

Central limits and homogenization in random media

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March 9, 2009

Abstract

We consider the perturbation of elliptic operators of the form $P(\mathbf{x}, \mathbf{D})$ by random, rapidly varying, sufficiently mixing, potentials of the form $q(\frac{\mathbf{x}}{\varepsilon}, \omega)$. We analyze the source and spectral problems associated to such operators and show that the properly renormalized difference between the perturbed and unperturbed solutions may be written asymptotically as $\varepsilon \rightarrow 0$ as explicit Gaussian processes. Such results may be seen as central limit corrections to the homogenization (law of large numbers) process. Similar results are derived for more general elliptic equations in one dimension of space.

The results are based on the availability of a rapidly converging integral formulation for the perturbed solutions and on the use of classical central limit results for random processes with appropriate mixing conditions.

keywords: Homogenization, central limit, mixing coefficients, partial differential equations with random coefficients, random oscillatory integrals.

AMS: 35R60, 35J05, 35P20, 60H05.

1 Introduction

There are many practical applications of partial differential equations with coefficients that oscillate at a faster scale than the scale of the domain on which the equation is solved. In such settings, it is often necessary to model the rapidly oscillatory coefficients as random processes, whose properties are known only at a statistical level. The numerical simulation of the resulting partial differential equation with random coefficients becomes a daunting task.

Two simplifications are then typically considered. The first simplification consists in assuming that the coefficients oscillate very rapidly and replacing the equation with random coefficients by a homogenized equation with deterministic (effective medium)

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coefficients. The homogenization of many linear and nonlinear partial differential equations with periodic [7, 40] and random highly oscillatory coefficients has been obtained to date [11, 16, 30, 35, 37, 38, 43].

The solution to the equation with random equations may also be interpreted as a functional of an infinite number of random variables and expanded in polynomial chaos [17, 48]. A second simplification consists then in discretizing the randomness in the coefficients over sufficiently low dimensional subspaces -primarily by Galerkin projection- so that the partial differential equation with random coefficients may be fully solved numerically. We refer the reader to e.g. [3, 27, 28, 29, 39, 50] for references on this active area of research. Such problems, which are posed in domains of dimension $d + Q$, where d is spatial dimension and Q the dimension of the random space, are computationally very intensive, although they have the main advantage of providing realistic statistical fluctuations of the random solution, which are absent in the homogenization approximation.

The two aforementioned approaches can hardly be reconciled. Homogenization arises in a limit where the law of large numbers applies and the solution becomes asymptotically a deterministic quantity. The number of random variables describing the random coefficients thus tends to infinity, a limit that is difficult to obtain by polynomial chaos-type expansions.

In several practical settings such as e.g. the analysis of geological basins or the manufacturing of composite materials, one may be interested in an intermediate situation. We may observe experimental fluctuations in the random solution which are not accounted for by the homogenized solution, and yet, may be in the presence of a sufficiently rich random environment so that full solutions of the equation with random coefficients may not be feasible. This is the type of settings that motivate the studies of this paper. Our main objective will be to characterize the statistical structure of the corrector to the homogenized, deterministic, limit. Whereas the deterministic limit may be seen as a law of large number effect, we are interested in characterizing the next order term, which arises as an application of the central limit theory.

In most practical cases of interest, starting with the elliptic operator $\nabla \cdot a_\varepsilon(\mathbf{x}, \omega) \nabla$, with $\mathbf{x} \in D \subset \mathbb{R}^d$ and $\omega \in \Omega$ the space of random realizations, the calculation of the homogenized tensor is difficult and does not admit analytic expressions except in very simple cases [30]. The amplitude of the corrector to homogenization, let alone its statistical description, remains largely open. The best estimates currently available in spatial dimension $d \geq 2$ may be found in [49]; see also [19, 21], [18] for discrete equations, and [4] for applications of such error estimates. Only in one dimension of space do we have an explicit characterization of the effective diffusion coefficient and of the corrector [13]. Unlike the case of periodic media, where the corrector is proportional to the size of the cell of periodicity ε , the random corrector to the homogenized solution is an explicitly characterized Gaussian process of order $\sqrt{\varepsilon}$ when the random coefficient has integrable correlation [13]. In the case of correlations that are non integrable and of the form $R(t) \sim t^{-\alpha}$ for some $0 < \alpha < 1$, the corrector may be shown to be still an explicitly characterized Gaussian process, but now of order $\varepsilon^{\frac{\alpha}{2}}$ [5].

The reason why explicit characterizations of the correctors may be obtained in [5, 13] is that the solution to the heterogeneous elliptic equation may be written explicitly. Correctors to homogenization have been obtained in more general settings. The analysis

of homogenized solutions and central limit correctors to evolution equations with time dependent randomly varying coefficients is well known; see e.g. [9, 23, 25, 33, 36, 42] for reference on the Markov diffusion approximation and the method of the perturbed test function. In the context of the one-dimensional Helmholtz equation, this would correspond to solving the equation on an interval $(0, a)$ with initial conditions of the form $u_\varepsilon(0)$ and $u'_\varepsilon(0)$ known. The asymptotic limit of boundary value problems, which corresponds in our example to prescribing $u_\varepsilon(0)$ and $u_\varepsilon(a)$, requires somewhat different mathematical techniques. We refer the reader to [26, 47] for results in the setting of one-dimensional problems. Note that in the case of a much stronger potential, in dimension $d = 1$ of the form $\varepsilon^{-\frac{1}{2}}q_\varepsilon$ instead of q_ε in the above Helmholtz operator, the deterministic homogenization limit no longer holds. Somewhat surprisingly, the solution of a corresponding evolution equation still converges to a well identified limit; see [44].

