\perp g(t,\theta) = g(t,\theta)\cos\Big(\frac{H^\perp\breve{a}(t,\theta)}{2}\Big),\quad C_s^\perp g(t,\theta) = g(t,\theta)\sin\Big(\frac{H^\perp\breve{a}(t,\theta)}{2}\Big).\end{aligned} \tag{46}$$

We verify that $R_{a,\theta}^\perp$ and $R_{a,\theta}^{\perp *}$ are indeed in duality for the inner products on $L^2(\mathbb{R})$ and $L^2(D;dm_g(z))$.

We then have the following results. First, the data need to satisfy the following compatibility condition

$$\frac{1}{2\pi}\int_0^{2\pi} P(e^{-i\theta}z)\varphi^\perp(z,e^{i\theta})d\theta = 0, \qquad \text{for all } z\in D. \tag{47}$$

Moreover the source term $f(z)$ is uniquely determined by its (weighted) integrals along horocycles and is given by

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(e^{-i\theta}z)X(e^{i\theta})R_{-a,\theta}^{\perp *}H_a^\perp[R_{a,\theta}^\perp f](z,\theta)d\theta. \tag{48}$$

In the case of vanishing absorption, the above formula simplifies to

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(e^{-i\theta}z)\Big[\frac{\partial}{\partial t}H^\perp\check{f}\Big](t(e^{-i\theta}z),\theta)d\theta. \tag{49}$$

We note that the inversions of the geodesic and horocyclic ray transforms have very similar structures and are also quite close to their counterparts in Euclidean geometry [3, 24].

4 Ray transform and inversion

We now come to the derivation of the inversion formulae. This section deals with the inversion of the geodesic ray transform introduced in (21).

4.1 Complexification of the vector fields

In Euclidean geometry, the geodesic vector field is given by $\boldsymbol{\theta}\cdot\nabla_{\mathbf{x}}$, where $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ in Cartesian coordinates. Introducing the complex number $\lambda = e^{i\theta}$, we may then recast $\boldsymbol{\theta}\cdot\nabla_{\mathbf{x}} = \lambda\partial + \lambda^{-1}\bar{\partial}$ in the $(z, \bar{z})$ variables. For $\lambda \in T$, the unit circle, the latter differential operator is hyperbolic (with proper vanishing conditions at infinity). However, for $\lambda \in \mathbb{C}\backslash T$ (and $\lambda \neq 0$), the operator becomes elliptic, and therefore can be associated a fundamental solution $G_{\mathrm{Eucl}}(z;\lambda)$ that will be analytic on the disc $D^+ = \{\lambda \in \mathbb{C}; |\lambda| < 1\}$ and on $D^- = \{\lambda \in \mathbb{C}; |\lambda| > 1\}$. This behavior is the central element of Novikov's inversion formula for the Euclidean ray (or Radon) transform; we refer to [3, 24, 25] for the details. The question now is how this complexification of the vector field $\boldsymbol{\theta}\cdot\nabla_{\mathbf{x}} = \theta^i \frac{\partial}{\partial x^i}$ can be extended to non-Euclidean geometry.

Using the Christoffel symbols associated to the hyperbolic geometry [16], we could write the geodesic vector field (on the tangent bundle) in a given system of coordinates as

$$G_{x,\theta} = \theta^i \frac{\partial}{\partial x^i} + \theta^i \theta^j \Gamma^k_{ij} \frac{\partial}{\partial \theta^k}.$$

Because direction is no longer constant along a geodesic, it becomes more difficult to see which quantity could be extended into the complex plane. Direct complex extensions based on the above description of the vector field do not look very promising.

A description of the geodesic vector field amenable to complexification is precisely the one we have introduced in section 2. The quantity that we wish to extend to the complex plane is no longer the direction of propagation along a geodesic, but rather the point of convergence of the geodesic on the sphere at infinity (i.e., as $t \to +\infty$ in the parameterization introduced in section 2). Note that this notion also makes sense in Euclidean geometry, where the main direction of propagation at infinity is well-defined.

We recall that the geodesic vector field $X(e^{i\theta})$ is defined in (15). We define $\lambda = e^{i\theta}$ (so that $e^{-i\theta} = \lambda^{-1}$) and want to view λ as an arbitrary complex number and no longer an element of the unit circle T. For each complex number λ, we define the function (still denoted by λ)

$$\lambda : (z, \bar{z}) \mapsto \lambda(z, \bar{z}) = (\lambda z, \lambda^{-1}\bar{z}). \tag{50}$$

This is a function formally defined on $\mathbb{CC} = \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}^2$, the complexification of $\mathbb{C}$. Note that the second component is no longer the complex conjugate of the first component unless $\lambda \in T$, the unit circle. The Jacobian of the transformation is uniformly equal to one so that it defines a diffeomorphism on $\mathbb{CC} \cong \mathbb{C}^2$. Its inverse is given by $\lambda^{-1}(z, \bar{z}) = (\lambda^{-1}z, \lambda\bar{z})$. We then define the "complexified" vector field $X(\lambda)$ as the push-forward of $X(e^{i0})$ by the function λ:

$$X(\lambda) = \lambda_* X(e^{i0}). \tag{51}$$

The definition for $\lambda \in T$ is consistent with (14). The vector field is now defined on $\mathbb{C}TD = TD \otimes \mathbb{C} \cong D \otimes \mathbb{C}^2$, the complexification of the tangent bundle on D. The rules for the change of variables being $\lambda^{-1}z \leftarrow z$, $\lambda\bar{z} \leftarrow \bar{z}$, $\lambda\partial \leftarrow \partial$, and $\lambda^{-1}\bar{\partial} \leftarrow \bar{\partial}$, we obtain the following vector field

$$X(\lambda) = (1-|z|^2)\Big(\frac{\lambda - z}{1-\lambda\bar{z}}\,\partial + \frac{1-\lambda\bar{z}}{\lambda - z}\,\bar{\partial}\Big), \tag{52}$$

in the $(z, \bar{z})$ system of coordinates. The above vector field never vanishes and has poles at $\lambda = z$ and $\lambda = \bar{z}^{-1}$ that will be exploited in the inversion formulae. We verify the symmetry relation

$$\bar{X}(\overline{\lambda^{-1}}) = X(\lambda), \tag{53}$$

and can thus concentrate on the case $|\lambda| < 1$, i.e., $\lambda \in D^+$.

We also define the generalization of the parallel geodesic coordinates

$$P(z,\lambda) = \lambda_* P(z), \quad s(z,\lambda) = \lambda_* s(z), \quad t(z, e^{i\theta}) = e^{i\theta}_* t(z), \tag{54}$$

where the push-forward of functions λ_* is defined by $\lambda_* f = f \circ \lambda^{-1}$. More explicitly, we have

$$P(z,\lambda) = \frac{1 - z\bar{z}}{(\lambda - z)(\lambda^{-1} - \bar{z})}, \qquad s(z,\lambda) = \frac{1}{2i}\Big(\frac{1}{1-\lambda\bar{z}} - \frac{\lambda}{\lambda - z}\Big). \tag{55}$$

For $\lambda \in D^+$, these functions are defined for $(z, \bar{z}) \in S_\lambda = D\backslash\{\lambda\}$.

