Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions

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Abstract

This paper concerns the reconstruction of possibly complex-valued coefficients in a second-order scalar elliptic equation posed on a bounded domain from knowledge of several solutions of that equation. We show that for a sufficiently large number of solutions and for an open set of corresponding boundary conditions, all coefficients can be uniquely and stably reconstructed up to a well characterized gauge transformation. We also show that in some specific situations, a minimum number of such available solutions equal to $I_n = \frac{1}{2}n(n+3)$ is sufficient to uniquely and globally reconstruct the unknown coefficients. This theory finds applications in several coupled-physics medical imaging modalities including photo-acoustic tomography, transient elastography, and magnetic resonance elastography.

1 Introduction

We consider the general second-order elliptic equation:

$$\nabla \cdot a\nabla u + b \cdot \nabla u + cu = 0 \quad \text{in } X, \qquad u = f \quad \text{on } \partial X, \tag{1}$$

with complex-valued coefficients such that a is a symmetric, complex-valued tensor verifying the ellipticity condition for $\alpha_0 > 0$:

$$\alpha_0|\xi|^2 \leq \xi \cdot \Re a\xi \leq \alpha_0^{-1}|\xi|^2, \tag{2}$$

and with (b, c) such that the above equation admits a unique solution in $H^1(X)$ for $f \in H^{\frac{1}{2}}(\partial X)$. Here X is an open bounded domain in $\mathbb{R}^n$ with smooth boundary ∂X. We assume throughout this paper that b, c and $\nabla \cdot a$ are of class $C^{0,\alpha}(\bar{X})$ for some $\alpha > 0$ and that a is of class $C^{0,1}(\bar{X})$. Elliptic regularity results [20, Theorem 6.3.7] then ensure that the solution u is a strong solution of class $C^{2,\alpha}(\bar{X})$ when the boundary condition is of class $C^{2,\alpha}(\partial X)$.

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We assume knowledge of internal functionals given by the complex valued solutions $u_j(x)$ of (1) for a given set of boundary conditions $f = f_j$ for $1 \leq j \leq I$. The main objective of this paper is to show that the coefficients (a, b, c) can be uniquely and stably reconstructed up to a natural gauge transformation from knowledge of the corresponding $(u_j)_{1\leq j\leq I}$, provided that the gradients and Hessians of the latter satisfy some rank maximality conditions. We show that the latter conditions are satisfied provided that I is sufficiently large and the $(f_j)_{1\leq j\leq I}$ belong to an appropriate open set.

In some specific situations, for instance when (a, b, c) is close to $(a_0, 0, 0)$ for a_0 a constant complex-valued matrix satisfying (2), we can guaranty that the reconstructions can be achieved with $I = I_n = \frac{1}{2}n(n+3)$, which corresponds to the (complex) dimension of the unknown coefficients (a, b, c) up to the (one parameter) gauge transformation, provided that the $(f_j)_{1\leq j\leq I_n}$ belong to an appropriate open set.

The mathematical description of the measurement operator considered here and the main results of the paper are presented in section 2. Each solution u_j in (1) generates a linear constraint on (a, b, c). When these constraints are of maximal rank, an explicit inversion procedure provides a global, unique and stable reconstruction of (a, b, c) up to a natural gauge transformation. The main technical work of the paper is to show that these maximal rank conditions are satisfied uniformly on the domain X under the sufficient (but by no means necessary) conditions that I is sufficiently large and $(f_j)_{1\leq j\leq I}$ are chosen in an appropriate, but often poorly characterized, open set, which may depend on the unknown coefficients. The proofs of the main results are detailed in sections 3 and 4. Their salient features are summarized in Remark 4.9.

The theory presented in this paper finds applications in several recent coupled-physics (also called hybrid or multi-wave) imaging modalities that are extensively studied in the bio-engineering community. These methods arose as an attempt to devise new imaging modalities that combine high resolution with high contrast. In section 5, we consider three such modalities: Photo-Acoustic Tomography (PAT), Transient Elastography (TE) and Magnetic Resonance Elastography (MRE).

Our results show that for some models, all the medium parameters can be reconstructed uniquely and stably for these imaging modalities. The reason is that these coefficients have a structure that allows us to uniquely reconstruct the gauge as well. In PAT, the main novelty of our result is that general anisotropic diffusion coefficients can be reconstructed. In a scalar model for TE and MRE, the main novelty is that we can not only reconstruct anisotropic coefficients but also complex-valued coefficients that account for possible dispersion and attenuation effects. An extension of our analysis to more general models of PAT, TE, and MRE may be found in [5, 10].

The reconstruction of anisotropic coefficients a from boundary value measurements of u (as in, e.g., the Calderón problem) can be performed only up to a very large class of changes of variables [28]. Moreover, the corresponding stability estimates are of logarithmic type, which corresponds to drastic amplifications of measurement noise during the reconstruction. The availability of internal functionals such as those in PAT, TE, and MRE, allows one to reconstruct a larger class of coefficients and with significantly better stability estimates. General real-valued anisotropic tensors were reconstructed in the context of ultrasound modulation, another hybrid modality, in dimension $n = 2$ in [19].

For additional information on quantitative PAT and TE and MRE, we refer the

reader to e.g. [7, 8, 9, 12, 13, 29] and [3, 16, 18, 21] and their lists of references, respectively. For general references to the theory of hybrid inverse problems devoted to the mathematical analysis of similar coupled-physics imaging modalities, we refer the reader to e.g. [2, 6, 26].

2 Main results

Gauge transform. The elliptic equation (1) may be recast as

$$a : \nabla^{\otimes 2} u + (\nabla \cdot a + b) \cdot \nabla u + cu = 0. \tag{3}$$

Upon multiplying through by the scalar function $\tau \neq 0$, we get

$$a_\tau : \nabla^{\otimes 2} u + (\nabla \cdot a_\tau + b_\tau) \cdot \nabla u + c_\tau u = 0, \tag{4}$$

with

$$a_\tau = \tau a, \quad b_\tau = \tau b - a \nabla \tau, \quad c_\tau = \tau c. \tag{5}$$

This shows that the coefficients (a, b, c) can be reconstructed at most up to the above gauge transformation. We say that $(a, b, c) \sim (a_\tau, b_\tau, c_\tau)$ belong to the same equivalence class if there exists a non vanishing (sufficiently smooth with smoothness depending on context) function τ such that (5) holds. Then we say that $(a, b, c) \in \mathfrak{c}$ the equivalence class.

Measurement operator. For $f \in H^{\frac{1}{2}}(\partial X)$, we obtain a solution $u \in H^1(X)$ and we can define the solution operator

$$\mathfrak{S}_{\mathfrak{c}} : \begin{array}{rcl} H^{\frac{1}{2}}(\partial X) & \to & H^1(X) \\ f & \mapsto & u = \mathfrak{S}_{\mathfrak{c}} f. \end{array} \tag{6}$$

Note that the solution operator is independent of the element $(a, b, c) \in \mathfrak{c}$.

The main measurement operator we consider in this paper is defined as follows. Let $I \in \mathbb{N}^*$ and $f_i \in H^{\frac{1}{2}}(\partial X)$ for $1 \leq i \leq I$ be a given set of I boundary conditions. Define $\mathfrak{f} = (f_1, \ldots, f_I)$. The measurement operator $\mathfrak{M}_{\mathfrak{f}}$

$$\mathfrak{M}_{\mathfrak{f}} : \begin{array}{rcl} \mathfrak{X} & \to & \mathfrak{Y}^I \\ \mathfrak{c} & \mapsto & \mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) = (\mathfrak{S}_{\mathfrak{c}} f_1, \ldots, \mathfrak{S}_{\mathfrak{c}} f_I). \end{array} \tag{7}$$

Here, $\mathfrak{X}$ is a subset of a Banach space in which the unknown coefficients are defined. That space will depend on the context. Also $\mathfrak{Y}$ is a subset of $H^1(X)$ where the solutions to (1) are defined. The main objective of this paper is to consider settings in which $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$ for an appropriate choice of $\mathfrak{f}$ uniquely and stably determines $\mathfrak{c}$.

Main results. Since coefficients in $\mathfrak{c}$ are determined up to an arbitrary multiplicative constant, the latter must be normalized to obtain stability estimates in the reconstruction. We denote by $a_{\mathfrak{c}}$ the unique element such that $(a_{\mathfrak{c}}, b_{\mathfrak{c}}, c_{\mathfrak{c}}) \in \mathfrak{c}$ and normalized so that $\mathrm{Tr}(a_{\mathfrak{c}} a_{\mathfrak{c}}^*) = 1$.

The main results of this paper are summarized in the following Theorem:

Theorem 2.1 *Let $\mathfrak{c}$ and $\tilde{\mathfrak{c}}$ be two classes of coefficients with $(a, b, c, \nabla \cdot a)$ of class $C^{m,\alpha}(\bar{X})$ for $\alpha > 0$ and $m = 0$ or $m = 1$, and a of class $C^{0,1}(\bar{X})$. We assume that (1) is well posed for the coefficients $\mathfrak{c}$.*

Then for I sufficiently large and for an open set (for instance in the topology of $C^{2,\alpha}(\bar{X})$) of boundary conditions $\mathfrak{f} = (f_j)_{1\leq j\leq I}$, then $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$ uniquely determines $\mathfrak{c}$. Moreover, for any other $\tilde{\mathfrak{c}}$ as given above, we have the stability results

$$\begin{aligned} \|(a_{\mathfrak{c}}, b_{\mathfrak{c}} + \nabla \cdot a_{\mathfrak{c}}, c_{\mathfrak{c}}) - (a_{\tilde{\mathfrak{c}}}, b_{\tilde{\mathfrak{c}}} + \nabla \cdot a_{\tilde{\mathfrak{c}}}, c_{\tilde{\mathfrak{c}}})\|_{W^{m,\infty}(X)} &\leq C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{m+2,\infty}(X)}, \\ \|b_{\mathfrak{c}} - b_{\tilde{\mathfrak{c}}}\|_{L^\infty(X)} &\leq C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{3,\infty}(X)}, \end{aligned} \tag{8}$$

for $m = 0, 1$.