In spatial dimensions two and higher, a methodology to compute the Gaussian fluctuations for boundary value problems of the form $-\Delta u_\varepsilon + F(u_\varepsilon, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) = f(\mathbf{x})$ was developed in [24]. An explicit expression for the fluctuations was obtained and proved for the linear equation $(-\Delta + \lambda + q(\frac{\mathbf{x}}{\varepsilon}))u_\varepsilon(\mathbf{x}) = f(\mathbf{x})$ in dimension $d = 3$. In this paper, we revisit the problem and generalize it to linear problems of the form $P(\mathbf{x}, \mathbf{D})u_\varepsilon + q_\varepsilon(\mathbf{x})u_\varepsilon = f(\mathbf{x})$ with an unperturbed equation $P(\mathbf{x}, \mathbf{D})u = f$, which admits a Green's function $G(\mathbf{x}, \mathbf{y})$ that is more than square integrable (see (4) below). The prototypical example of interest is the operator $P(\mathbf{x}, \mathbf{D}) = -\nabla \cdot a(\mathbf{x})\nabla + q_0(\mathbf{x})$ with sufficiently smooth (deterministic) coefficients $a(\mathbf{x})$ and $q_0(\mathbf{x})$ posed on a bounded domain with, say, Dirichlet boundary conditions, for which the Green's function is more than square integrable in dimensions $1 \leq d \leq 3$.

Under appropriate mixing conditions on the random process $q_\varepsilon(\mathbf{x}, \omega)$, we will show that arbitrary spatial moments of the correctors

$$\left(\frac{u_\varepsilon - u_0}{\varepsilon^{\frac{d}{2}}}, M \right),$$

where u_ε and u_0 are the solutions to perturbed and unperturbed equations, respectively, and where M is a smooth function, converge in distribution to Gaussian random variables, which admit a convenient and explicit representation as a stochastic integral with respect to a standard (multi-parameter) Wiener process. If we denote by u_1 the weak limit of $w_{1\varepsilon} = \varepsilon^{-\frac{d}{2}}(u_\varepsilon - u_0)$, we observe, for $1 \leq d \leq 3$, that $\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x}, \omega)\}$ converges to $\mathbb{E}\{u_1^2(\mathbf{x}, \omega)\}$, where $v_{1\varepsilon}$ is the leading term in $w_{1\varepsilon}$ up to an error term we prove is of order $O(\varepsilon^d)$ in $L^1(\Omega \times D)$. This shows that the limiting process u_1 captures all the fluctuations of the corrector to homogenization. This result is in sharp contrast to the cases $d \geq 4$ and to homogenization in periodic media in arbitrary dimension, where the weak limit of the corrector captures a fraction of the energy of that corrector. We thus see that corrections to homogenization are somewhat different for $1 \leq d \leq 3$ and $d \geq 4$ so that the square integrability of the Green's function is a natural condition in the framework of homogenization in random media.

We obtain similar expressions for the spectral elements of the perturbed elliptic equation. We find that the correctors to the eigenvalues and the spatial moments of the correctors to the corresponding eigenvectors converge in distribution to Gaussian variables as the correlation length ε vanishes. In the setting $d = 1$, we obtain similar result for more general elliptic operators of the form $-\frac{d}{dx}(a_\varepsilon \frac{d}{dx}) + q_0 + q_\varepsilon$ by appropriate

use of harmonic coordinates [35]. The extension to similar operators in dimension $d \geq 2$ remains open.

As was already mentioned, the theory developed here allows us to characterize the statistical properties of the solutions to equations with random coefficients in the limit where the correlation length (the scale of the heterogeneities) is small compared to the overall size of the domain. In many practical problems, it is a good approximation to the statistical structure of the solution of the equation and possibly be all that one is interested in.

Asymptotically explicit expressions for the correctors may also find applications in the testing of numerical algorithms. Several numerical schemes have been developed to estimate the heterogeneous solution accurately in the regime of validity of homogenization by using discretizations with a length scale h that is large compared to the correlation length: $h \gg \varepsilon$; see e.g. [1, 2, 21, 22, 41]. A possible application of the explicit expression for the correctors is to see whether these algorithms can capture the central limit correction to the solutions to the random partial differential equations.

Another application concerns the reconstruction of the constitutive parameters of a differential equation from various measurements, for instance the reconstruction of the potential in a Helmholtz equation from spectral measurements [31, 45]. In such cases, reconstruction algorithms provide lower-variance reconstructions when the cross-correlations are known and used optimally in the inversion; see e.g. [46]. Provided that q_0 is the deterministic quantity that we wish to reconstruct, higher-frequency components that we may not hope to reconstruct still influence available data. The correctors obtained in this paper provide asymptotic estimates for the cross-correlation of the measured data, which allow us to obtain lower-variance reconstructions for q_0 ; see [6].