The main interest of the above vector field is that it generates an elliptic operator for $\lambda \in \mathbb{C}\backslash T$, and thus admits a fundamental solution, in the sense that the following holds

$$X(\lambda)G(z;\lambda,z_0) = \delta_g(z - z_0) = (1-|z_0|^2)^2\delta(z - z_0). \tag{56}$$

This needs to be augmented by conditions at infinity (i.e., as $|z| \to 1$), which will be apparent below.

4.2 A Riemann-Hilbert problem

Analyticity properties. Our first objective in this section is to show that $G(z;\lambda,z_0)$ is analytic in the λ variable on D^+ and D^-. This will be the premise to show that the complexification of the geodesic transport equation

$$X(\lambda)u(z,\lambda) = f(z), \tag{57}$$

with appropriate boundary conditions admits a unique solution given by

$$u(z,\lambda) = \int_D G(z;\lambda,z_0)f(z_0)dm_g(z_0), \tag{58}$$

which is analytic in the λ variable on D^+ and D^-.

We first derive an explicit expression for the fundamental solution. Thanks to (53), it is enough to consider the case $\lambda \in D^+$. Let $z_0 \in D$ and consider first $\lambda \in D^+\backslash\{0, z_0\}$. The central idea in the derivation is that the parallel geodesic coordinates (t, s) form an orthogonal basis when $\theta = 0$ and that s is constant on each geodesic. This property survives after complexification and we verify that

$$X(\lambda)s(z,\lambda) = 0, \qquad X(\lambda)P(z,\lambda) = 2P(z,\lambda), \tag{59}$$

on $(z,\bar z)\in S_\lambda = D\backslash\{\lambda\}$. Both relations can be proved similarly. The first one, that of interest to us in the sequel, follows from the observation

$$X(\lambda)s(z,\lambda)=\lambda_* X(e^{i0})\lambda_* s(z)=\lambda_*(X(e^{i0})s(z))=(\lambda\circ z)_*\frac{\partial s}{\partial t}=0.$$

The second one follows from the calculation $\frac{\partial}{\partial t}e^{2t}=2e^{2t}$. The results can also be directly obtained from

$$\frac{\partial s}{\partial z}=-\frac{\lambda}{2i(\lambda-z)^2},\qquad \frac{\partial s}{\partial \bar z}=\frac{\lambda}{2i(1-\lambda\bar z)^2},\qquad (z,\bar z)\in S_\lambda. \tag{60}$$

We also verify that

$$\frac{\partial \bar s}{\partial z}=\frac{-\bar\lambda}{2i(1-\bar\lambda z)^2},\quad \frac{\partial \bar s}{\partial \bar z}=\frac{\bar\lambda^{-1}}{2i(1-\bar\lambda^{-1}\bar z)^2},\quad \partial s(z)=\frac{|\lambda|^2}{4}\Big(\frac{1}{|z-\lambda|^4}-\frac{1}{|1-\lambda\bar z|^4}\Big)>0,$$

where $\partial s(z)$ is the Jacobian (positive for $0<|\lambda|<1$ and $|z|<1$) of the transformation (still denoted by s):

$$s(\lambda):(z,\bar z)\mapsto(s,\bar s)=\Big(\frac{1}{2i}\Big(\frac{1}{1-\lambda\bar z}-\frac{\lambda}{\lambda-z}\Big),\frac{1}{2i}\Big(\frac{\bar\lambda}{\bar\lambda-\bar z}-\frac{1}{1-\bar\lambda z}\Big)\Big). \tag{61}$$

The change of variables generates the vector field on S_λ:

$$\begin{aligned}
s_*X(\lambda) &= X(\lambda)\overline{s(z,\lambda)}\frac{\partial}{\partial\bar s}=(1-|z|^2)\Big(\frac{\lambda-z}{1-\lambda\bar z}\frac{\partial\bar s}{\partial z}+\frac{1-\lambda\bar z}{\lambda-z}\frac{\partial\bar s}{\partial\bar z}\Big)\frac{\partial}{\partial\bar s}\\
&= \frac{1-|z|^2}{2i}\bar\lambda(\lambda-z)(1-\lambda\bar z)\Big(\frac{-1}{|1-\lambda\bar z|^4}+\frac{1}{|z-\lambda|^4}\Big)\frac{\partial}{\partial\bar s}\\
&= \frac{2}{i\lambda}(1-|z|^2)(\lambda-z)(1-\lambda\bar z)(\partial s(z))\frac{\partial}{\partial\bar s}.
\end{aligned}$$

Now we are interested in the fundamental solution of

$$s_*X(s_*G)=(1-|z_0|^2)^2 s_*\delta(z-z_0)=(1-|z_0|^2)^2|\partial s(z_0)|\delta(s-s(z_0,\lambda)), \tag{62}$$

or using the above expression for s_*X,

$$\frac{\partial}{\partial\bar s}(s_*G)=\frac{-\lambda}{2i}\frac{1-|z_0|^2}{(\lambda-z_0)(1-\lambda\bar z_0)}\delta(s-s(z_0,\lambda))=\frac{-P(z_0,\lambda)}{2i}\delta(s-s(z_0,\lambda)).$$

This shows that the fundamental solution of (56) is given by

$$G(z;\lambda,z_0)=\frac{-P(z_0,\lambda)}{2i\pi}\frac{1}{s(z,\lambda)-s(z_0,\lambda)},\qquad \text{on } S_\lambda. \tag{63}$$

Here, we have chosen the fundamental solution of the $\bar\partial$ problem on $\mathbb{C}$

$$\frac{\partial}{\partial\bar\zeta}\phi(\zeta)=\delta(\zeta)\qquad\text{as}\qquad \phi(\zeta)=\frac{1}{\pi\zeta}. \tag{64}$$