We say that $\mathfrak{c}$ is in the vicinity of $\tilde{\mathfrak{c}}$ if $(a_{\mathfrak{c}}, b_{\mathfrak{c}}, c_{\mathfrak{c}}, \nabla \cdot a_{\mathfrak{c}})$ is in the $C^{0,\alpha}(\bar{X})$ vicinity of $(a_{\tilde{\mathfrak{c}}}, b_{\tilde{\mathfrak{c}}}, c_{\tilde{\mathfrak{c}}}, \nabla \cdot a_{\tilde{\mathfrak{c}}})$. Let us assume that $\mathfrak{c}$ in the vicinity of either:

(i) $(a_0, 0, 0)$ for some constant diffusion tensor a_0;

(ii) $(\gamma(x) Id_n, 0, c(x))$ for some scalar coefficients $\gamma \in H^{\frac{n}{2}+4+\epsilon}(\bar{X})$ and $c \in H^{\frac{n}{2}+2+\epsilon}(\bar{X})$ for $\epsilon > 0$ (with Id_n the $n \times n$ identity matrix);

(iii) $(\gamma(x), 0, c(x))$ for an arbitrary diffusion tensor of class $H^{5+\epsilon}(X)$ and c of class $H^{3+\epsilon}(X)$ in dimension $n = 2$.

Then for $I = I_n = \frac{1}{2}n(n + 3)$ and an open set of boundary conditions $\mathfrak{f}$, we have that $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$ uniquely determines $\mathfrak{c}$. Moreover, (8) holds.

The proof of the different results described in the above theorem is presented in detail in sections 3 and 4.

Reconstruction of the gauge. In some situations the gauge in $\mathfrak{c}$ can be uniquely and stably determined. Let us for instance consider the specific, practically important, case of coefficients $(a, b, c) = (\gamma, 0, c)$. Then we have the following result:

Corollary 2.2 *Under the hypotheses of the preceding theorem, and in the setting where $b = 0$, we have that $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$ uniquely determines $(\gamma, 0, c)$. Then we have the following stability result:*

$$\|\gamma - \tilde{\gamma}\|_{L^\infty(X)} \leq C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{2,\infty}(X)}. \tag{9}$$

The proof of the corollary from the results stated in Theorem 2.1 may be found in section 3.2.

3 Local reconstruction

3.1 Reconstruction of equivalence classes

Let n be the spatial dimension and define $I_n = \frac{1}{2}n(n + 3)$. We wish to reconstruct an element in the class $\mathfrak{c} = (a_\tau, b_\tau, c_\tau)$ for τ an arbitrary non-vanishing function from knowledge of u_i for $1 \leq i \leq I_n$. We perform the reconstruction locally in the vicinity of a point x_0. We assume that we have constructed a solution u_1 such that w.l.o.g. $u_1(x_0) = 1$ and by continuity $u_1 \neq 0$ in the vicinity of x_0. We then define

$$v_j = \frac{u_{j+1}}{u_1}, \quad 1 \leq j \leq I_n - 1, \qquad \alpha = u_1^2 a, \qquad \beta = \nabla \cdot (a u_1^2) + u_1^2 b \tag{10}$$

and find after some algebra that

$$\alpha : \nabla^{\otimes 2} v_j + \beta \cdot \nabla v_j = 0, \tag{11}$$

in the vicinity of x_0.

We assume that $(\nabla v_1, \ldots, \nabla v_n)$ forms a frame in the vicinity of x_0. Then $H = (H_{ij})_{1 \le i,j \le n}$ with $H_{ij} = \nabla v_i \cdot \nabla v_j$ is a boundedly invertible symmetric matrix with H^{ij} the coefficients of H^{-1}. Let us define $M_n := \frac{1}{2} n(n+1) - 1$. Note that $I_n = 1 + n + M_n$. We then write for $1 \le m \le M_n$:

$$-\nabla v_{m+n} = \Theta_j^m \nabla v_j, \qquad \Theta_j^m = -H^{jk} \nabla v_{m+n} \cdot \nabla v_k. \tag{12}$$

Here, we use the convention of summation over repeated indices. We then define

$$\theta_j^m = \begin{cases} \Theta_j^m & 1 \le j \le n \\ 1 & j = m+n \\ 0 & \text{otherwise} \end{cases}, \qquad 1 \le j \le I_n - 1, \quad 1 \le m \le M_n. \tag{13}$$

Note that we have constructed the complex-valued coefficients θ_j^m in such a way that

$$\sum_{j=1}^{I_n - 1} \theta_j^m \nabla v_j = 0, \qquad 1 \le m \le M_n. \tag{14}$$

We next construct the symmetric matrices

$$M^m = \sum_{j=1}^{I_n - 1} \theta_j^m \nabla^{\otimes 2} v_j, \quad 1 \le m \le M_n. \tag{15}$$

We assume that the matrices M^m are linearly independent. Since the (complex) dimension of symmetric matrices equals $M_n + 1$, this implies the existence of a unique symmetric, complex-valued, matrix M^0 such that

$$M^0 : M^m = \mathrm{Tr}(M^0 M^m) = 0, \qquad 1 \le m \le M_n, \tag{16}$$

normalized so that $\mathrm{Tr}(M^0 (M^0)^*) = 1$. In other words, there exists a unique normalized matrix $(M^0)^*$ that is orthogonal to the constructed M^m matrices for the inner product $(A, B) = \mathrm{Tr}(A^* B)$. The construction of M^0 can be obtained as follows. We anticipate the fact that the identity matrix Id_n cannot be in the span of the matrices M^m. We can thus use the Gram Schmidt procedure to orthogonalize $(Id_n, (M^m)_{1 \le m \le M_n})$ and construct the matrix M^0.

Multiplying (11) by θ_j^m and summing over j yields the M_n constraints

$$\alpha : M^m = 0, \qquad 1 \le m \le M_n.$$

This shows that α is proportional to the now known matrix $M^0(x)$. Since the real part of a is positive definite, the matrix α cannot be orthogonal to Id_n for the inner product $(\cdot, \cdot)$. This justifies the fact that $(Id_n, (M^m)_{1 \le m \le M_n})$ form a free family. Let us define

$$\alpha(x) := M^0(x). \tag{17}$$

Since the matrix α can be reconstructed up to the gauge transformation corresponding to the multiplication by an arbitrary complex-valued function, we choose the gauge as prescribed above, i.e., such that $\mathrm{Tr}(\alpha^*\alpha) = 1$.

The vector field β is then uniquely determined using (11) by the explicit formula:

$$\beta = -H^{ij}(M^0 : \nabla^{\otimes 2} v_j)\nabla v_i. \tag{18}$$

Once (α, β) are known (up to the gauge transformation), we define

$$a = \frac{1}{u_1^2}\alpha, \quad b = \frac{1}{u_1^2}(\beta - \nabla\cdot\alpha), \quad c = -\frac{1}{u_1}(\nabla\cdot a\nabla u_1 + b\cdot\nabla u_1). \tag{19}$$

This provides an explicit reconstruction of $(a, b, c) \in \mathfrak{c}$, the equivalence class, which is therefore now known. The above results may be recast as follows

$$a = \frac{1}{u_1^2}\alpha, \quad \nabla\cdot a + b = \frac{1}{u_1^2}(\beta - a\nabla u_1^2), \quad c = -\frac{1}{u_1}\Big((\nabla\cdot a + b)\cdot\nabla u_1 + a : \nabla^{\otimes 2} u_1\Big). \tag{20}$$

We summarize the above results in the following lemma.

Lemma 3.1 *Let u_i for $1 \le i \le I_n$ be solutions of the elliptic equation with boundary conditions $u_i = f_i$ on ∂X. Let us define $v_i = u_1^{-1}u_{i+1}$ for $1 \le i \le I_n - 1$ and assume that:*

(i) in the vicinity X_0 of a point x_0, we have that $u_1 \neq 0$ on X and $u_1(x_0) = 1$,

(ii) the vectors $(\nabla v_1, \ldots, \nabla v_n)$ form a frame on X_0 with a matrix $H_{ij} = \nabla v_i \cdot \nabla v_j$ with uniform bounded inverse on $\overline{X_0}$;

(iii) the matrices M^m for $1 \le m \le M_n$ constructed above in (12)-(15) are linearly independent.

Then the class of gauge equivalence $\mathfrak{c}$ is uniquely determined by $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) = (u_i)_{1\le i\le I_n}$ on X_0 in the sense that $u_i = \tilde{u}_i$ for $1 \le i \le I$ implies that $\mathfrak{c} = \tilde{\mathfrak{c}}$ where $\tilde{u}_i$ are the internal functionals obtained by replacing one element in $\mathfrak{c}$ by one element in $\tilde{\mathfrak{c}}$.