An outline for the rest of the paper is as follows. Section 2 considers the convergence of the corrector to the homogenized solution for the Helmholtz equation with source term in dimensions $1 \leq d \leq 3$. The proof is based on showing the rapid convergence of a Lippman-Schwinger-type integral formulation (see (8) below), and on applying central limit theorems to random oscillatory integrals. The behavior of the oscillatory integrals is considered in arbitrary dimensions in section 2.4, where a comparison between homogenization in random and periodic environments is also considered. The generalization to a more general one-dimensional elliptic source problem is detailed in section 3. The results on the correctors obtained for source problems are then extended to correctors for spectral problems in section 4. The proof is based on adapting classical results [32] on the convergence of the spectra of operators that converge on average in the uniform norm. The results obtained for the spectral problems are then briefly applied to the analysis of evolution equations. Some concluding remarks are presented in section 5.

2 Correctors for Helmholtz equations

Consider an equation of the form:

$$\begin{aligned} P(\mathbf{x}, \mathbf{D})u_\varepsilon + q_\varepsilon u_\varepsilon &= f, & \mathbf{x} \in D \\ u_\varepsilon &= 0 & \mathbf{x} \in \partial D, \end{aligned} \tag{1}$$

where $P(\mathbf{x}, \mathbf{D})$ is a (deterministic) self-adjoint, elliptic, pseudo-differential operator and D an open bounded domain in \mathbb{R}^d . We assume that $P(\mathbf{x}, \mathbf{D})$ is invertible with symmetric and “more than square integrable” Green’s function. More precisely, we assume that the equation

$$\begin{aligned} P(\mathbf{x}, \mathbf{D})u &= f, & \mathbf{x} &\in D \\ u &= 0 & \mathbf{x} &\in \partial D, \end{aligned} \quad (2)$$

admits a unique solution

$$u(\mathbf{x}) = \mathcal{G}f(\mathbf{x}) := \int_D G(\mathbf{x}, \mathbf{y})f(\mathbf{y})d\mathbf{y}, \quad (3)$$

and that the real-valued and non-negative (to simplify notation) symmetric kernel $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x})$ has more than square integrable singularities so that

$$\mathbf{x} \mapsto \left(\int_D |G|^{2+\eta}(\mathbf{x}, \mathbf{y})d\mathbf{y} \right)^{\frac{1}{2+\eta}} \quad \text{is bounded on } D \text{ for some } \eta > 0. \quad (4)$$

The assumption is typically satisfied for operators of the form $P(\mathbf{x}, D) = -\nabla \cdot a(\mathbf{x})\nabla + \sigma(\mathbf{x})$ for $a(\mathbf{x})$ uniformly bounded and coercive, $\sigma(\mathbf{x}) \geq 0$, and in dimension $d \leq 3$, with $\eta = +\infty$ when $d = 1$ (i.e., the Green’s function is bounded), $\eta < \infty$ for $d = 2$, and $\eta < 1$ for $d = 3$.

Let $\tilde{q}_\varepsilon(\mathbf{x}, \omega) = q(\frac{\mathbf{x}}{\varepsilon}, \omega)$ be a mean zero, strictly stationary, process defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ [15]. The process $\tilde{q}_\varepsilon(\mathbf{x}, \omega)$ will be modified in the sequel as the process $q_\varepsilon(\mathbf{x}, \omega)$ appearing in (1) to ensure that solutions to the Helmholtz equation exist. We assume that $q(\mathbf{x}, \omega)$ has an integrable correlation function:

$$R(\mathbf{x}) = \mathbb{E}\{q(\mathbf{y}, \omega)q(\mathbf{y} + \mathbf{x}, \omega)\}, \quad (5)$$

where \mathbb{E} is mathematical expectation associated to \mathbb{P} . The above expression is independent of \mathbf{y} by stationarity of the process $q(\mathbf{x}, \omega)$. We also assume that $q(\mathbf{x}, \omega)$ is strongly mixing in the following sense. For two Borel sets $A, B \subset \mathbb{R}^d$, we denote by \mathcal{F}_A and \mathcal{F}_B the sub- σ algebras of \mathcal{F} generated by the field $q(\mathbf{x}, \omega)$ for $\mathbf{x} \in A$ and $\mathbf{x} \in B$, respectively. Then we assume the existence of a (ρ -) mixing coefficient $\varphi(r)$ such that

$$\left| \frac{\mathbb{E}\{(\eta - \mathbb{E}\{\eta\})(\xi - \mathbb{E}\{\xi\})\}}{(\mathbb{E}\{\eta^2\}\mathbb{E}\{\xi^2\})^{\frac{1}{2}}} \right| \leq \varphi(2d(A, B)) \quad (6)$$

for all (real-valued) random variables η on $(\Omega, \mathcal{F}_A, \mathbb{P})$ and ξ on $(\Omega, \mathcal{F}_B, \mathbb{P})$. Here, $d(A, B)$ is the Euclidean distance between the Borel sets A and B . The multiplicative factor 2 in (6) is here only for convenience. Moreover, we assume that $\varphi(r)$ is bounded and decreasing. We will impose additional restrictions on the process to ensure that the equation (1) admits a solution.

We formally recast (1) as

$$u_\varepsilon = \mathcal{G}(f - q_\varepsilon u_\varepsilon), \quad (7)$$

where $\mathcal{G} = P(\mathbf{x}, D)^{-1}$, and after one more iteration as

$$u_\varepsilon = \mathcal{G}f - \mathcal{G}q_\varepsilon \mathcal{G}f + \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_\varepsilon. \quad (8)$$

This is the integral equation we aim to analyze.