This provides us implicitly with boundary conditions at ∂D for the transport equations (56) and (57). It remains to show that (63) holds for $z \in D$. We verify that

$$\frac{\partial}{\partial \bar{z}} \frac{1}{s(z,\lambda)-s(z_0,\lambda)} = -\frac{\dfrac{\partial s(z,\lambda)}{\partial \bar{z}}}{(s(z,\lambda)-s(z_0,\lambda))^2} = R(z,\lambda,z_0) - \frac{\lambda\pi}{2i}\frac{\delta(z-\lambda)}{(s(z,\lambda)-s(z_0,\lambda))^2},$$

where $R(z,\lambda,z_0)$ is the smoother term obtained from (60). Since $s(z,\lambda)=O((z-\lambda)^{-1})$, we deduce from the above calculation that (56) also holds at $z=\lambda$ as $(\lambda-z)\delta(\lambda-z)=0$. The same calculation shows that $G(z;\lambda,z_0)$ tends to 0 as λ approaches z. So for $z \neq z_0$, we deduce that $G(z;\lambda,z_0)$ is analytic on $\lambda \in D^+\backslash\{0,z_0\}$. Asymptotic expansions show that $G(z;\lambda,z_0)$ converges to $\frac{m(z_0)}{m(z)-m(z_0)}$ where $m(z)=\frac{1-|z|^2}{z}$ as $\lambda \to 0$ and converges to $-\pi^{-1}$ as $\lambda \to z_0$ for $z \neq z_0$. This implies that $G(z;\lambda,z_0)$ is analytic for $\lambda \in D^+$ when $z \neq z_0$. This allows us to conclude that after integration in z_0 against a smooth function, $u(z,\lambda)$ defined in (58) is analytic on D^+ as in the Euclidean case [24]. Thanks to the symmetry (53), we verify that

$$G(z;\lambda,z_0) = \frac{\mathrm{sign}\big(|\lambda|-1\big)}{2\pi i}\frac{P(z,\lambda)}{s(z,\lambda)-s(z_0,\lambda)}. \tag{65}$$

This shows the following result:

> $u(z,\lambda)$ solution of (58) is analytic on $D^+ \cup D^-$.

Note that since, $G(z;z,z_0)=0$ and $u(z,z)=0$, the point $\lambda=z$ plays the same role as the point $\lambda=0$ in Euclidean geometry (see [3]).

Jump conditions. Sectionally analytic functions are determined by the values they take where they are not analytic [1]. We thus need to analyze the behavior of $G(z;\lambda,z_0)$ and $u(z,\lambda)$ as λ converges to T. By rotational invariance, it is sufficient to understand the case $\lambda \to 1^-$. We thus define $\lambda = 1-\varepsilon$ and let $\varepsilon \to 0^+$. The factor $P(z_0,\lambda)$ simply tends to $P(z_0)$. The difficulties come from the behavior of $s(z,\lambda)-s(z_0,\lambda)$. We find that

$$s(z,1) = \frac{1}{2i}\Big(\frac{1}{1-\bar{z}} - \frac{1}{1-z}\Big), \qquad \frac{\partial s}{\partial \lambda}(z,1) = \frac{1}{2i}\Big(\frac{\bar{z}}{(1-\bar{z})^2} + \frac{z}{(1-z)^2}\Big).$$

We thus obtain for $\lambda = 1-\varepsilon$ that

$$\begin{aligned}\frac{1}{s(z,\lambda)-s(z_0,\lambda)} &= \frac{1}{\big[s(z,1)-s(z_0,1)\big] - \varepsilon\Big[\dfrac{\partial s}{\partial\lambda}(z,1) - \dfrac{\partial s}{\partial \lambda}(z_0,1)\Big] + O(\varepsilon^2)} \\ &\xrightarrow{\varepsilon\to 0^+} \frac{1}{s(z,1)-s(z_0,1)} - i\pi\delta\big(s(z,1)-s(z_0,1)\big)\,\mathrm{sign}\Big(i\frac{\partial s}{\partial \lambda}(z,1) - i\frac{\partial s}{\partial\lambda}(z_0,1)\Big),\end{aligned}$$

in the sense of distributions. This comes from the fact that $(ix+\varepsilon)^{-1}$ converges to $(ix)^{-1}+\pi\,\mathrm{sign}(\varepsilon)\delta(x)$ in the same sense.

Let us define the function $\nu(t,s)$ such that

$$i\frac{\partial s}{\partial \lambda}(z,1) = z_*\nu(z), \qquad \text{i.e.,} \quad \nu(t,s) = z^*\big(i\frac{\partial s}{\partial\lambda}\big)(t,s).$$

Here $z^* = z_*^{-1}$. Then we have $X(e^{i0})i\frac{\partial s}{\partial\lambda}(z,1) = z_*\big(\frac{\partial}{\partial t}\nu\big)$ by construction. The above function is positive (in both coordinate systems) since

$$X(e^{i0})i\frac{\partial s}{\partial\lambda}(z,1) = \frac{1-|z|^2}{2}\Big(\frac{1+z}{|1-z|^2(1-z)} + \frac{1+\bar z}{|1-z|^2(1-\bar z)}\Big) = \frac{(1-|z|^2)^2}{|1-z|^4} > 0.$$

This implies that

$$\operatorname{sign}\Big(i\frac{\partial s}{\partial\lambda}(z,1) - i\frac{\partial s}{\partial\lambda}(z_0,1)\Big) = \operatorname{sign}\big(t(z)-t(z_0)\big), \tag{66}$$

where we recall that $t(z)$ is the first geodesic coordinate.

Since $\bar G(\bar\lambda^{-1}) = G(\lambda)$ thanks to (53), we deduce that

$$G_\pm(z;e^{i0},z_0) = \lim_{\varepsilon\to0^+} G(z;1\mp\varepsilon,z_0) = \mp\frac{P(z_0)}{2\pi i}\frac{1}{s-s_0} + \frac{P(z_0)}{2}\operatorname{sign}(t-t_0)\delta(s-s_0). \tag{67}$$

Here, $(t,s) = (t(z),s(z))$ and $(t_0,s_0) = (t(z_0),s(z_0))$ are the geodesic coordinates of z and z_0 on $\xi(s,0)$, respectively. By rotational invariance we verify that $G_\pm(z;e^{i\theta},z_0) = G_\pm(e^{-i\theta}z;e^{i0},e^{-i\theta}z_0)$ so that:

$$\begin{aligned} G_\pm(z;e^{i\theta},z_0) &= \mp\frac{P(e^{-i\theta}z_0)}{2\pi i}\frac{1}{s(e^{-i\theta}z)-s(e^{-i\theta}z_0)} \\ &\quad +\frac{P(e^{-i\theta}z_0)}{2}\operatorname{sign}\big(t(e^{-i\theta}z)-t(e^{-i\theta}z_0)\big)\delta\big(s(e^{-i\theta}z)-s(e^{-i\theta}z_0)\big). \end{aligned} \tag{68}$$

We finally deduce from (58) that

$$u_\pm(z,e^{i\theta}) = \int_D G_\pm(z;e^{i\theta},z_0)f(z_0)dm_g(z_0).$$

For $z\in D$ and $0\le\theta<2\pi$, we recall that $s(z,e^{i\theta}) = s(e^{-i\theta}z)$ defined in (55) is the second geodesic coordinate of z on the unique geodesic converging to $e^{i\theta}$. Thanks to (68) and the fact that the measure dm_g is preserved by rotation (so that $(e^{i\theta})^*dm_g = dm_g$), we find that

$$\begin{aligned} u_\pm(z,e^{i\theta}) &= \int_{\mathbb{R}^2} P(z(t_0,s_0))f\big(e^{i\theta}z(t_0,s_0)\big)\Big(\mp\frac{1}{2\pi i}\frac{1}{s(e^{-i\theta}z)-s_0} \\ &\qquad +\frac{\operatorname{sign}\big(t(e^{-i\theta}z)-t_0\big)\delta\big(s(e^{-i\theta}z)-s_0\big)}{2}\Big)e^{-2t_0}dt_0ds_0 \\ &= \int_{\mathbb{R}^2}\mp\frac{1}{2\pi i}\frac{1}{s(e^{-i\theta}z)-s(z_0)}f\big(e^{i\theta}z(t_0,s_0)\big)dt_0ds_0 \\ &\qquad +\int_{\mathbb{R}}\frac{\operatorname{sign}\big(t(e^{-i\theta}z)-t_0\big)}{2}f\big(e^{i\theta}z(t_0,s(e^{-i\theta}z))\big)dt_0 \\ &= \frac{\mp1}{2i}(H\hat f)(s(e^{-i\theta}z),\theta) + D_\theta f(z). \end{aligned} \tag{69}$$

Here, we have used that $P(z(t_0,s_0)) = e^{2t_0}$. The Hilbert transform and the symmetrized beam transform have been defined in (26) and (24), respectively.