Moreover the reconstruction is stable in the sense that

$$\|(a_{\mathfrak{c}}, b_{\mathfrak{c}} + \nabla\cdot a_{\mathfrak{c}}, c_{\mathfrak{c}}) - (a_{\tilde{\mathfrak{c}}}, b_{\tilde{\mathfrak{c}}} + \nabla\cdot a_{\tilde{\mathfrak{c}}}, c_{\tilde{\mathfrak{c}}})\|_{L^\infty(X_0)} \le C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{2,\infty}(X_0)}. \tag{21}$$

The stability result is clear by inspection of the reconstruction procedure: two derivatives on v are taken in the reconstruction of the matrices M^m and hence of M^0 for instance by the Gram-Schmidt procedure, a multi-linear operation that preserves errors in the uniform norm. The same loss of derivatives is observed in the reconstruction of (α, β), and hence in a, $\nabla\cdot a + b$ and c as can be seen in (20). Note that we similarly reconstruct the above coefficients in the $W^{1,\infty}$ sense, and hence b in the L^∞ sense, when errors are bounded in the $W^{3,\infty}$ sense as indicated in Theorem 2.1.

Remark 3.2 Note that the procedure described in Lemma 3.1 allows one to reconstruct $I_n = \frac{1}{2}n(n+3)$ *complex-valued* coefficients from I_n *complex-valued* internal functionals of the form u_j above or to reconstruct I_n *real-valued* coefficients from I_n *real-valued* functionals of the same form.

3.2 Reconstruction of the gauge

Let us assume that we can reconstruct $(a_0, b_0, c_0) \in \mathfrak{c}$ on a domain X, which may be a part of the domain such as X_0 above or the whole domain if local reconstructions of $\mathfrak{c}$ are patched together to provide a global reconstruction. Let us then define

$$(a_\tau, b_\tau, c_\tau) = (\tau a_0, \tau b_0 - a_0 \nabla \tau, \tau c_0) \in \mathfrak{c}$$

an other element of the equivalence class. In this section, we show that prior information about the unknown coefficient allows us to uniquely determine the unknown gauge τ.

Note that another way to represent the gauge transform is to realize that

$$(a_\tau, a_\tau^{-1} b_\tau, c_\tau) = (\tau a_0, a_0^{-1} b_0 - \nabla \ln \tau, \tau c_0).$$

In other words, $a_\tau^{-1} b_\tau - a_0^{-1} b_0$ is a gradient. If $a_\tau^{-1} b_\tau$ is seen as the 1-form $(a_\tau^{-1})_{ij} b_j dx_i$, then $d(a_\tau^{-1} b_\tau)$ is independent of the element in $\mathfrak{c}$ and characterizes the equivalence class on a simply connected domain X.

Let us assume that $\nabla \cdot (a_\tau^{-1} b_\tau) = \Phi$ is known. Then we observe that

$$-\Delta \ln \tau = \Phi - \nabla \cdot (a_0^{-1} b_0),$$

so that τ is uniquely defined on X if it is known at the boundary ∂X.

As another practical assumption to reconstruct the gauge, let us assume that $b = b_\tau$ is divergence free so that $\nabla \cdot (\tau b_0 - a_0 \nabla \tau) = 0$, or in other words

$$-\nabla \cdot (a_0 \nabla \tau) + \nabla \cdot (\tau b_0) = 0.$$

Note that $a_0 = u_1^{-2} M^0$. This is an elliptic equation. Provided that all coefficients are real-valued and that a_0 is uniformly elliptic, then this equation admits a unique solution for τ when τ is known on ∂X. This is a consequence of the maximum principle [14] that does not (always) apply in the case of complex-valued coefficients.

Let us assume the stronger constraint that $b = b_\tau = 0$. This corresponds to $\Phi = 0$ above. In fact, this provides the redundant system of transport equations for τ:

$$\nabla \tau = a_0^{-1} b_0 \tau, \qquad \text{or equivalently} \quad \nabla \ln \tau = a_0^{-1} b_0,$$

which admits a unique solution (on a simply connected domain since $d(a_0^{-1} b_0) = 0$ for $a_0^{-1} b_0$ seen as a 1-form) provided that τ is known at one point . This reconstruction applies for arbitrary complex valued coefficients (a, b, c). In the PAT, TE, and MRE applications considered in section 5, the natural setting is with $b = 0$ so that the gauge can indeed be reconstructed.

Consider the specific example of

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } X, \qquad u = f \text{ on } \partial X.$$

Define $\gamma = \tau M^0$ so that

$$\alpha : \nabla^{\otimes 2} u + \beta \cdot \nabla u = 0, \qquad \alpha = M^0, \quad \beta = \tau^{-1} \nabla \cdot (\tau M^0),$$

with α and β reconstructed as in the preceding section. This shows that

$$\nabla \ln \tau = \alpha^{-1} (\beta - \nabla \cdot \alpha).$$

This is a redundant system of first-order equations for $\ln \tau$, which is one derivative more regular than the source $\alpha^{-1}(\beta - \nabla \cdot \alpha)$. The reconstruction of β involves differentiating the available functionals (u_j) twice while the reconstruction of $\nabla \cdot \alpha$ involves differentiating them thrice. We thus obtain the unique reconstruction of γ with the stability estimate

$$\|\gamma - \tilde{\gamma}\|_{L^\infty(X)} \leq C \|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{2,\infty}(X)}. \tag{22}$$

This proves Corollary 2.2. However, when the anisotropy $\alpha = M^0$ is known a priori, then we obtain the better stability estimate

$$\|\gamma - \tilde{\gamma}\|_{W^{1,\infty}(X)} \leq C \|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{2,\infty}(X)}. \tag{23}$$

The reconstruction of anisotropic tensors is less stable than that of the isotropic coefficients. This is consistent with similar results obtained for the ultrasound modulation problem; see [19].

3.3 Variations of the coefficients

The above procedure shows that reconstructions are stable with respect to fluctuations in the measurements $\mathfrak{M}(\mathfrak{c})$ when the hypotheses of Lemma 3.1 are satisfied. We show that such hypotheses are stable with respect to small changes in the parameters $\mathfrak{c}$.

Lemma 3.3 *Let u and $\tilde{u}$ be solutions of*

$$\nabla \cdot a \nabla u + b \cdot \nabla u + cu = \nabla \cdot \tilde{a} \nabla \tilde{u} + \tilde{b} \cdot \nabla \tilde{u} + \tilde{c} \tilde{u} = 0 \text{ on } X_0, \tag{24}$$

with Dirichlet conditions $u = \tilde{u} = f$ for f of class $C^{m,\alpha}(\partial X_0)$. Then

$$\|u - \tilde{u}\|_{C^{m+2,\alpha}(X_0)} \leq C \|(a, b, c, \nabla \cdot a) - (\tilde{a}, \tilde{b}, \tilde{c}, \nabla \cdot \tilde{a})\|_{C^{m,\alpha}(X_0)}, \tag{25}$$

for some positive constant C independent of $(\tilde{a}, \tilde{b}, \tilde{c}, \nabla \cdot \tilde{a})$ sufficiently close to $(a, b, c, \nabla \cdot a)$ in $C^{m,\alpha}$.

Proof. Let $w = \tilde{u} - u$. We find

$$\nabla \cdot (a - \tilde{a}) \nabla u + (b - \tilde{b}) \cdot \nabla u + (c - \tilde{c}) u = \nabla \cdot \tilde{a} \nabla w + \tilde{b} \cdot \nabla w + \tilde{c} w.$$

The proof then follows from standard regularity results; see [14] for the case of real-valued coefficients and [20, Theorem 6.3.7] for the case of complex-valued coefficients. □

With $m = 0$, we deduce that when the hypotheses of Lemma 3.1 are verified for the coefficients $\mathfrak{c}$, then they are verified with the same boundary conditions for all coefficients $\tilde{\mathfrak{c}}$ that are sufficiently close to $\mathfrak{c}$ in the sense given above.

4 Global reconstruction

We now consider several settings in which global reconstructions of $\mathfrak{c}$ are possible. We have seen in the preceding section that $\mathfrak{c}$ was globally and uniquely reconstructed provided that the solutions $(u_i)_{1 \leq i \leq I}$ satisfied some rank maximality conditions. We provide sufficient conditions on the choice of I and the boundary conditions $(f_i)_{1 \leq i \leq I}$ for the rank maximality conditions to hold.

4.1 Global reconstructions close to constant tensor

We start with reconstructions in the vicinity of $a = Id_n$, $b = 0$, and $c = 0$, where Id_n is the identity matrix in dimension n. The main interest of this result is that the boundary conditions f_i are explicit and particularly simple. Moreover, the following result shows that the $I_n = \frac{1}{2}n(n+3)$ unknown complex-valued (real-valued) coefficients may uniquely and stably be reconstructed from exactly I_n complex-valued (real-valued) internal functionals; see remark 3.2.

Theorem 4.1 *Let $\mathfrak{X}$ be the space of $\mathfrak{c}$ sufficiently close to $(Id_n, 0, 0)$ in the sense of Lemma 3.3 with $m = 0$. Let $I_n = \frac{1}{2}n(n+3)$ and f_i be the traces on ∂X of I_n homogeneous harmonic polynomials of degree equal to 1 or 2 (see the proof for the description of the polynomials). Then $\mathfrak{M}_{\mathfrak{f}}$ from $\mathfrak{X}$ to $\mathfrak{Y}^{I_n}$ is injective. Moreover, the stability result* (21) *holds for $X_0 = X$.*

This is a global (in space) stability result from knowledge of I_n functionals but only for classes $\mathfrak{c}$ that admit an element sufficiently close to $(Id_n, 0, 0)$. In fact, the reconstruction works for $\mathfrak{c}$ close to $(a_0, 0, 0)$ for a_0 arbitrary elliptic as we observe in Theorem 4.2 below.