2.1 Existence and error estimates

In order for the above equation to admit a unique solution, we need to ensure that $(I - \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon)$ is invertible \mathbb{P} -a.s. We modify the process $\tilde{q}_\varepsilon(\mathbf{x}, \omega)$ defined above on a set of measure of order ε^d so that $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon$ has spectral radius bounded by $\rho < 1$ \mathbb{P} -a.s. To do so and to estimate the source term $\mathcal{G}f - \mathcal{G}q_\varepsilon\mathcal{G}f$ in (8), we need a few lemmas.

Lemma 2.1 *Let $q(\mathbf{x}, \omega)$ be strongly mixing so that (6) holds and such that $\mathbb{E}\{q^6\} < \infty$. Then, we have:*

$$|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}| \lesssim \sup_{\{\mathbf{y}_k\}_{1 \leq k \leq 4} = \{\mathbf{x}_k\}_{1 \leq k \leq 4}} \varphi^{\frac{1}{2}}(|\mathbf{y}_1 - \mathbf{y}_3|) \varphi^{\frac{1}{2}}(|\mathbf{y}_2 - \mathbf{y}_4|) \mathbb{E}\{q^6\}^{\frac{2}{3}}. \quad (9)$$

Here, we use the notation $a \lesssim b$ when there is a positive constant C such that $a \leq Cb$.

Proof. Let \mathbf{y}_1 and \mathbf{y}_2 be two points in $\{\mathbf{x}_k\}_{1 \leq k \leq 4}$ such that $d(\mathbf{y}_1, \mathbf{y}_2) \geq d(\mathbf{x}_i, \mathbf{x}_j)$ for all $1 \leq i, j \leq 4$ and such that $d(\mathbf{y}_1, \{\mathbf{z}_3, \mathbf{z}_4\}) \leq d(\mathbf{y}_2, \{\mathbf{z}_3, \mathbf{z}_4\})$, where $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{z}_3, \mathbf{z}_4\} = \{\mathbf{x}_k\}_{1 \leq k \leq 4}$.

Let us call \mathbf{y}_3 a point in $\{\mathbf{z}_3, \mathbf{z}_4\}$ closest to \mathbf{y}_1 . We call \mathbf{y}_4 the remaining point in $\{\mathbf{x}_k\}_{1 \leq k \leq 4}$. We have, using (6) and $\mathbb{E}\{q\} = 0$, that:

$$|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}| \lesssim \varphi(2|\mathbf{y}_1 - \mathbf{y}_3|) (\mathbb{E}\{q^2\})^{\frac{1}{2}} (\mathbb{E}\{(q(\mathbf{y}_2)q(\mathbf{y}_3)q(\mathbf{y}_4))^2\})^{\frac{1}{2}}.$$

The last two terms are bounded by $\mathbb{E}\{q^6\}^{\frac{1}{6}}$ and $\mathbb{E}\{q^6\}^{\frac{1}{2}}$, respectively, using Hölder's inequality. Because $\varphi(r)$ is assumed to be decreasing, we deduce that

$$|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}| \lesssim \varphi(|\mathbf{y}_1 - \mathbf{y}_3|) \mathbb{E}\{q^6\}^{\frac{2}{3}}. \quad (10)$$

If \mathbf{y}_4 is (one of) the closest point(s) to \mathbf{y}_2 , then the same arguments show that

$$|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}| \lesssim \varphi(|\mathbf{y}_2 - \mathbf{y}_4|) \mathbb{E}\{q^6\}^{\frac{2}{3}}. \quad (11)$$

Otherwise, \mathbf{y}_3 is the closest point to \mathbf{y}_2 , and we find that

$$|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}| \lesssim \varphi(2|\mathbf{y}_2 - \mathbf{y}_3|) \mathbb{E}\{q^6\}^{\frac{2}{3}}.$$

However, by construction, $|\mathbf{y}_2 - \mathbf{y}_4| \leq |\mathbf{y}_1 - \mathbf{y}_2| \leq |\mathbf{y}_1 - \mathbf{y}_3| + |\mathbf{y}_3 - \mathbf{y}_2| \leq 2|\mathbf{y}_2 - \mathbf{y}_3|$, so (11) is still valid (this is the only place where the factor 2 in (6) is used).

Combining (10) and (11), the result follows from $a \wedge b \leq (ab)^{\frac{1}{2}}$ for $a, b \geq 0$, where $a \wedge b = \min(a, b)$. \square

Lemma 2.2 *Let q_ε be a stationary process $q_\varepsilon(\mathbf{x}, \omega) = q(\frac{\mathbf{x}}{\varepsilon}, \omega)$ with integrable correlation function in (5). Let f be a deterministic square integrable function on D . Then we have:*

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|_{L^2(D)}^2\} \lesssim \varepsilon^d \|f\|_{L^2(D)}^2. \quad (12)$$

Let q_ε satisfy one of the following additional hypotheses:

[H1] *There is a constant C such that $|q(\mathbf{x}, \omega)| < C$ $d\mathbf{x} \times \mathbb{P}$ -a.s.*

[H2] *$\mathbb{E}\{q^6\} < \infty$ and $q(\mathbf{x}, \omega)$ is strongly mixing with mixing coefficient in (6) such that $\varphi^{\frac{1}{2}}(r)$ is bounded and $r^{d-1}\varphi^{\frac{1}{2}}(r)$ is integrable on \mathbb{R}^+ .*