Note that $D_\theta f(z)$ cannot directly be written in terms of the measured data $\hat f(s,\theta)$. However, the difference

$$\varphi(z,e^{i\theta}) = u_+(z,e^{i\theta}) - u_-(z,e^{i\theta}) = i(H\hat f)(s(e^{-i\theta}z),\theta), \tag{70}$$

is explicitly calculated from the ray transform of $f(z)$.

Riemann-Hilbert problem. We have obtained so far that $u(z,\lambda)$ is analytic in λ on $D^+ \cup D^-$ and have characterized the jump of $u(z,\lambda)$ across $\lambda \in T$. This does not fully characterize $u(z,\lambda)$ yet as $u(z,\lambda)$ does not converge to 0 as $\lambda \to \infty$. We have observed however that $u(z,\lambda = z) = \bar{u}(z, \lambda = \bar{z}^{-1}) = 0$. This constraint may be used to uniquely determine $u(z,\lambda)$ from its jump conditions across T as follows.

Let us define the conformal mapping

$$\mu : \lambda \mapsto \mu(\lambda) = \frac{\lambda - z}{1 - \lambda \bar{z}}. \tag{71}$$

We verify that μ sends z to 0 and preserves the unit circle T (but not its points individually). Define now

$$\tilde{u}(z,\mu) = \mu_* u(z,\mu) = u(z,\lambda), \quad \text{and} \quad \tilde{\varphi}(z,\nu) = \mu_* \varphi(z,\nu) = \varphi(z,\tau), \tag{72}$$

for $\nu = \mu(\tau)$ (and τ) on the unit circle. Since μ is conformal, we verify that

$$\tilde{\varphi}(z,\nu) = \tilde{u}_+(z,\nu) - \tilde{u}_-(z,\nu),$$

the jump of $\tilde{u}(z,\cdot)$ across T. Note that $\bar{z}^{-1}$ is mapped to ∞ by μ and recall that $\bar{u}(z,\bar{\lambda}^{-1}) = u(z,\lambda)$ so that $\tilde{u}(z,\mu)$ converges to 0 as $\mu \to \infty$. Since in addition $\tilde{u}(z,\mu)$ is analytic in μ on $D^+ \cup D^-$, $\tilde{u}(z,\mu)$ is the unique solution of a Riemann Hilbert problem [1] given by the following Cauchy formula:

$$\tilde{u}(z,\mu) = \frac{1}{2\pi i} \int_T \frac{\tilde{\varphi}(z,\nu)}{\nu - \mu} d\nu. \tag{73}$$

Asymptotic expansions in the vicinity of $\mu = 0$ yield

$$\tilde{u}(z,\mu) = \frac{1}{2\pi i} \int_T \frac{\tilde{\varphi}(z,\nu)}{\nu} d\nu + \mu \frac{1}{2\pi i} \int_T \frac{\tilde{\varphi}(z,\nu)}{\nu^2} d\nu + O(\mu^2). \tag{74}$$

Because $\tilde{u}(z,\mu)$ vanishes at $\mu = 0$, we deduce that the data need to satisfy the following compatibility condition

$$\frac{1}{2\pi i} \int_T \frac{\tilde{\varphi}(z,\nu)}{\nu^2} d\nu = 0. \tag{75}$$

Let us now define the field

$$\tilde{X}(\mu) = (1 - |z|^2)\Big(\mu\partial + \frac{1}{\mu}\bar{\partial}\Big) = \mu_* X(\mu) = X(\lambda); \quad \mu = \mu(\lambda). \tag{76}$$

Using the transport equation (57) in the μ variable $\tilde{X}(\mu)\tilde{u}(z,\mu) = f(z)$, we deduce from asymptotic expansions in the vicinity of $\mu = 0$ that

$$\frac{1 - |z|^2}{2\pi i} \int_T \frac{\bar{\partial}\tilde{\varphi}(z,\nu)}{\nu^2} d\nu = f(z). \tag{77}$$

Using the automorphism of T (the restriction of μ on T):

$$\nu = \frac{\tau - z}{1 - \tau\bar{z}}, \quad \frac{d\nu}{\nu} = \frac{1 - |z|^2}{(\tau - z)(1 - \tau\bar{z})} d\tau, \quad \frac{d\nu}{\nu^2} = \frac{1 - |z|^2}{(\tau - z)^2} d\tau, \tag{78}$$

we deduce that

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_T P(\tau^{-1}z)\varphi(z,\tau)\frac{d\tau}{\tau} = \frac{1}{2\pi}\int_0^{2\pi} P(e^{-i\theta}z)\varphi(z,e^{i\theta})d\theta, \\ f(z) &= \frac{(1-|z|^2)^2}{2\pi i}\int_T \frac{\bar\partial\varphi(z,\tau)}{(\tau-z)^2}d\tau. \end{aligned} \tag{79}$$

Since $\varphi(z,\tau)$ is explicitly given in terms of the geodesic ray transform of $f(z)$ in (70), we obtain an explicit reconstruction formula for the source term $f(z)$.