Proof. Let $u_1 = 1$ be the constant solution. Let then $v_j = x_j$ for $1 \leq j \leq n$ be the homogeneous polynomials of degree 1. Finally, let us denote by $v_{ij} = x_i x_j$ and $w_i(x) = \frac{1}{2}(x_i^2 - x_{i+1}^2)$ the homogeneous harmonic polynomials of degree two for $1 \leq i < j \leq n$. The other such polynomials, such as for instance $\frac{1}{2}(x_1^2 - x_3^2)$ can be constructed by linear combination of the polynomials w_i. We have thus constructed $1 + n + \frac{1}{2}n(n-1) + n - 1 = \frac{1}{2}n(n+3) = I_n$ harmonic homogeneous polynomials of degree less than or equal to 2.

We verify that

$$\nabla u_1 = 0, \quad \nabla v_i = e_i, \quad \nabla v_{ij} = x_j e_i + x_i e_j, \quad \nabla w_i = x_i e_i - x_{i+1} e_{i+1}.$$

Moreover,

$$\nabla^{\otimes 2} v_{ij} = e_i \otimes e_j + e_j \otimes e_i, \qquad \nabla^{\otimes 2} w_i = e_i \otimes e_i - e_{i+1} \otimes e_{i+1}.$$

Let us define Θ_k^{ij} for $1 \leq i < j \leq n$ and $1 \leq k \leq n$ such that

$$-\nabla v_{ij} = \Theta_k^{ij} \nabla v_k, \qquad \text{i.e.,} \qquad \Theta_k^{ij} = \begin{cases} -x_j & \text{when } k = i \\ -x_i & \text{when } k = j \\ 0 & \text{otherwise.} \end{cases}$$

Then we find that the corresponding matrices M^m are defined by

$$M^{ij} = \nabla^{\otimes 2} v_{ij} = e_i \otimes e_j + e_j \otimes e_i.$$

Let us now define Θ_k^i for $1 \leq i \leq n-1$ and $1 \leq k \leq n$ such that

$$-\nabla w_i = \Theta_k^i \nabla v_k, \qquad \text{i.e.,} \qquad \Theta_k^i = \begin{cases} -x_i & \text{when } k = i \\ x_{i+1} & \text{when } k = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then we find that the corresponding matrices M^m are defined by

$$M^i = \nabla^{\otimes 2} w_i = e_i \otimes e_i - e_{i+1} \otimes e_{i+1}.$$

The matrices M^{ij} for $1 \leq i < j \leq n$ and M^i for $1 \leq i \leq n-1$ form a free family of dimension $\frac{1}{2}n(n-1) + n - 1 = \frac{1}{2}n(n+1) - 1 = M_n$ as can easily be verified. They are orthogonal to the matrix $M^0 = Id_n$.

Let now $\mathfrak{c}$ be close to $(Id_n, 0, 0)$ in the sense given in Lemma 3.3 and let the functions u_i for $1 \leq i \leq I_n$ be solutions of the equation (1) with boundary conditions f_i that are the traces of the harmonic polynomials constructed above. (This means that $f_1 = 1$, $f_2 = x_1$, and so on on ∂X.)

By continuity of the solution to (1) stated in Lemma 3.3, the linear independence of the vectors ∇v_j for $1 \leq j \leq n$ still holds. The linear combinations θ_j^m in (13) and the matrices M^m in (15) constructed by continuity from the case $\mathfrak{c} = (Id_n, 0, 0)$ still satisfy (14) and the fact that the matrices M^m are linearly independent. This ensures the existence of a matrix M^0 close to the identity matrix such that $\alpha = \tau M^0$ for some unknown scalar quantity τ. We may then apply Lemma 3.1. This concludes the proof of the theorem. □

We presented the above result for γ in the vicinity of Id_n in order to obtain a simple proof of a construction that satisfies the hypotheses of Lemma 3.1 and because the construction also appears in a later section. In fact, the result may be generalized as follows.

Theorem 4.2 *The results of Theorem 4.1 hold for $\mathfrak{X}$ the space of $\mathfrak{c}$ sufficiently close to $(a_0, 0, 0)$, where a_0 is an arbitrary constant symmetric matrix satisfying* (2)*.*

Proof. The proof is very similar to that of the preceding theorem and is in some sense included in the proof of Theorem 4.7 below, to which we refer for the details. The construction of $u_1 = 1$ and $v_j = x_j$ is the same as that of Theorem 4.1. The solutions v_{n+m} are then constructed as

$$v_{n+m} = \frac{1}{2} Q_m x \cdot x, \qquad 1 \leq m \leq \frac{1}{2}n(n+1) - 1,$$

with Q_m forming a family of $M_n = \frac{1}{2}n(n+1) - 1$ linearly independent matrices that are orthogonal to a_0^*, or in other words, such that $a_0 : Q_{ij} = 0$. The linear combinations θ_j^m are then constructed as in Theorem 4.1 with the matrices $M^m = Q_m$ since $\nabla^{\otimes 2} v_j = 0$ for $1 \leq j \leq n$. This allows us to verify the hypotheses of Lemma 3.1 globally on $X = X_0$ for boundary conditions equal to the traces of the polynomials 1, x_j, $\frac{1}{2}Q_m x \cdot x$, and by continuity for an open set of boundary conditions and for all coefficients $\mathfrak{c}$ sufficiently close to $(a_0, 0, 0)$. □

4.2 Global reconstructions close to isotropic tensor

Let us generalize the above result by assuming that a is in the vicinity of $\gamma(x) Id_n$ where γ is a scalar real-valued (hence positive) diffusion coefficient. We still assume that b is in the vicinity of 0. Also, c is an arbitrary complex-valued potential so that (1) is uniquely solvable. Then we have the following result.

Theorem 4.3 *Let $\gamma(x) \in H^{\frac{n}{2}+4+\epsilon}(X)$ and $c(x) \in H^{\frac{n}{2}+2+\epsilon}(X)$ for $\epsilon > 0$ with $\mathfrak{S}_{\mathfrak{c}}$ in* (6) *bounded. Let $\mathfrak{X}$ be the space of $\mathfrak{c}$ sufficiently close to $(\gamma(x)Id_n, 0, c(x))$ in the sense of Lemma 3.3 with $m = 0$. Let $I_n = \frac{1}{2}n(n+3)$. There there exists an open set of $(f_i)_{1\leq i\leq I_n}$ (in any topology of sufficiently smooth functions on ∂X) such that $\mathfrak{M}_{\mathfrak{f}}$ from $\mathfrak{X}$ to $\mathfrak{Y}^{I_n}$ is injective. Moreover, the stability result* (21) *holds for $X_0 = X$.*

Proof. The proof is based on the construction of complex geometrical optics solutions of the form

$$u(x;\rho) = \frac{1}{\sqrt{\gamma(x)}} e^{\rho\cdot x}(1 + \psi_\rho(x)), \tag{26}$$

with ρ a complex-valued vector such that $\rho \cdot \rho = 0$. We know that for γ and c with the aforementioned regularity and for $|\rho|$ sufficiently large, then ψ_ρ is of order $|\rho|^{-1}$ in $C^2(\bar{X})$ [9].

In the construction preceding Lemma 3.1, we need to consider derivatives of ratios of solutions. We find that

$$\frac{u(x;\tilde{\rho})}{u(x;\rho)}\nabla\frac{u(x;\rho)}{u(x;\tilde{\rho})} = \rho - \tilde{\rho} + \varphi, \quad \frac{u(x;\tilde{\rho})}{u(x;\rho)}\nabla^{\otimes 2}\frac{u(x;\rho)}{u(x;\tilde{\rho})} = (\rho - \tilde{\rho})^{\otimes 2} + \phi, \tag{27}$$

with the vector φ bounded independent of $(\rho, \tilde{\rho})$ and matrix ϕ of order $\max(|\rho|, |\tilde{\rho}|)$ uniformly in $x \in X$.

Let us define $\rho_{ij} = \mathfrak{k}(e_i + ie_j)$ and define $u_{ij} = u(\cdot;\rho_{ij})$ as well as $\tilde{u}_{ij} = u(\cdot;\rho^*_{ij})$. Note that $\tilde{u}_{ij}$ is asymptotically close to u^*_{ij} as $|\mathfrak{k}| \to \infty$ but since $c(x)$ may be complex valued, is not necessarily equal to u^*_{ij}.

We also define $\tilde{\rho}_{ij} = \varepsilon\rho_{ij}$ as well as $\rho_1 = \varepsilon^2\rho_{12}$ and $u_1 = u(\cdot;\rho_1)$, with $\varepsilon^2\mathfrak{k}$ sufficiently large that contributions such as φ and ϕ above remain negligible for the forthcoming constructions but ε sufficiently small that $\tilde{\rho} = \rho_1$ or $\tilde{\rho} = \tilde{\rho}_{ij}$ in (27) is so small that it does not modify the independence of the matrices M^m constructed below.