Then we find the following bound for the operator $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon$:

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}(L^2(D))}^2\} \lesssim \varepsilon^d. \quad (13)$$

Remark 2.3 Note the assumption [H2] combined with $\varphi(r)$ decreasing together impose that $\varphi(r) = o(r^{-2d})$. For otherwise, we would have an increasing sequence $r_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $\varphi^{\frac{1}{2}}(r_n) \geq Cr_n^{-d}$ for some $C > 0$, and then, since $\varphi^{\frac{1}{2}}$ is also decreasing,

$$\int_0^\infty r^{d-1}\varphi^{\frac{1}{2}}(r)dr \geq \sum_n \int_{r_n}^{r_{n+1}} \frac{r^{d-1}dr}{r_{n+1}^d} = \sum_n \frac{r_{n+1}^d - r_n^d}{dr_{n+1}^d} \geq \sum_n \frac{r_{n+1} - r_n}{dr_{n+1}}.$$

Now if there is an infinite number of terms n such that $r_{n+1} \geq 2r_n$, then there is an infinite number of terms such that $\frac{r_{n+1}-r_n}{dr_{n+1}} \geq \frac{1}{2d}$ and the above sum is infinite. If there is a finite number of such terms, then for all $n \geq n_0$ for n_0 sufficiently large, we have $r_{n+1} \leq 2r_n$ so that

$$\int_0^\infty r^{d-1}\varphi^{\frac{1}{2}}(r)dr \geq \sum_{n \geq n_0} \frac{r_{n+1} - r_n}{dr_{n+1}} \geq \sum_{n \geq n_0} \frac{r_{n+1} - r_n}{2dr_n} \geq \frac{1}{2d} \int_{r_{n_0}}^\infty \frac{dx}{x} = +\infty.$$

Our assumptions then impose that $\varphi(r)$ decay faster than r^{-2d} .

Proof [Lemma 2.2]. We denote $\|\cdot\| = \|\cdot\|_{L^2(D)}$ and calculate

$$\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x}) = \int_D \left(\int_D G(\mathbf{x}, \mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y}, \mathbf{z})d\mathbf{y} \right) f(\mathbf{z})d\mathbf{z},$$

so that by the Cauchy-Schwarz inequality, we have

$$|\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x})|^2 \leq \|f\|^2 \int_D \left(\int_D G(\mathbf{x}, \mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y}, \mathbf{z})d\mathbf{y} \right)^2 d\mathbf{z}.$$

By definition of the correlation function, we thus find that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2 \int_{D^4} G(\mathbf{x}, \mathbf{y})G(\mathbf{x}, \boldsymbol{\zeta})R\left(\frac{\mathbf{y} - \boldsymbol{\zeta}}{\varepsilon}\right)G(\mathbf{y}, \mathbf{z})G(\boldsymbol{\zeta}, \mathbf{z})d\mathbf{x}d\mathbf{y}d\boldsymbol{\zeta}d\mathbf{z}. \quad (14)$$

Extending $G(\mathbf{x}, \mathbf{y})$ by 0 outside $D \times D$, we find in the Fourier domain that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2 \int_{D^2} \int_{\mathbb{R}^d} |G(\mathbf{x}, \cdot)\widehat{G}(\mathbf{z}, \cdot)|^2(\mathbf{p})\varepsilon^d \hat{R}(\varepsilon\mathbf{p})d\mathbf{p}d\mathbf{x}d\mathbf{z}.$$

Here $\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}f(\mathbf{x})d\mathbf{x}$ is the Fourier transform of $f(\mathbf{x})$. Since $R(\mathbf{x})$ is integrable, then $\hat{R}(\varepsilon\mathbf{p})$ (which is always non-negative by e.g. Bochner's theorem) is bounded by a constant we call R_0 so that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2 \varepsilon^d R_0 \int_{D^3} G^2(\mathbf{x}, \mathbf{y})G^2(\mathbf{z}, \mathbf{y})d\mathbf{x}d\mathbf{y}d\mathbf{z} \lesssim \|f\|^2 \varepsilon^d R_0,$$

by the square-integrability assumption on $G(\mathbf{x}, \mathbf{y})$. This yields (12). Let us now consider (13). We denote by $\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}}$ the norm $\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}(L^2(D))}$ and calculate that

$$\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\phi(\mathbf{x}) = \int_D \left(\int_D G(\mathbf{x}, \mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y}, \mathbf{z})d\mathbf{y} \right) q_\varepsilon(\mathbf{z})\phi(\mathbf{z})d\mathbf{z}.$$

Therefore,

$$\left(\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\phi(\mathbf{x})\right)^2 \leq \int_D \left(\int_D G(\mathbf{x}, \mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y}, \mathbf{z})q_\varepsilon(\mathbf{z})d\mathbf{y}\right)^2 d\mathbf{z} \int_D \phi^2(\mathbf{z})d\mathbf{z},$$

by Cauchy-Schwarz. This shows that

$$\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}}^2(\omega) \leq \int_{D^2} \left(\int_D G(\mathbf{x}, \mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y}, \mathbf{z})d\mathbf{y}\right)^2 q_\varepsilon^2(\mathbf{z})d\mathbf{z}d\mathbf{x}.$$

When $q_\varepsilon(\mathbf{z}, \omega)$ is bounded \mathbb{P} -a.s., the above proof leading to (12) applies and we obtain (13) under hypothesis [H1].