4.3 Reconstruction formulae

The relation (79) provides us with an inversion formula for the ray transform on the two-dimensional hyperbolic manifold. We may render it more explicit as follows. We recall that the orthogonal vector fields $X^\perp(e^{i\theta})$ and $\check{X}^\perp(e^{i\theta})$ are defined in (16) and (18), respectively. Since the ray transform preserves the real-valuedness of $f(z)$, we can assume without loss of generality that $f(z)$ is real-valued. Since from (70), $\varphi \in i\mathbb{R}$, we deduce that $\bar\varphi = -\varphi$. Using the change of variables $\tau = e^{i\theta}$ so that $d\tau = ie^{i\theta}d\theta$, we deduce that

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-|z|^2)^2}{(1-e^{-i\theta}z)^2}e^{-i\theta}\bar\partial\varphi d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-|z|^2)^2}{(1-e^{i\theta}\bar z)^2}e^{i\theta}(-\partial\varphi)d\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi}\frac{(1-|z|^2)^2}{(1-e^{-i\theta}z)(1-e^{i\theta}\bar z)}\Big(\frac{1-e^{i\theta}\bar z}{1-e^{-i\theta}z}e^{-i\theta}\bar\partial - \frac{1-e^{-i\theta}z}{1-e^{i\theta}\bar z}e^{i\theta}\partial\Big)\varphi d\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi} P(e^{-i\theta}z)\check{X}^\perp(e^{i\theta})(-i\varphi)(z,e^{i\theta})d\theta. \end{aligned}$$

This yields the reconstruction formula

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(e^{-i\theta}z)\check{X}^\perp(e^{i\theta})(H\hat f)(s(e^{-i\theta}z),\theta)d\theta. \tag{80}$$

The above formula may be recast as follows

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P^2(e^{-i\theta}z)\Big(H\frac{\partial}{\partial s}\hat f\Big)(s(e^{-i\theta}z),\theta)d\theta. \tag{81}$$

Indeed we verify that

$$X^\perp(e^{i\theta})\Big(H\hat f(s(e^{-i\theta}))\Big) = (e^{i\theta}z)_*\frac{\partial}{\partial s}(e^{i\theta}z)_*H\hat f = (e^{i\theta}z)_*\Big(\frac{\partial}{\partial s}H\hat f\Big) = (e^{i\theta}z)_*\Big(H\frac{\partial}{\partial s}\hat f\Big).$$

This proves (29), and using the definition (27) of the dual transform R^*, the formula (30). It remains to justify the compatibility condition in (79). We verify that $X(e^{i\theta})\varphi(z,e^{i\theta}) = 0$ by construction. This implies that $\varphi \equiv \varphi(s(e^{-i\theta}z),\theta)$ is independent of the geodesic variable t. The compatibility condition in (79) is then equivalent to (28).

5 Attenuated ray transform and inversion

Generalizations of the reconstruction formulae obtained for the ray transform (21) to the attenuated ray transform (32) are now relatively straightforward. The transport equation (22) is replaced by (31). Its extension into the complex plane is then

$$X(\lambda)u(z,\lambda) + a(z)u(z,\lambda) = f(z). \tag{82}$$

The solution is given by

$$u(z,\lambda) = \int_D G(z;\lambda,\zeta)e^{h(\zeta,\lambda)-h(z,\lambda)}f(\zeta)dm_g(\zeta), \tag{83}$$

where we have defined

$$h(z,\lambda) = \int_D G(z;\lambda,\zeta)a(\zeta)dm_g(\zeta), \tag{84}$$

the solution of $X(\lambda)h = a$. We verify that the limits as $\lambda \to T$ are given by

$$h_\pm(z,e^{i\theta}) = \mp\frac{1}{2i}(HR_\theta a)(s(e^{-i\theta}z),\theta) + D_\theta a(z).$$

The limits of the transport equation are thus given by

$$\begin{aligned} u_\pm(z,e^{i\theta}) &= e^{-h_\pm(z,e^{i\theta})}\Big[\frac{\mp 1}{2i}HR_\theta\Big(e^{\mp\frac{1}{2i}H\hat{a}(s(e^{-i\theta}\zeta))}e^{D_\theta a(\zeta)}f(\zeta)\Big)(s(e^{-i\theta}z),\theta)\Big] \\ &\quad + e^{-h_\pm(z,e^{i\theta})}D_\theta\Big(e^{\mp\frac{1}{2i}H\hat{a}(s(e^{-i\theta}\zeta))}e^{D_\theta a(\zeta)}f(\zeta)\Big)(z). \end{aligned} \tag{85}$$

Since D_θ and R_θ are operators acting on the t variable only, and not on the s variable, in geodesic coordinates, we deduce that

$$R_\theta e^{\mp\frac{1}{2i}H\hat{a}(s(e^{-i\theta}\zeta))} = e^{\mp\frac{1}{2i}H\hat{a}(s(e^{-i\theta}\zeta))}R_\theta, \qquad D_\theta e^{\mp\frac{1}{2i}H\hat{a}(s(e^{-i\theta}\zeta))} = e^{\mp\frac{1}{2i}H\hat{a}(s(e^{-i\theta}\zeta))}D_\theta.$$

This allows us to recast $u_\pm$ as

$$\begin{aligned} u_\pm(z,e^{i\theta}) &= e^{-h_\pm(z,e^{i\theta})}\Big(\frac{\mp 1}{2i}He^{\mp\frac{1}{2i}H\hat{a}(s(e^{-i\theta}\zeta))}[R_a f]\Big)(s(e^{-i\theta}z),\theta) \\ &\quad + e^{-D_\theta a(z)}D_\theta\Big(e^{D_\theta a(\zeta)}f(\zeta)\Big)(z). \end{aligned} \tag{86}$$

Let us now define the jump across T:

$$\varphi(z,e^{i\theta}) = (u_+ - u_-)(z,e^{i\theta}). \tag{87}$$

We observe that

$$\varphi(z,e^{i\theta}) = iP^{-1}(e^{-i\theta}z)R^*_{-a,\theta}H_a R_{a,\theta}f(z), \tag{88}$$

where the operators involved in the above formula are defined in (34). We can now use the Cauchy formula (73) to conclude the reconstruction. The transport equation (82) may be recast after conformal mapping by (71) as

$$\big(\tilde{X}(\mu) + a(z)\big)\tilde{u}(z,\mu) = f(z). \tag{89}$$

The expressions (74) and (76) still hold in the attenuated case and yield the same formulae (79) as in the non-attenuated case, except that $\varphi(z, e^{i\theta})$ is now given by (88). Note that the compatibility condition

$$\frac{1}{2\pi}\int_0^{2\pi} P(e^{-i\theta}z)\varphi(z, e^{i\theta})d\theta = 0, \tag{90}$$

can no longer be written as a back-projection as in (28) since φ solves $(X(e^{i\theta})+a)\varphi = 0$ and thus depends on both variables t and s non trivially.

As before the reconstruction formula in (79) can be written more explicitly as

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(e^{-i\theta}z)X^{\perp}(e^{i\theta})(-i\varphi)(z, e^{i\theta})d\theta, \tag{91}$$

which yields after using (88):

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} \check{X}^{\perp}(e^{i\theta})\big(R^*_{-a,\theta}H_a[R_{a,\theta}f]\big)(z, e^{i\theta})d\theta, \tag{92}$$

where $[R_{a,\theta}f]$ is the measured attenuated ray transform. This is (36), which may be recast as (37).