Let us define v_j as follows

$$v_1 = \frac{u(\cdot;\tilde{\rho}^*_{12})}{u_1}, \quad v_j = \frac{u(\cdot;\tilde{\rho}_{j-1,j})}{u_1}, \quad j \geq 2.$$

Since all solutions v_j do not vanish for $\mathfrak{k}$ sufficiently large, it is clear that $(\nabla v_1, \ldots, \nabla v_n)$ form a basis with $H_{ij} = \nabla v_i \cdot \nabla v_j$ a matrix with a uniformly bounded inverse for $x \in X$ (with a bound that depends on $\mathfrak{k}$ and ε). Moreover, we find that

$$\frac{1}{v_1}\nabla v_1 \sim \varepsilon\rho^*_{12}, \ \frac{1}{v_1}\nabla^{\otimes 2}v_1 \sim \varepsilon^2(\rho^*_{12})^{\otimes 2}, \quad \frac{1}{v_j}\nabla v_j \sim \varepsilon\rho_{j-1,j}, \ \frac{1}{v_j}\nabla^{\otimes 2}v_j \sim \varepsilon^2\rho^{\otimes 2}_{j-1,j}, \ j \geq 2.$$

Here and below, we denote by $\sim$ equalities up to terms such as φ and ϕ above that are asymptotically negligible as $|\mathfrak{k}| \to \infty$ as well as terms that are lower order in ε.

Now for $1 \leq i < j \leq n$, we define

$$v_{ij} = \frac{u_{ij}}{u_1}, \ \text{so that} \ \frac{1}{v_{ij}}\nabla v_{ij} \sim \rho_{ij}, \quad \frac{1}{v_{ij}}\nabla^{\otimes 2}v_{ij} \sim \rho^{\otimes 2}_{ij}.$$

For $1 \leq j \leq n-1$, we construct

$$\tilde{v}_j = \frac{\tilde{u}_{j,j+1}}{u_1}, \ \text{so that} \ \frac{1}{\tilde{v}_j}\nabla\tilde{v}_j \sim \rho^*_{j,j+1}, \quad \frac{1}{\tilde{v}_j}\nabla^{\otimes 2}\tilde{v}_j \sim (\rho^*_{j,j+1})^{\otimes 2}.$$

Each of the vectors ∇v_{ij} and $\nabla \tilde{v}_j$ can uniquely be written in terms of the vectors ∇v_j. Let us define $\hat{\rho}_1 = \rho_{12}^*$ and $\hat{\rho}_j = \rho_{j-1,j}$. Note that $(\hat{\rho}_j)_{1\le j\le n}$ form a basis of $\mathbb{C}^n$. Let us then introduce

$$-\rho_{ij} = \varepsilon \tilde{\Theta}_k^{ij} \hat{\rho}_k, \qquad -\rho_{j,j+1}^* = \varepsilon \tilde{\Theta}_k^{j} \hat{\rho}_k.$$

Here, the summation is over the index k. We find that all coefficients $\tilde{\Theta}$ are of order ε^{-1}. Then we find that

$$-\nabla v_{ij} = \Theta_k^{ij} \nabla v_k, \ -\nabla \tilde{v}_j = \Theta_k^j \nabla v_k \quad \text{for} \quad \Theta_k^{ij} \sim \frac{v_{ij}}{v_k} \tilde{\Theta}_k^{ij}, \qquad \Theta_k^j \sim \frac{\tilde{v}_j}{v_k} \tilde{\Theta}_k^j.$$

Now for these choices, we find that

$$M^{ij} := \nabla^{\otimes 2} v_{ij} + \Theta_k^{ij} \nabla^{\otimes 2} v_k = v_{ij} \Big(\rho_{ij}^{\otimes 2} - \tilde{\Theta}_k^{ij} \varepsilon^2 \hat{\rho}_k^{\otimes 2} \Big) \sim v_{ij} \rho_{ij}^{\otimes 2}.$$

Similarly, we have

$$M^j := \nabla^{\otimes 2} \tilde{v}_j + \Theta_k^j \nabla^{\otimes 2} v_k = \tilde{v}_j \Big((\rho_{j,j+1}^*)^{\otimes 2} - \tilde{\Theta}_k^j \varepsilon^2 \hat{\rho}_k^{\otimes 2} \Big) \sim \tilde{v}_j (\rho_{j,j+1}^*)^{\otimes 2}.$$

Note that

$$\frac{\rho_{ij}^{\otimes 2}}{|\mathfrak{k}|^2} = i(e_i \otimes e_j + e_j \otimes e_i) + (e_i \otimes e_i - e_j \otimes e_j), \quad \frac{(\rho_{ij}^*)^{\otimes 2}}{|\mathfrak{k}|^2} = -i(e_i \otimes e_j + e_j \otimes e_i) + (e_i \otimes e_i - e_j \otimes e_j).$$

Therefore the matrices M^{ij} and M^j constructed above are indeed linearly independent and as in the proof of Theorem 4.1 span a subspace of the vector space of symmetric matrices of dimension $\frac{1}{2}n(n-1)+n-1 = \frac{1}{2}n(n+1)-1$. The above result obtained in the limit $\varepsilon \to 0$ still holds for ε sufficiently small. Moreover, once ε is fixed, several terms of the form ϕ and φ above become negligible when $|\mathfrak{k}|$ is sufficiently large. Therefore, for $|\mathfrak{k}|$ sufficiently large, there exists an open set of boundary conditions (f_i) such that all the hypotheses of Lemma 3.1 are satisfied for all $x \in X$.

This proves the result for $\tilde{\mathfrak{c}}$ of the form $(\gamma(x) Id_n, 0, c(x))$. Now by continuity and Lemma 3.3, the same boundary conditions can be used to satisfy the requirements of Lemma 3.1 for all $\mathfrak{c}$ sufficiently close to $\tilde{\mathfrak{c}}$. □

4.3 Global reconstructions in two dimensions

The above reconstruction procedure has been proved to hold in the vicinity of $(Id_n, 0, 0)$ or $(\gamma(x) Id_n, 0, c(x))$. In this section, we generalize the result to proving that global reconstructions are possible for coefficients in the vicinity of $(\gamma(x), 0, c(x))$ where γ is an arbitrary real-valued second-order elliptic tensor in dimension $n = 2$ and $c(x)$ is a complex-valued potential:

Theorem 4.4 *Let $\mathfrak{X}$ be the space of $\mathfrak{c}$ sufficiently close to $(\gamma, 0, c)$ in the sense of Lemma 3.3 with $m = 0$ with γ of class $H^{5+\epsilon}(\bar{X})$ and c of class $H^{3+\epsilon}(\bar{X})$ for $\epsilon > 0$. Let $I_2 = 5$. Then there exists an open set of boundary conditions $\mathfrak{f} = (f_i)_i$ such that $\mathfrak{M}_{\mathfrak{f}}$ from $\mathfrak{X}$ to $\mathfrak{Y}^{I_2}$ is injective. Moreover, the stability result* (21) *holds for $X_0 = X$.*

Proof. To simplify the notation, we set $c \equiv 0$ and leave the details to the reader to consider the case $c \neq 0$ as was done in the proof of Theorem 4.3.

We prove that we can apply Lemma 3.1 for $X_0 = X$ and $(a, b, c) = (\gamma, 0, 0)$ for an open set of boundary conditions $\mathfrak{f}$. Then by continuity, the hypotheses of Lemma 3.1 still hold for $\mathfrak{c}$ sufficiently close to $(\gamma, 0, 0)$. As we did in the proof of the preceding theorem, we can assume that γ is smooth since by an application of Lemma 3.3, the result can be extended to any γ satisfying the regularity hypotheses of Lemma 3.3 with $m = 0$.

The global reconstruction applies only in two dimensions of space and for real valued tensors γ. The reason is that global complex geometrical optics solutions can be constructed in two dimensions by means of an appropriate quasiconformal map, which we now describe. Such results do not hold in general in dimension $n \geq 3$.

We first extend γ to a smooth real-valued tensor on $\mathbb{R}^2 \sim \mathbb{C}$, which remains uniformly positive definite and equal to I outside of a compact domain. For $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ a diffeomorphism of the plane with differential $D\varphi$, we denote by

$$\varphi_* \gamma(y) = \frac{D\varphi(x)\gamma(x)(D\varphi)^t(x)}{|\det(D\varphi)|}\Big|_{x=\varphi^{-1}(y)},$$

the push-forward of γ by the diffeomorphism φ. A result of quasi-conformal theory [4] implies the existence of a unique such diffeomorphism φ normalized such that $\varphi(z) = z + O(\frac{1}{z})$ as $|z| \to \infty$ and such that

$$\varphi_* \gamma(x) = \Big(\det\gamma \circ \varphi^{-1}(x)\Big)^{\frac{1}{2}} Id_2. \tag{28}$$

In other words, the tensor-valued conductivity γ is conformal to the Euclidean conductivity Id_2, or equivalently the Riemannian metric $g = (\det\gamma)^{\frac{1}{2}}\gamma^{-1}$ is conformal to the Euclidean metric. As we did in the proof of Theorem 4.3, we may thus construct CGO solutions of the form

$$v(y) = \frac{1}{\sqrt{\varphi_* \gamma(y)}} e^{\rho \cdot y}(1 + \phi_\rho(y)),$$

with $|\rho|\phi_\rho$ bounded in the $C^2(\varphi(X))$ sense. As a consequence, we construct

$$u_\rho(x) = \frac{1}{\sqrt{\varphi_* \gamma \circ \varphi(x)}} e^{\rho \cdot \varphi(x)}(1 + \psi_\rho(x)), \tag{29}$$

with $|\rho|\psi_\rho$ bounded in $C^2(X)$.