Using Lemma 2.1, we obtain that

$$\mathbb{E}\{q_\varepsilon(\mathbf{y})q_\varepsilon(\boldsymbol{\zeta})q_\varepsilon^2(\mathbf{z})\} \lesssim \varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\right)\varphi^{\frac{1}{2}}(0) + \varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\right)\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{z}-\boldsymbol{\zeta}|}{\varepsilon}\right).$$

Under hypothesis [H2], we thus obtain that

$$\begin{aligned} \mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}}^2\} &\lesssim \int_{D^4} G(\mathbf{x}, \mathbf{y})G(\mathbf{x}, \boldsymbol{\zeta})\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\right)G(\mathbf{y}, \mathbf{z})G(\boldsymbol{\zeta}, \mathbf{z})d\mathbf{y}d\boldsymbol{\zeta}d\mathbf{x}d\mathbf{z} \\ &\quad + \int_{D^2} \left(\int_D G(\mathbf{x}, \mathbf{y})\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\right)G(\mathbf{y}, \mathbf{z})d\mathbf{y}\right)^2 d\mathbf{x}d\mathbf{z}. \end{aligned}$$

Because $r^{d-1}\varphi^{\frac{1}{2}}(r)$ is integrable, then $\mathbf{x} \mapsto \varphi^{\frac{1}{2}}(|\mathbf{x}|)$ is integrable as well and the bound of the first term above under hypothesis [H2] is done as in (14) by replacing $R(\mathbf{x})$ by $\varphi^{\frac{1}{2}}(|\mathbf{x}|)$. As for the second term, it is bounded, using the Cauchy-Schwarz inequality, by

$$\int_D \left(\int_D \left(\int_D G^2(\mathbf{x}, \mathbf{y})d\mathbf{x}\right)G^2(\mathbf{y}, \mathbf{z})d\mathbf{y}\right) \left(\int_D \varphi\left(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\right)d\mathbf{y}\right) d\mathbf{z} \lesssim \varepsilon^d,$$

since $\mathbf{x} \mapsto \varphi(|\mathbf{x}|)$ is integrable, D is bounded, and (4) holds. \square

Applying the previous result to the process $\tilde{q}_\varepsilon(\mathbf{x}, \omega) = q\left(\frac{\mathbf{x}}{\varepsilon}, \omega\right)$, we obtain from the Chebyshev inequality that

$$\mathbb{P}(\omega; \|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\tilde{q}_\varepsilon\|_{\mathcal{L}} > \rho) \lesssim \frac{\mathbb{E}\{\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\tilde{q}_\varepsilon\|_{\mathcal{L}}^2\}}{\rho^2} \lesssim \varepsilon^d. \quad (15)$$

On the set $\Omega_\varepsilon \subset \Omega$ of measure $\mathbb{P}(\Omega_\varepsilon) \lesssim \varepsilon^d$ where $\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\tilde{q}_\varepsilon\|_{\mathcal{L}} > \rho$, we modify the potential \tilde{q}_ε and set it to e.g. 0. We thus construct

$$q_\varepsilon(\mathbf{x}, \omega) = \begin{cases} \tilde{q}_\varepsilon(\mathbf{x}, \omega) & \Omega \setminus \Omega_\varepsilon, \\ 0 & \Omega_\varepsilon. \end{cases} \quad (16)$$

We have

Lemma 2.4 *The results obtained for $\tilde{q}_\varepsilon(\mathbf{x}, \omega) = q\left(\frac{\mathbf{x}}{\varepsilon}, \omega\right)$ in Lemma 2.2 hold for $q_\varepsilon(\mathbf{x}, \omega)$ constructed in (16).*

Proof. For instance,

$$\begin{aligned}\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} &= \mathbb{E}\{\chi_{\Omega^\varepsilon}(\omega)\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} + \mathbb{E}\{\chi_{\Omega\setminus\Omega^\varepsilon}(\omega)\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \\ &= \mathbb{E}\{\chi_{\Omega\setminus\Omega^\varepsilon}(\omega)\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}f\|^2\} \leq \mathbb{E}\{\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}f\|^2\} \lesssim \varepsilon^d\|f\|^2.\end{aligned}$$

The same proof holds for the second bound (13). \square

We need to ensure that the oscillatory integrals studied in subsequent sections are not significantly modified when $q(\frac{\mathbf{x}}{\varepsilon}, \omega)$ is replaced by the new $q_\varepsilon(\mathbf{x}, \omega)$. Let

$$I_\varepsilon(\omega) = \left\| \frac{1}{\varepsilon^{\frac{d}{2}}} \left(q\left(\frac{\mathbf{x}}{\varepsilon}, \omega\right) - q_\varepsilon(\mathbf{x}, \omega) \right) \right\| \quad (17)$$

Then, under [H1] or [H2], we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}\{I_\varepsilon\} \equiv \lim_{\varepsilon \rightarrow 0} \mathbb{E}\left\{ \chi_{\Omega^\varepsilon}(\omega) \left\| \frac{1}{\varepsilon^{\frac{d}{2}}} q\left(\frac{\mathbf{x}}{\varepsilon}, \omega\right) \right\| \right\} = 0. \quad (18)$$

Indeed, by Hölder's inequality, boundedness of D and stationarity of q , we have

$$\mathbb{E}\{I_\varepsilon\} \lesssim \mathbb{E}\{\chi_{\Omega^\varepsilon}(\omega)\}^{\frac{1}{p'}} \varepsilon^{-\frac{d}{2}} \mathbb{E}\{|q|^p\}^{\frac{1}{p}} \lesssim \varepsilon^{d(\frac{1}{2} - \frac{1}{p})} (\mathbb{E}|q|^p)^{\frac{1}{p}},$$

for $\frac{1}{p} + \frac{1}{p'} = 1$. The result follows when $p > 2$.