6 Dual transforms and inversions

The derivation of inversion formulae for the horocyclic transform is very similar to that for the geodesic transform. We highlight the differences. The complexification of the normalized vector field along horocycle is given by

$$\check{X}^{\perp}(\lambda) = i(1-|z|^2)\Big(-\frac{\lambda - z}{1-\lambda\bar{z}}\partial + \frac{1-\lambda\bar{z}}{\lambda - z}\bar{\partial}\Big). \tag{93}$$

The transport equation involved in the definition of the attenuated ray transform along horocycles of a source term $f(z)$ is then

$$\check{X}^{\perp}(\lambda)u^{\perp}(z,\lambda) + a(z)u^{\perp}(z,\lambda) = f(z). \tag{94}$$

The fundamental solution to the above equation is thus defined by

$$\check{X}^{\perp}(\lambda)G^{\perp}(z;\lambda, z_0) = \delta_g(z - z_0). \tag{95}$$

In order to solve the above equation, we first verify that

$$\check{X}^{\perp}(\lambda)P(z,\lambda) = (\lambda\circ z)_*(e^{2t}\frac{\partial}{\partial s}e^{2t}) = 0.$$

We thus define the transformation (still denoted by P):

$$P(\lambda) : (z,\bar{z}) \mapsto (P,\bar{P}) = \left(\frac{1-z\bar{z}}{(1-\lambda^{-1}z)(1-\lambda\bar{z})}, \frac{1-z\bar{z}}{(1-\bar{\lambda}^{-1}\bar{z})(1-\bar{\lambda}z)}\right). \tag{96}$$

We compute for $z \in S_\lambda = D\backslash\{\lambda\}$:

$$\frac{\partial P}{\partial z} = \frac{\lambda}{(\lambda - z)^2}, \quad \frac{\partial P}{\partial \bar z} = \frac{\lambda}{(1-\lambda \bar z)^2}, \qquad \frac{\partial \bar P}{\partial z} = \frac{\bar\lambda}{(1-\bar\lambda z)^2}, \quad \frac{\partial \bar P}{\partial \bar z} = \frac{\bar\lambda}{(\bar\lambda - \bar z)^2}.$$

This implies that the Jacobian of the transformation P is given by

$$\partial P(z) = |\lambda|^2 \left(\frac{1}{|\lambda - z|^4} - \frac{1}{|1-\lambda \bar z|^4} \right) > 0.$$

We now have

$$\begin{aligned} P_* \check X^\perp(\lambda) &= i(1-|z|^2) \left(-\frac{\lambda - z}{1-\lambda\bar z} \frac{\partial \bar P}{\partial z} + \frac{1-\lambda \bar z}{\lambda - z} \frac{\partial \bar P}{\partial \bar z} \right) \frac{\partial}{\partial \bar P} \\ &= i(1-|z|^2)\lambda^{-1}(\lambda - z)(1-\lambda \bar z)(\partial P(z)) \frac{\partial}{\partial \bar P}. \end{aligned}$$

The fundamental solution $G^\perp$ thus satisfies

$$P_*\check X^\perp(\lambda) P_* G^\perp = (1-|z|^2)^2 P_* \delta(z - z_0) = (1-|z_0|^2)^2 |\partial P(z_0)| \delta(P - P(z_0, \lambda)),$$

which implies that

$$\frac{\partial}{\partial \bar P}(P_* G^\perp) = -i \frac{\lambda(1-|z_0|^2)}{(\lambda - z_0)(1-\lambda \bar z_0)} \delta(P - P(z_0,\lambda)).$$

Using (64), the solution to the above equation is finally given by

$$G^\perp(z;\lambda,z_0) = \frac{P(z_0,\lambda)}{i\pi} \frac{1}{P(z,\lambda) - P(z_0,\lambda)}. \tag{97}$$

We verify as in (63) that $G^\perp(z;\lambda,z_0)$ is analytic in $\lambda \in D^+$ when $z \neq z_0$.

We now characterize the limit of $G^\perp(\lambda)$ as $\lambda \to 1^-$. We verify that

$$P(z) = \frac{1-|z|^2}{|1-z|^2}, \qquad \frac{\partial P}{\partial \lambda}(z,1) = \frac{(1-|z|^2)(\bar z - z)}{|1-z|^4} = 2is(z)P(z).$$

Since $P(z) > 0$ on the horocycle $P(z) = P(z_0)$, we obtain that $\mathrm{sign}(i\frac{\partial P}{\partial \lambda}(z,1) - i\frac{\partial P}{\partial \lambda}(z_0,1)) = -\mathrm{sign}(s(z) - s(z_0))$. Following the same calculation as in (67), we deduce that

$$G^\perp_\pm(z;e^{i0},z_0) = \frac{\pm P(z_0)}{i\pi} \frac{1}{P(z) - P(z_0)} + P(z_0)\delta(P(z) - P(z_0))\mathrm{sign}(s(z) - s(z_0)). \tag{98}$$

We may thus recast the above limit as

$$G^\perp_\pm(z;e^{i0},z_0) = \frac{\pm P(z_0)}{i\pi} \frac{1}{P(z) - P(z_0)} + \frac{1}{2}\delta(t(z) - t(z_0))\mathrm{sign}(s(z) - s(z_0)). \tag{99}$$

Let us define

$$h^\perp(z,\lambda) = \int_D G^\perp(z;\lambda,z_0) a(z_0) dm_g(z_0),$$

the solution to $\check{X}^\perp(\lambda)h^\perp = a$. We recall that $z^* dm_g = e^{-2t} dt ds$. The limit as $\lambda \to 1$ is given by

$$\begin{aligned} & h^\perp_\pm(z, e^{i0}) \\ = \;& \pm \int_D \frac{P(z_0)}{i\pi(P(z) - P(z_0))} a(z_0) dm_g(z_0) + \int_D \frac{\delta(t(z) - t_0)}{2} \mathrm{sign}(s(z) - s_0) a(z_0) dm_g(z_0) \\ = \;& \pm \frac{1}{i\pi} \int_{\mathbb{R}^2} \frac{a(z(t_0, s_0))}{e^{2t(z)} - e^{2t_0}} dt_0 ds_0 + \frac{e^{-2t(z)}}{2} \int_{\mathbb{R}} \mathrm{sign}(s(z) - s_0) a(z(t(z), s_0)) ds_0 \\ = \;& \pm \frac{1}{i\pi} \int_{\mathbb{R}^2} \frac{a(z(t_0, s_0)) e^{-2t_0}}{e^{2(t(z) - t_0)} - 1} dt_0 ds_0 + \frac{e^{-2t(z)}}{2} \int_{\mathbb{R}} \mathrm{sign}(s(z) - s_0) a(z(t(z), s_0)) ds_0. \end{aligned}$$

Thanks to the invariance of the family of horocycles by rotation, we finally obtain the following limit

$$h^\perp_\pm(z, e^{i\theta}) = \pm \frac{1}{2i} (H^\perp R^\perp_\theta a)(t(e^{-i\theta} z), \theta) + D^\perp_\theta a(z). \tag{100}$$