Let $\rho_1 = k(ie_1 + e_2)$ and $\rho_2 = k(ie_2 - e_1)$. Let $u_j = u_{\rho_j}$, $j = 1, 2$. As in the preceding section, we also define $\tilde{u}_j = u_{\rho_j^*}$ for $j = 1, 2$. Then we find

$$k^{-1}\nabla u_1 = (i\nabla\varphi_1 + \nabla\varphi_2)u_1 + \zeta_1, \qquad k^{-1}\nabla u_2 = i(i\nabla\varphi_1 + \nabla\varphi_2)u_2 + \zeta_2,$$

with $|\rho|u_j^{-1}\zeta_j$ bounded for $j = 1, 2$. We thus find

$$k^{-1}u_2^{-1}\nabla u_2 = ik^{-1}u_1^{-1}\nabla u_1 + o(1) = -\nabla\varphi_1 + i\nabla\varphi_2 + o(1).$$

Here, we are decomposing ∇u_2 over ∇u_1 and $\nabla\tilde{u}_1$, which form a basis for k sufficiently large. Then

$$\begin{aligned} M &= ik^{-2}u_1^{-1}\nabla \otimes \nabla u_1 - k^{-2}u_2^{-1}\nabla \otimes \nabla u_2 \\ &\sim (i+1)\big(\nabla\varphi_2^{\otimes 2} - \nabla\varphi_1^{\otimes 2}\big) + (i-1)\big(\nabla\varphi_1 \otimes \nabla\varphi_2 + \nabla\varphi_2 \otimes \nabla\varphi_1\big), \end{aligned}$$

in the limit $k \to \infty$. In the same way that we have decomposed ∇u_2 over ∇u_1 and $\nabla \tilde{u}_1$ above, we can decompose $\nabla \tilde{u}_2$ over ∇u_1 and $\nabla \tilde{u}_1$ as well. In the limit $k \to \infty$, the matrix M^* will thus be given by the complex conjugation of the above matrix. This proves that in the limit $k \to \infty$, the matrices M^m defined in section 3 are given by the real and imaginary parts of M:

$$M_\pm = \big(\nabla\varphi_2^{\otimes 2} - \nabla\varphi_1^{\otimes 2}\big) \pm \big(\nabla\varphi_1 \otimes \nabla\varphi_2 + \nabla\varphi_2 \otimes \nabla\varphi_1\big).$$

After change of coordinates, we obtain the two matrices:

$$M_1 = \nabla\varphi_2^{\otimes 2} - \nabla\varphi_1^{\otimes 2}, \qquad M_2 = \nabla\varphi_1 \otimes \nabla\varphi_2 + \nabla\varphi_2 \otimes \nabla\varphi_1$$

which we want to be non trivial and linearly independent. The above matrices $M_{1,2}$ are those obtained in the limit $k \to \infty$. This means that for k sufficiently large, the two constructed matrices $M_{1,2}$ from u_1 and u_2 will be close to their limits and hence satisfy the same properties of linear independence.

Now we observe that $a = \nabla\varphi_1$ and $b = \nabla\varphi_2$ are linearly independent since φ is a diffeomorphism. And $a \otimes a$, $b \otimes b$, $a \otimes b + b \otimes a$ are basis elements for symmetric matrices. Thus M_1 has coordinates $(-1, 1, 0)$ while M_2 has coordinates $(0, 0, 1)$ in that basis. As a consequence, both matrices M_1 and M_2 are linearly independent, in the limit $k \to \infty$ as well as for k sufficiently large. Note that the independence is uniform in $x \in X$ for k sufficiently large.

This shows that the hypotheses of Lemma 3.1 are satisfied for $X_0 = X$. Such a calculation holds for any set of coefficients close to $(\gamma, 0, 0)$. A very similar proof applies to $\mathfrak{c}$ in the vicinity of $(\gamma, 0, c)$ as stated in the theorem. This proves the theorem. □

4.4 Global reconstructions with redundant measurements

In this section, we show that reconstructions are possible for essentially arbitrary (sufficiently smooth) coefficients $\mathfrak{c}$. However, the proof of linear independence of the matrices M^m becomes (spatially) local. We thus need to use a number of internal functionals I that is potentially much larger than I_n, although we do not expect this large number of coefficients to be necessary in practical inversions.

As for the preceding results, the conditions of linear independence will be satisfied for well-chosen illuminations f_j on the boundary ∂X. The control of the linear independence from the boundary is obtained by means of a Runge approximation; see Lemma 4.8 below. This step requires that the operator $L = \nabla \cdot a\nabla + b \cdot \nabla + c$ satisfy a unique continuation principle, which we state as follows:

Property 4.5 (Unique Continuation) *We say that L satisfies the unique continuation principle when $Lu = 0$ on $X \backslash X_0$ with $u = 0$ on ∂X and $n \cdot \nabla u = 0$ on ∂X implies that $u = 0$ on $X \backslash X_0$, where X_0 is an arbitrary sufficiently smooth open domain $X_0 \subset\subset X$.*

For unique continuation results, we refer the reader to [11, 23, 24, 30] and the theoretical results we shall use here [15, Theorem 17.2.1]. The latter result states that L satisfies the unique continuation principle 4.5 when the principal symbol of L given by $p(x, \xi) = a(x)\xi \cdot \xi$ is such that:

(i) $a(x)$ is Lipschitz continuous,

(ii) For $\xi, N \in \mathbb{R}^n \backslash \{0\}$, the quadratic equation $p(x, \xi + \tau N) = 0$ in the variable $\tau \in \mathbb{C}$ admits a double root τ if and only if $\xi + \tau N = 0$.

Then we have the following lemma:

Lemma 4.6 *Let $p(x, \xi) = a(x)\xi \cdot \xi$ be the principal symbol of L, which we assume is elliptic.*
In dimension $n \geq 3$, the quadratic equation $p(x, \xi + \tau N) = 0$ for $\xi, N \in \mathbb{R}^n \backslash \{0\}$ never admits a double root τ unless $\xi + \tau N = 0$.
In dimension $n = 2$, the same result holds when in addition (2) is satisfied.
In all these cases, L thus satisfies Property 4.5 when a is Lipschitz continuous.

Proof. The proof is essentially given in [15, Lemma 17.2.5]. In dimension $n \geq 3$, the equation $p(x, \xi + \tau N)$ has one root with $\Im \tau > 0$ and one root with $\Re \tau > 0$. In dimension $n = 2$, the equation $p(x, \xi + \tau N)$ has a double root at a fixed point $x \in X$ if and only if we have $p(x, \xi) = (l(x) \cdot \xi)^2$ for some complex-valued vector $l = l_r + i l_i \in \mathbb{C}^n$ (with $l_r = \Re l$ and $l_i = \Im l$) to preserve ellipticity (note that such quadratic forms cannot be elliptic in dimension $n \geq 3$). But then $p(x, \xi) = (l_r \cdot \xi)^2 - (l_i \cdot \xi)^2 + i l_r \cdot \xi l_i \cdot \xi$ so that the real part of a is not elliptic. This proves the lemma. □

With this result, we can now state the main theorem of the paper.

Theorem 4.7 *Let $\mathfrak{X}$ be the space of coefficients $\mathfrak{c}$ such that $(b, c, \nabla \cdot a)$ are of class $C^{0,\alpha}(X)$, a is of class $C^{0,1}(X)$, and such that (2) holds. Then there exists $I \geq I_n$ and an open set (for the topology of $C^{2,\alpha}(\partial X)$) of boundary conditions $\mathfrak{f} = (f_i)_{1 \leq i \leq I}$ such that $\mathfrak{M}_{\mathfrak{f}}$ from $\mathfrak{X}$ to $\mathfrak{Y}^I$ is injective. Moreover, the stability result (21) holds for $X_0 = X$.*

Proof. We decompose the proof into three steps: we first construct local solutions assuming that the coefficients are constant. We then extend the local constructions to the case of non-constant coefficients. We finally apply the Runge approximation to obtain an open set of boundary conditions such that the hypotheses of Lemma 3.1 are satisfied locally. Local constructions are then patched together to provide global stable and unique reconstructions.

Problem with constant coefficients. Let first x_0 be a point inside X, which by change of coordinates we call 0. Let us define $(a_0, b_0, c_0) = (a(0), b(0) + \nabla \cdot a(0), c(0))$. We then look for solutions of the constant coefficient equation

$$L_0 \mathfrak{u} := a_0 : \nabla^{\otimes 2} \mathfrak{u} + b_0 \cdot \nabla \mathfrak{u} + c_0 \mathfrak{u} = 0. \tag{30}$$

We look for solutions approximately of the form

$$\mathfrak{p} = \frac{1}{2} Q x \cdot x + \rho \cdot x + d, \qquad \nabla \mathfrak{p} = Q x + \rho, \qquad \nabla^{\otimes 2} \mathfrak{p} = Q.$$

In order for $\mathfrak{p}$ to satisfy the equation at $x = 0$, we need to find (Q, ρ, d) such that

$$a_0 : Q + b_0 \cdot \rho + c_0 d = 0.$$

We construct $I_n = 1 + n + \frac{1}{2}n(n+1) - 1$ such solutions below. We then realize that $L_0\mathfrak{p} = O(x)$. Let r_0 be sufficiently small and let us define

$$L_0\mathfrak{u} = 0 \quad \text{in} \quad B(0, r_0), \qquad \mathfrak{u} = \mathfrak{p} \quad \text{on } \partial B(0, r_0). \tag{31}$$

For r_0 sufficiently small, the derivatives up to order two of $\mathfrak{u}$ and $\mathfrak{p}$ are very close. The linear independence of the structures constructed below with the polynomials $\mathfrak{p}$ at $x = 0$ therefore still holds for the corresponding structures constructed with the elliptic solutions $\mathfrak{u}$ in (31).