With the modified potential, (8) admits a unique solution \mathbb{P} -a.s. and we find that

$$\|u_\varepsilon\|(\omega) \lesssim \|\mathcal{G}f\| + \|\mathcal{G}q_\varepsilon\mathcal{G}f\| \quad \mathbb{P} - \text{ a.s.}, \quad (19)$$

where $\|\cdot\|$ denotes $L^2(D)$ norm. Using the first result of Lemma 2.2, we find that

$$\mathbb{E}\{\|u_\varepsilon\|^2\} \lesssim \|f\|^2. \quad (20)$$

Now we can address the behavior of the correctors. We define

$$u_0 = \mathcal{G}f, \quad (21)$$

the solution of the unperturbed problem. We find that

$$(I - \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon)(u_\varepsilon - u_0) = -\mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f. \quad (22)$$

Using the results of Lemma 2.2, we obtain that

Lemma 2.5 *Let u_ε be the solution to the heterogeneous problem (1) and u_0 the solution to the corresponding homogenized problem. Then we have that*

$$\left(\mathbb{E}\{\|u_\varepsilon - u_0\|^2\}\right)^{\frac{1}{2}} \lesssim \varepsilon^{\frac{d}{2}}\|f\|. \quad (23)$$

Note that if we write $u_\varepsilon = A_\varepsilon f$ and $u_0 = A_0 f$, with A_ε and A_0 the solution operators of the heterogeneous and homogeneous equations, respectively, then we have just shown that

$$\mathbb{E}\{\|A_\varepsilon - A_0\|^2\} \lesssim \varepsilon^d. \quad (24)$$

Now $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)$ is bounded by ε^d in $L^1(\Omega; L^2(D))$ by Cauchy-Schwarz:

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)\|\} \leq \left(\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}}^2\}\right)^{\frac{1}{2}} \left(\mathbb{E}\{\|u_\varepsilon - u_0\|^2\}\right)^{\frac{1}{2}} \lesssim \varepsilon^d.$$

We need the following estimate:

Lemma 2.6 Under hypothesis [H2] of Lemma 2.2, we find that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \varepsilon^{2d\frac{1+\eta}{2+\eta}}\|f\|^2 \ll \varepsilon^d\|f\|^2, \quad (25)$$

where η is such that $\mathbf{y} \mapsto \left(\int_D |G|^{2+\eta}(\mathbf{x}, \mathbf{y})d\mathbf{x}\right)^{\frac{1}{2+\eta}}$ is uniformly bounded on D .

Proof. By Cauchy-Schwarz,

$$|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x})|^2 \leq \|f\|^2 \int_D \left(\int_{D^2} G(\mathbf{x}, \mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y}, \mathbf{z})q_\varepsilon(\mathbf{z})G(\mathbf{z}, \mathbf{t})d\mathbf{y}d\mathbf{z}\right)^2 d\mathbf{t}.$$

So we want to estimate

$$A = \mathbb{E}\left\{\int_{D^6} G(\mathbf{x}, \mathbf{y})G(\mathbf{x}, \zeta)q_\varepsilon(\mathbf{y})q_\varepsilon(\zeta)G(\mathbf{y}, \mathbf{z})G(\zeta, \xi)q_\varepsilon(\mathbf{z})q_\varepsilon(\xi)G(\mathbf{z}, \mathbf{t})G(\xi, \mathbf{t})d[\xi\zeta\mathbf{y}\mathbf{z}\mathbf{x}\mathbf{t}]\right\}.$$

We now use (9) to obtain that $A \lesssim A_1 + A_2 + A_3$, where

$$\begin{aligned} A_1 &= \int_{D^6} G(\mathbf{x}, \mathbf{y})G(\mathbf{x}, \zeta)\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\zeta|}{\varepsilon}\right)G(\mathbf{y}, \mathbf{z})G(\zeta, \xi)\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{z}-\xi|}{\varepsilon}\right)G(\mathbf{z}, \mathbf{t})G(\xi, \mathbf{t})d[\xi\zeta\mathbf{y}\mathbf{z}\mathbf{x}\mathbf{t}], \\ A_2 &= \int_{D^2} \left(\int_{D^2} G(\mathbf{x}, \mathbf{y})G(\mathbf{y}, \mathbf{z})\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\right)G(\mathbf{z}, \mathbf{t})d\mathbf{y}d\mathbf{z}\right)^2 d\mathbf{t}d\mathbf{x}, \\ A_3 &= \int_{D^6} G(\mathbf{x}, \mathbf{y})G(\xi, \mathbf{t})G(\mathbf{x}, \zeta)G(\mathbf{z}, \mathbf{t})\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\xi|}{\varepsilon}\right)G(\mathbf{y}, \mathbf{z})G(\zeta, \xi)\varphi^{\frac{1}{2}}\left(\frac{|\zeta-\mathbf{z}|}{\varepsilon}\right)d[\xi\zeta\mathbf{y}\mathbf{z}\mathbf{x}\mathbf{t}]. \end{aligned}$$