We recall that the weighted Hilbert transform $H^\perp$ is defined in (44). This implies that the limit of the transport solution as $\lambda \to T$ is given by

$$\begin{aligned} u^\perp_\pm(z, e^{i\theta}) \;=\;& e^{-h^\perp_\pm(z, e^{i\theta})} \Big[\frac{\pm 1}{2i} H^\perp \Big(e^{\frac{\pm 1}{2i} H^\perp \check{a}(t(e^{-i\theta}\zeta))} [R^\perp_a f] \Big) \Big] (t(e^{-i\theta} z), \theta) \\ & + e^{-D^\perp_\theta a(z)} D^\perp_\theta \Big(e^{D^\perp_\theta a(\zeta)} f(\zeta) \Big)(z). \end{aligned} \tag{101}$$

We have used here that for any smooth function $\varphi(t(z))$,

$$R^\perp_\theta \varphi(t(e^{-i\theta}\zeta)) = \varphi(t(e^{-i\theta}\zeta)) R^\perp_\theta, \qquad D^\perp_\theta \varphi(t(e^{-i\theta}\zeta)) = \varphi(t(e^{-i\theta}\zeta)) D^\perp_\theta,$$

since $R^\perp_\theta$ and $D^\perp_\theta$ act only on the s-geodesic coordinate. Let us define

$$\varphi^\perp(z, e^{i\theta}) = (u^\perp_+ - u^\perp_-)(z, e^{i\theta}). \tag{102}$$

We observe that

$$i\varphi^\perp(z, e^{i\theta}) = R^{\perp *}_{-a,\theta} H^\perp_a R^\perp_{a,\theta} f(z), \tag{103}$$

where the operators in the above equality are defined in (46). As in the case dealing with integration along geodesics, we verify that $u(z, \lambda)$ vanishes for $\lambda = z$ since $|P(z, \lambda)|$ becomes infinite there. We can thus use the conformal map (71) and define $\tilde{u}^\perp(z, \mu) = \mu_* u^\perp(z, \mu) = u^\perp(z, \lambda)$. The following Cauchy formula then applies

$$\tilde{u}^\perp(z, \mu) = \frac{1}{2\pi i} \int_T \frac{\tilde{\varphi}^\perp(z, \nu)}{\nu - \mu} d\nu.$$

Using the change of variables defined in (78), this yields as before the compatibility condition

$$\frac{1}{2\pi i} \int_T \frac{\tilde{\varphi}^\perp(z, \nu)}{\nu} d\nu = \frac{1}{2\pi i} \int_T P(z, \tau) \frac{\varphi^\perp(z, \tau)}{\tau} d\tau = 0. \tag{104}$$

We now verify that

$$\mu_* \check{X}^\perp(\mu) = \tilde{X}^\perp(\mu) = i(1 - |z|^2)(-\mu\partial + \frac{1}{\mu}\bar{\partial}).$$

We deduce from the asymptotic expansion in the vicinity of $\mu = 0$ of the transport equation $(\tilde{X}^\perp(\mu) + a)\tilde{u}^\perp(z,\mu) = f(z)$ that

$$f(z) = \frac{(1-|z|^2)}{2\pi}\int_T \frac{\bar\partial\tilde\varphi^\perp(z,\nu)}{\nu^2}d\nu = (1-|z|^2)^2\frac{1}{2\pi}\int_T \frac{\bar\partial\varphi^\perp(z,\tau)}{(\tau-z)^2}d\tau. \qquad (105)$$

Using that $i\varphi^\perp$ is real-valued when $f(z)$ is real-valued, we deduce that

$$\begin{aligned} f(z) &= (1-|z|^2)^2\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-i\theta}}{(1-e^{-i\theta}z)^2}\bar\partial(i\varphi^\perp)d\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi}\frac{1-|z|^2}{(1-e^{-i\theta}z)(1-e^{i\theta}\bar z)}(1-|z|^2)\Big(\frac{1-e^{-i\theta}z}{1-e^{i\theta}\bar z}e^{i\theta}\partial + \frac{1-e^{i\theta}\bar z}{1-e^{-i\theta}z}e^{-i\theta}\bar\partial\Big)(i\varphi^\perp)d\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi} P(e^{-i\theta}z)X(e^{i\theta})(i\varphi^\perp)d\theta. \end{aligned}$$

It remains to replace $i\varphi^\perp$ by its expression (103) in terms of the data to obtain the final reconstruction formulae (48) and (49).

7 Vectorial ray transform

So far we have considered the integration of scalar source terms. We now consider the reconstruction of vectorial source terms from their geodesic ray transforms; see [23, 33] for references on vector tomography in Euclidean geometry.

Let $F(z)$ be a vector field on TD. We consider the transport equation

$$X(e^{i\theta})u(z,e^{i\theta}) + a(z)u(z,e^{i\theta}) = \langle X(e^{i\theta}), F\rangle, \qquad (106)$$

with the usual vanishing conditions at $-\infty$. The attenuated ray transform of the vector field is given by

$$R_aF(s,\theta) \equiv R_{a,\theta}F(s) = \int_{\xi(s,\theta)} e^{D_\theta a}(z,e^{i\theta})\langle X(e^{i\theta}),F\rangle dm_g(z). \qquad (107)$$

This is again the limit of $e^{D_\theta a}u$ as $t\to+\infty$ along the geodesic $\xi(s,\theta)$. The question is then whether F can be reconstructed from R_aF. The answer is yes on the support of a positive absorption term $a(z)$. In the absence of absorption, all one can reconstruct about F is its solenoidal component, or equivalently its curl defined by $*dF^\flat$; see [2] for the notation.

The inversion formulae are based on the analysis of $u(z,\lambda)$ for λ in the vicinity of z. Let $F^\flat$ be the co-vector (one form) associated to the vector F. The complexified transport equation reads

$$X(\lambda)u(z,\lambda) + a(z)u(z,\lambda) = F^\flat(z)X(\lambda). \qquad (108)$$

We recall the definition $\tilde{X}(\mu) = \mu_* X(\mu)$ in (76) and define the expansion

$$\tilde{X}(\mu) = \frac{1}{\mu}\tilde{X}_0 + O(\mu), \qquad \tilde{X}_0 = (1-|z|^2)\bar\partial. \qquad (109)$$

Since $F^\flat$ is independent of μ, we deduce that $\tilde{u}(z,\mu)$ is bounded (but no longer converges to 0) in the vicinity of $\mu=0$. We thus have

$$\tilde{u}(z,\mu) = u_0(z) + \mu u_1(z) + O(\mu^2).$$

Upon plugging these expansions into the transport equation and equating like powers of μ, we get

$$\begin{aligned} \tilde{X}_0 u_0(z) &= F^\flat(z)\tilde{X}_0 \\ \tilde{X}_0 u_1(z) + a u_0(z) &= 0. \end{aligned} \tag{110}$$

Let us return to the Riemann-Hilbert problem. Since $\tilde{u}(z,\mu)$ is bounded and analytic on $D^+\cup D^-$, we can apply the Cauchy formula as in (73) and obtain that

$$\tilde{u}(z,\mu) = \frac{1}{2\pi i}\int_T \frac{\tilde{\varphi}(z,\nu)}{\nu-\mu} d\nu + \bar{u}_0(z). \tag{111}$$