We call the first solution $\mathfrak{u}_0$ obtained by defining

$$d = 1, \qquad \rho = 0, \qquad Q = -\frac{c_0 a_0^*}{a_0 : a_0^*}.$$

Note that $\mathfrak{u}_0$ does not vanish in a sufficiently small neighborhood of 0 (and can be normalized so that $\mathfrak{u}_0(0) = 1$). We next define the solutions $\mathfrak{u}_j$ for $1 \le j \le n$. The vector $b_0 = b_{0r} + ib_{0i}$ is after a rotation if necessary in the span of e_1 and e_2. We thus write $b_0 = \mu e_1 + \nu e_2$ for μ and ν in $\mathbb{C}$. For $j = 1, 2$, we define

$$d_j = 0, \qquad \rho_j = e_j, \qquad Q_1 = -\frac{\mu a_0^*}{a_0 : a_0^*}, \qquad Q_2 = -\frac{\nu a_0^*}{a_0 : a_0^*}.$$

For $j \ge 3$, we define

$$d = 0, \qquad \rho = e_j, \qquad Q = 0.$$

The solutions $\mathfrak{u}_j$ are therefore constructed such that $\nabla\mathfrak{p}_j = e_j$ at $x = 0$. Moreover, we find that $\nabla\frac{\mathfrak{p}_j}{\mathfrak{p}_0} = e_j$ at $x = 0$ as well since $\nabla\mathfrak{p}_0 = 0$ at $x = 0$. We thus obtain that $\nabla\frac{\mathfrak{u}_j}{\mathfrak{u}_0}$ form a basis of $\mathbb{R}^n$ in a sufficiently small neighborhood of 0.

Finally, for $n+1 \le j \le n + \frac{1}{2}n(n+1) - 1$, we define $d = 0$ and $\rho = 0$ and choose the matrices Q_j such that they form a free family of symmetric matrices that are orthogonal to a_0^*, the complex conjugate of a_0. This free family has dimension $\frac{1}{2}n(n+1) - 1$. This implies that $(a_0 : Q_j) = 0$.

In the construction preceding Lemma 3.1, it is $\nabla^{\otimes 2} v_{j+m}$ for $v_j = \mathfrak{u}_0^{-1}\mathfrak{u}_j$ that is used to form a free family of symmetric matrices of dimension $\frac{1}{2}n(n+1) - 1$. We verify that

$$\nabla^{\otimes 2}\frac{\mathfrak{u}_{j+m}}{\mathfrak{u}_0} = \frac{1}{\mathfrak{u}_0}\nabla^{\otimes 2}\mathfrak{u}_{j+m} + \nabla\mathfrak{u}_{j+m} \otimes \nabla\frac{1}{\mathfrak{u}_0} + \nabla\frac{1}{\mathfrak{u}_0} \otimes \nabla\mathfrak{u}_{j+m} + \mathfrak{u}_{j+m}\nabla^{\otimes 2}\frac{1}{\mathfrak{u}_0}.$$

We verify that both sides equal Q_j at $x = 0$ when $\mathfrak{u}_j$ is replaced by $\mathfrak{p}_j$.

Let now θ_j^m and M^m be defined as in the construction preceding Lemma 3.1. We verify that M^m is close to Q_{j+m} at $x = 0$. By continuity, the matrices M^m are therefore linearly independent in a ball $X_0 = B(0, r_0)$ for $r_0 > 0$ sufficiently small. This shows that on that ball, the matrices M^m satisfy the hypotheses of Lemma 3.1. All other hypotheses of that Lemma are therefore satisfied for the family $\mathfrak{u}_j$.

Problem with non-constant coefficients. We now return to the original problem on X and look for solutions of the form:

$$a(x) : \nabla^{\otimes 2}u + (b + \nabla \cdot a)(x) \cdot \nabla u + c(x)u = 0 \quad \text{in } X_0, \qquad u = \mathfrak{u} \quad \text{on } \partial X_0, \tag{32}$$

where $X_0 = B(0, r_0) \cap X$ with $B(0, r_0)$ the ball whose radius r_0 is equal to or smaller than the value chosen in the construction of $\mathfrak{u}$. Let $w = \mathfrak{u} - u$. We find

$$a : \nabla^{\otimes 2} w + (b + \nabla \cdot a) \cdot \nabla w + cw = (a - a_0) : \nabla^{\otimes 2} \mathfrak{u} + (b + \nabla \cdot a - b_0) \cdot \nabla \mathfrak{u} + (c - c_0)\mathfrak{u}$$

on X_0 with $w = 0$ on ∂X_0. By assumption on the coefficients and $\mathfrak{u}$, the above right-hand side is bounded in $C^{0,\beta}(X_0)$ by $C_\beta r_0^{\alpha-\beta}$ for some $0 < \beta < \alpha$. We deduce from elliptic regularity results for complex-valued coefficients [20, Chapter 6] that

$$\|w\|_{C^{2,\beta}(X_0)} \leq C r_0^{\alpha-\beta},$$

for some positive constant C. Thus the functions u_j defined as solutions for (32) with boundary conditions $\mathfrak{u}_j$ are arbitrarily close to $\mathfrak{u}_j$ in the C^2 sense for r_0 sufficiently small. This proves that u_0 remains non-vanishing and close to 1 on X_0, that $\nabla \frac{u_j}{u_0}$ for $1 \leq j \leq n$ remain linearly independent, and that the matrices M^m constructed above Lemma 3.1 satisfy the independence properties stated in that lemma.

Continuation to the boundary. So far, we have constructed solutions u that are defined on $X_{0,k} = B(x_k, r_0) \cap X$ for $x_k \in X$ with $B(x_k, r_0)$ the open ball centered at x_k with radius r_0. We find a sequence of points x_k for $1 \leq k \leq K$ such that $X \subset \cup_{k=1}^K X_{0,k}$, and such that either $\overline{B(x_k, r_0)} \subset X$ or $x_k \notin X$. In other words, we cover X with balls either strictly included in X or such that their center lies outside X. Such a covering is possible for X bounded and smooth ∂X.

We also define $X_{\mu,k} = B(x_k, \mu r_0) \cap X$ and find $0 < \mu < 1$ sufficiently close to 1 such that $X \subset \cup_{k=1}^K X_{\mu,k}$. We need to construct solutions on the whole domain X such that their restrictions on $X_{\mu,k}$ is a sufficiently accurate approximation of u constructed in (32). We need the following Runge approximation property, following [22]; see also [17].

Lemma 4.8 (Runge approximation) *Let L be an operator satisfying the unique continuation property of Cauchy data on X as described above.*

Let u_0 be a solution of $Lu_0 = 0$ on $X_{0,k}$ with $1 \leq k \leq K$ fixed. Then for each $\varepsilon > 0$, there is a function $f_\varepsilon \in H^{\frac{1}{2}}(\partial X)$ such that the solution of $Lu_\varepsilon = 0$ on X with $u_\varepsilon = f_\varepsilon$ on ∂X is such that

$$\|u_\varepsilon - u_0\|_{C^{2,\alpha}(X_{\mu,k})} \leq \varepsilon. \tag{33}$$

Proof. [Runge Lemma]. We drop the dependency in k and denote $X_0 = X_{0,k}$. Let $E = \{u \in H^1(X_0),\, Lu = 0 \text{ in } X_0,\, u = 0 \text{ on } \partial X \cap \partial X_0\}$ and $F = \{u_{|X_0},\, u \in H^1(X),\, Lu = 0 \text{ in } X,\, u = 0 \text{ on } \partial X \cap \partial X_0\}$ be linear subspaces of $L^2(X_0)$. Note that $\partial X \cap \partial X_0 = \emptyset$ when X_0 is a ball. We wish to prove that $\bar{F} = E$ for the strong L^2 topology. By Hahn Banach, this means that for all $\phi \in L^2(X_0)$, then $(\phi, u) = 0$ for all $u \in F$ implies that $(\phi, u) = 0$ for all $u \in E$.

Let us extend ϕ by 0 outside X_0 and still call ϕ the extension on X. Define then

$$L^* v = \phi \quad \text{in} \quad X, \qquad v = 0 \quad \text{on} \quad \partial X.$$

Here $L^* = \nabla \cdot a^* \nabla - b^* \cdot \nabla + c^*$ is the formal adjoint to L. Note that v is well-defined since $\mathfrak{S}_\mathfrak{c}$ in (6) is assumed to be bounded. Integrations by parts show that

$$0 = (Lu, v) - (u, L^* v) = \int_{\partial X} (an \cdot \nabla u v^* - an \cdot \nabla v^* u - b \cdot n u v^*) d\sigma = \int_{\partial X} an \cdot \nabla v^* u d\sigma.$$

Since this holds for any function $u \in H^1(X)$ and hence for any $u_{|\partial X} \in H^{\frac{1}{2}}(\partial X)$, we deduce that $a^* n \cdot \nabla v = 0$ on ∂X. We thus find

$$L^* v = 0 \quad \text{in} \quad X_1, \qquad v = 0 \text{ and } a^* n \cdot \nabla v = 0 \quad \text{on} \quad \partial X_1,$$

where X_1 is the interior of $X \backslash X_0$. We use the unique continuation assumption to deduce that $v \equiv 0$ in $H^1(X_1)$ so that $v = 0$ in the $H^{\frac{1}{2}}(\partial X_1)$-sense and $a^* n \cdot \nabla v = 0$ in the $H^{-\frac{1}{2}}(\partial X_1)$-sense. For any $u \in E$, we thus find that $(\phi, u) = 0$, which thus proves that $\bar{F} = E$. This shows that $u - u_0$ is arbitrarily small in $L^2(X_0)$.

Let us first assume that X_0 is a ball, so that then, $\overline{X_0} \subset X$. Then regularity results as they are written for instance in [20, Theorem 6.2.5] for elliptic problems with complex coefficients such that $(a, b, c, \nabla \cdot a)$ are of class $C^{0,\alpha}$ for $\alpha > 0$ (see also [15, Theorem 17.2.7]), allow us to conclude that (33) holds. Indeed, we have an equation $L(u_0 - u_\varepsilon) = 0$ on X_0. We then get the required interior regularity of $u_0 - u_\varepsilon$ in $C^{2,\alpha}(X_\mu)$ for $\mu < 1$.

Let X_0 be the intersection of a ball and X (which corresponds to points $x_k \notin X$). We separate $\partial X_0 = (\partial X_0 \cap \partial X) \cup \Gamma$, with Γ a (closed) spherical cap whose interior lies in X. Define $f = u_{0|\partial X_0 \cap \partial X}$ on $\partial X_0 \cap \partial X$ extended by 0 on the rest of ∂X and u_1 the solution of

$$L u_1 = 0 \quad \text{in } X, \qquad u_1 = f \quad \text{on } \partial X.$$

Define now $u_2 = u_0 - u_{1|X_0}$, which belongs to E. We find a sequence of solutions $u_{2,\varepsilon}$ approximating u_2 in $L^2(X_0)$. Now $u_\varepsilon = u_{2,\varepsilon} + u_1$ has the proper boundary conditions on $\partial X_0 \cap \partial X$ and approximates u in $L^2(X_0)$. The same type of regularity results as above with boundary conditions allow us to conclude that (33) holds. □

We now conclude the proof of Theorem 4.7. Uniqueness to the Cauchy problem is guaranteed by Lemma 4.6. We have obtained, using the Runge approximation, the construction of a family u_j for an open set of boundary conditions f_j such that the hypotheses of Lemma 3.1 are satisfied on $X_{\mu,k}$ for each $1 \leq k \leq K$. Since the open domains $X_{\mu,k}$ cover X, we can then apply Lemma 3.1 and obtain a unique and stable reconstruction of $\mathfrak{c}$ on X. □

Remark 4.9 The reconstruction of $\mathfrak{c}$ using the procedure presented in section 3.1 is *global*: provided that at every point $x_0 \in X$, we can find solutions $(u_j)_{1 \leq j \leq I}$ such that (after permutation of the indices if necessary) (i) $u_1 \neq 0$; (ii) $(\nabla v_j)_{1 \leq j \leq n}$ form a basis of $\mathbb{R}^n$ for v_j defined in (10); and (iii) M^m defined in (15) form a free family of symmetric matrices, then $\mathfrak{c}$ is *globally* uniquely reconstructed, i.e., $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) = \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})$ for any $\tilde{c}$ such that (1) is well-posed, implies that $\mathfrak{c} = \tilde{\mathfrak{c}}$. Moreover, the reconstruction is stable as indicated in Theorem 2.1.

Verifying (i)-(ii)-(iii) requires the control of gradients and Hessians of solutions to second-order elliptic equations, which in this paper is performed by using (locally defined) harmonic polynomials or (globally defined) complex geometric optics solutions when they are available. Such a control is guaranteed for an open set $\mathcal{F}$ of boundary conditions $\mathfrak{f} \in \mathcal{F}$ and I sufficiently large. Since $\mathcal{F}$ is open, we deduce for instance that for $(\varphi_n)_{n \geq 1}$ a basis of smooth functions in $L^2(\partial X)$, and for $f_{jk;N} = \frac{j}{N} \varphi_k$, we are guaranteed that $\mathfrak{f} = (f_{jk,N})_{-N \leq j \leq N, k \leq N} \in \mathcal{F}$ for N sufficiently large, and hence that $\mathfrak{c}$ will be uniquely and stably reconstructed.

However, conditions (i)-(ii)-(iii) are likely to hold for a very large class of boundary conditions $\mathfrak{f} = (f_1, \ldots f_I)$ for $I \geq I_n$. In numerical simulations in the setting of scalar

(isotropic) coefficients, it seems that good reconstructions are obtained for I (or N above) chosen quite small [8, 7]. Note that the reconstruction of anisotropic coefficients in a slightly different context in [19] is less stable than the reconstruction of scalar coefficients. This is consistent with the theoretical predictions in this paper; see (23) compared to (22).

The reconstruction of $\mathfrak{c}$ from $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$ requires that available data be differentiated twice even when (i)-(ii)-(iii) are satisfied. This differentiation is inherent to the hybrid inverse problem considered in this paper and cannot be avoided. This will undoubtedly limit the resolution on the reconstructed coefficients we can expect in the presence of significant noise in the acquisition of $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$. Nonetheless, the stability estimates obtained in Theorem 2.1 are much more favorable than estimates obtained for reconstructions from boundary measurements; see [28].

5 Applications to coupled-physics inverse problems

The salient feature of coupled-physics inverse problems (also known as hybrid inverse problems) is that they involve a high resolution modality and a high contrast modality to obtain a coupled (hybrid) modality imaging combining both high contrast with high resolution. We consider three such families of coupled physics inverse problems that may be modeled by the theory developed in the preceding sections; quantitative photo-acoustic tomography (QPAT), transient elastography (TE), and the mathematically similar modality called magnetic resonance elastography (MRE); see [6] for a review on hybrid inverse problems.

5.1 Quantitative Photo-Acoustic Tomography

The first modality we consider is called quantitative photo-acoustic tomography (QPAT). The high contrast modality is optical tomography. The ultimate objective of QPAT is the reconstruction of the optical coefficients in an elliptic equation.

Radiation propagation is modeled by the following equation

$$-\nabla \cdot \gamma \nabla u_j + \sigma u_j = 0 \quad \text{in } X, \qquad u_j = f_j \quad \text{on } \partial X. \tag{34}$$

Here, γ is the real-valued diffusion tensor and σ the real-valued absorption coefficient.

The high resolution modality is ultrasound. A first well posed inverse wave (ultrasound) problem is solved to reconstruct internal functionals of the unknown coefficients. This first step of QPAT provides access to the following internal functionals [8, 7, 9, 27]

$$H_j(x) = \Gamma(x)\sigma(x)u_j(x) \quad \text{in } X. \tag{35}$$

Here $\Gamma(x)$ is the Grüneisen coefficient, which is assumed to be known in this paper and, therefore, without loss of generality assumed to equal 1. We assume that $1 \leq j \leq I$, with I the number considered in the preceding sections. We assume that all coefficients are known on ∂X and that $f_1 > 0$ on ∂X so that $u_1 > 0$ by the maximum principle. Then multiplying the above equation for u_1 by u_j and for u_j by u_1 and subtracting the results, we get

$$-\nabla \cdot (\gamma u_1^2)\nabla \frac{u_j}{u_1} = -\nabla \cdot (\gamma u_1^2)\nabla \frac{H_j}{H_1} = 0.$$

Therefore if the $I-1$ boundary conditions $(f_2, \ldots, f_I)$ are chosen as in the preceding section, we obtain that γu_1^2 can be uniquely and stably reconstructed. Indeed, we are here in the setting where $b = 0$, which allows one to reconstruct the gauge and hence the whole diffusion tensor γ as indicated in Corollary 2.2.

Now the equation for u_1 may be recast as

$$-\nabla \cdot (\gamma u_1^2) \nabla \frac{1}{u_1} = H_1 \quad \text{in } X, \qquad \frac{1}{u_1} = \frac{1}{f_1} \quad \text{on } \partial X. \tag{36}$$

This uniquely determines u_1 and hence γ_1 in a stable fashion. Since $H_1 = \sigma u_1$, this also determines σ uniquely and stably. This concludes the derivation of the unique and stable reconstruction of (γ, σ) from QPAT measurements when the Grüneisen coefficient is known.

Note that the same elliptic equation (34) with $\sigma = 0$ has been used to reconstruct a scalar diffusion coefficient from knowledge of u by solving the transport equation (34) for γ with applications in underground water flows [1, 25].

5.2 Coupled-physics methods based on Elastography

In this section, the high contrast modality is elastography; see [18] and reference there for more details. The elastic (stiffness) properties of tissues are to be reconstructed. We assume here that the elastic displacements are modeled by a time-harmonic scalar equation of the form

$$\nabla \cdot \gamma(x) \nabla u_j + \omega^2 \rho(x) u_j = 0 \quad \text{in } X, \qquad u_j = f_j \quad \text{on } \partial X. \tag{37}$$

Here, γ is a tensor-valued, possibly complex-valued, Lamé parameter and ρ is a density that may also be complex-valued in full generality. The reason for these coefficients to be complex-valued is that elastic waves are attenuated by various dispersion effects. In the frequency domain, such attenuation effects take the form of complex-valued coefficients. Elastographic tomography was one of the main motivations to consider the reconstruction of complex-valued coefficients in the preceding section.

In transient elastography (TE), the high resolution modality is again ultrasound. As comparatively slow elastic waves propagate through the domain X of interest, ultrasound measurements are used to infer the internal displacements, i.e., the solution of the Helmholtz equation (37). In Magnetic resonance elastography (MRE), the high resolution modality is magnetic resonance. Along with elastic displacements, proton displacements occur that can be measured by an MRI machinery.

In both modalities, the internal functionals obtained by ultrasound in TE and by magnetic resonance imaging in MRE are given by the displacements:

$$H_j(x) = u_j(x) \quad \text{in } X. \tag{38}$$

This is exactly the setting considered in Theorem 2.1. Note, however, that in many applications of elastography, the scalar model considered here is not sufficiently accurate. Generalizations to more precise models of linear or nonlinear elasticity then need to be developed.

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