We verify that $\tilde{u}(z,\infty)=\bar{u}_0(z)$ thanks to (53). In (111), $\tilde{\varphi}(z,\nu)=\mu_*\varphi(z,\nu)$ is defined as in (72) and $\varphi(z,\tau)=u_+(z,\tau)-u_-(z,\tau)$ is defined in the vectorial context as

$$\varphi(z,e^{i\theta}) = iP^{-1}(e^{-i\theta}z)R^*_{-a,\theta}H_a[R_{a,\theta}F](z), \tag{112}$$

where the operators involved in the above formula are defined in (34) and (107). The derivation of the above formula is the same as in (103) as $X(\lambda)$ is analytic in the vicinity of T, whence does not jump across T. Since both sides in (111) are analytic in μ on D^+, we may perform an asymptotic expansion in μ. The first two terms in the expansion yield (we can verify [3] that higher-order terms in the expansion do not provide additional information):

$$\begin{aligned} u_0(z) - \bar{u}_0(z) &= \frac{1}{2\pi i}\int_T \frac{\tilde{\varphi}(z,\nu)}{\nu} d\nu = \frac{1}{2\pi}\int_0^{2\pi} P(e^{-i\theta}z)\varphi(z,e^{i\theta})d\theta \equiv \varphi_0(z) \\ u_1(z) &= \frac{1}{2\pi i}\int_T \frac{\tilde{\varphi}(z,\nu)}{\nu^2} d\nu = \frac{1}{2\pi i}\int_T \frac{\varphi(z,\tau)}{(\tau-z)^2} d\tau \equiv \varphi_1(z). \end{aligned} \tag{113}$$

Owing to (112), the terms $\varphi_0(z)$ and $\varphi_1(z)$ are uniquely determined by the data $R_{a,\theta}F(s)$.

Let us assume that $a\equiv 0$. Then the second equations in (110) and (113) do not provide any information about $F^\flat$. The first equations provide partial information about $F^\flat = F_1 dx + F_2 dy$. Indeed, we verify that the first equations in (110) and (113) are equivalent to the following system

$$\begin{aligned} \bar{\partial} u_0(z) &= \frac{F_1 + iF_2}{2} \\ 2i\Im\{u_0(z)\} &= \varphi_0(z). \end{aligned} \tag{114}$$

Upon applying ∂ to both sides of the first equation and $\Delta = 4\partial\bar{\partial}$ to the second equation, we get

$$\text{curl}F \equiv *dF^\flat = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{1}{2i}\Delta\varphi_0. \tag{115}$$

We can thus reconstruct the solenoidal component of F but not its gradient component.

Let us now assume that $a(z) > 0$ on the support of $F(z)$. We deduce from (110) that

$$F^\flat(z)\tilde{X}_0 = -\tilde{X}_0 \frac{1}{a(z)} \tilde{X}_0 u_1(z). \tag{116}$$

More explicitly, this yields

$$F_1(z) + iF_2(z) = -2\bar{\partial}\Big(\frac{1-|z|^2}{a(z)}\bar{\partial}\varphi_1(z)\Big). \tag{117}$$

Since the vectorial ray transform preserves the real-valuedness of the vector field, we may assume that F is real-valued. The above formula thus allows us to reconstruct both components of F. This is similar to the results obtained in Euclidean geometry [3, 8].

8 Half-plane geometry

The Poincaré disc model $(D, g_{1/2})$, where $g_{1/2}$ is defined in (19) and has constant curvature $K_{1/2} = -1$, is isometric [16] to the Poincaré half-plane model (H, h), where H is the upper half-plane $\mathbb{R} \times \mathbb{R}^+$ and h is the Riemannian metric defined by

$$ds_h^2 = \frac{dx^2 + dy^2}{y^2}, \qquad (x, y) \in \mathbb{R} \times \mathbb{R}^+. \tag{118}$$

The isometry may be realized by the conformal map

$$w : \begin{cases} D & \to & H \\ z & \mapsto & w(z) = i\dfrac{1-z}{1+z}, \end{cases} \tag{119}$$

such that 1 is mapped to $w(1) = 0$ and 0 is mapped to $w(0) = i$. The description of the geodesics presented in section 2 has an equivalent description on (H, h). We rather use the isometry $w(z)$ to obtain the properties we need on (H, h). For instance, $Y(0) = w_* X_{1/2}(e^{i0})$ is the geodesic vector field of geodesics converging to $(0, 0)$ on the real axis. After some algebra we find explicitly that

$$Y(0)_{|w} = iy\Big(\frac{w}{\bar{w}}\partial - \frac{\bar{w}}{w}\bar{\partial}\Big) \tag{120}$$

where we identify $w = x + iy$. Since (H, h) is invariant by translation along the $x-$axis, the geodesic vector field of geodesic integral curves converging to $(k, 0)$ for $k \in \mathbb{R}$ is given by

$$Y(k)_{|w} = iy\Big(\frac{w-k}{\bar{w}-k}\partial - \frac{\bar{w}-k}{w-k}\bar{\partial}\Big). \tag{121}$$

Here, the parameter k is related to $e^{i\theta}$ by $k = w(e^{i\theta})$.

The complexification of $Y(k)$ is now straightforward. The translation parameter k may be seen as a complex number $k \in \mathbb{C}$ rather than a real number. We verify that $Y(k)$ is an elliptic operator for $k \in \mathbb{C}\backslash\mathbb{R}$. The quantity being complexified is thus no longer the direction of propagation at infinity (all geodesics are almost parallel to the y-axis at infinity), but rather the point of convergence on the x-axis.

For $\Im(k) > 0$, we verify that $Y(k)$ is $\bar{\partial}$-like, which corresponds to the case $\lambda \in D^+$ on (D, g), with $\{\Im(k) > 0\} = w(D^+)$. We thus see how the theory developed on (D, g) may unfold on (H, h). Naturally, all the results obtained earlier in the paper may be directly stated on (H, h) via the conformal map $w(z)$.

9 Conclusions

The method of complexification of geodesic vector fields allows us to invert an important class of weighted ray transforms namely, the attenuated geodesic and horocyclic ray transforms in hyperbolic geometry. It involves a specific global parameterization of the geodesic vector field, based on the point of convergence at infinity of the geodesics.

The inversion of the ray transform in higher dimension is straightforward. Namely, hyperbolic manifolds of dimension higher than two can be foliated into totally geodesic two-dimensional manifolds that are isometric to (D, g). The inversion may then be performed on each leaf of the foliation independently as in Euclidean geometry [24].

For two-dimensional manifolds, it remains an open problem to characterize the class of metrics for which the method of complexification will provide explicit inversion formulae for ray transforms. One may be tempted to believe that real-analytic metrics with negative curvature (thus on geodesically complete manifolds) are in this class. More generally, one may ask which families of integral curves of vector fields of the form $X(e^{i\theta})$ permit the derivation of explicit inversion of ray transforms.

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