Denote $F_{\mathbf{x}, \mathbf{t}}(\mathbf{y}, \mathbf{z}) = G(\mathbf{x}, \mathbf{y})G(\mathbf{y}, \mathbf{z})G(\mathbf{z}, \mathbf{t})$. Then in the Fourier domain, we find that

$$A_1 \lesssim \int_{D^2} \int_{\mathbb{R}^{2d}} \varepsilon^{2d}\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{p})\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{q})|\widehat{F}_{\mathbf{x}, \mathbf{t}}(\mathbf{p}, \mathbf{q})|^2 d\mathbf{p}d\mathbf{q}d\mathbf{x}d\mathbf{t}.$$

Here $\widehat{\varphi^{\frac{1}{2}}}(\mathbf{p})$ is the Fourier transform of $\mathbf{x} \mapsto \varphi^{\frac{1}{2}}(|\mathbf{x}|)$. Since $\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{p})$ is bounded because $r^{d-1}\varphi^{\frac{1}{2}}(r)$ is integrable on \mathbb{R}^+ , we deduce that

$$A_1 \lesssim \varepsilon^{2d} \int_{D^4} G^2(\mathbf{x}, \mathbf{y})G^2(\mathbf{y}, \mathbf{z})G^2(\mathbf{z}, \mathbf{t})d\mathbf{x}d\mathbf{y}d\mathbf{z}d\mathbf{t} \lesssim \varepsilon^{2d},$$

using the integrability condition imposed on $G(\mathbf{x}, \mathbf{y})$.

Using $2ab \leq a^2 + b^2$ for $(a, b) = (G(\mathbf{x}, \mathbf{y}), G(\mathbf{x}, \zeta))$ and $(a, b) = (G(\xi, \mathbf{t}), G(\mathbf{z}, \mathbf{t}))$ successively, and integrating in \mathbf{t} and \mathbf{x} , we find that

$$A_3 \lesssim \int_{D^4} G(\mathbf{y}, \mathbf{z})G(\zeta, \xi)\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\xi|}{\varepsilon}\right)\varphi^{\frac{1}{2}}\left(\frac{|\zeta-\mathbf{z}|}{\varepsilon}\right)d[\mathbf{y}\zeta\mathbf{z}\xi],$$

thanks to (4). Now with $(a, b) = (G(\mathbf{y}, \mathbf{z}), G(\zeta, \xi))$, we find that

$$A_3 \lesssim \int_{D^4} G^2(\mathbf{y}, \mathbf{z})\varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y}-\xi|}{\varepsilon}\right)\varphi^{\frac{1}{2}}\left(\frac{|\zeta-\mathbf{z}|}{\varepsilon}\right)d[\mathbf{y}\zeta\mathbf{z}\xi] \lesssim \varepsilon^{2d},$$

since $\varphi^{\frac{1}{2}}$ is integrable and G is square integrable on the bounded domain D .

Let us now consider the contribution A_2 . We write the squared integral as a double integral over the variables $(\mathbf{y}, \boldsymbol{\zeta}, \mathbf{z}, \boldsymbol{\xi})$ and dealing with the integration in \mathbf{x} and \mathbf{t} using $2ab \leq a^2 + b^2$ as in the A_3 contribution, obtain that

$$A_2 \lesssim \int_{D^4} G(\mathbf{y}, \boldsymbol{\zeta}) \varphi^{\frac{1}{2}}\left(\frac{|\mathbf{y} - \boldsymbol{\zeta}|}{\varepsilon}\right) G(\mathbf{z}, \boldsymbol{\xi}) \varphi^{\frac{1}{2}}\left(\frac{|\mathbf{z} - \boldsymbol{\xi}|}{\varepsilon}\right) d[\mathbf{y}\boldsymbol{\zeta}\mathbf{z}\boldsymbol{\xi}].$$

Using Hölder's inequality, we obtain that

$$A_2 \lesssim \left(\left(\int_0^\infty \varphi^{\frac{p'}{2}}\left(\frac{r}{\varepsilon}\right) r^{d-1} dr \right)^{\frac{1}{p'}} \left(\int_{D^2} G^p(\mathbf{y}, \mathbf{z}) dy dz \right)^{\frac{1}{p}} \right)^2 \lesssim \varepsilon^{2d\frac{1+\eta}{2+\eta}},$$

with $p = 2 + \eta$ and $p' = \frac{2+\eta}{1+\eta}$ since $\varphi^{\frac{1}{2}}(r)r^{d-1}$, whence $\varphi^{\frac{p'}{2}}(r)r^{d-1}$, is integrable. \square

The above lemma applies to the stationary process $\tilde{q}_\varepsilon(\mathbf{x}, \omega)$, and using the same proof as in Lemma 2.4, for the modified process $q_\varepsilon(\mathbf{x}, \omega)$ in (16). We have therefore obtained that

$$\mathbb{E}\{\|u_\varepsilon - u + \mathcal{G}q_\varepsilon \mathcal{G}f\|\} \lesssim \varepsilon^{d\frac{1+\eta}{2+\eta}}. \quad (26)$$

For what follows, it is useful to recast the above result as:

Proposition 2.7 *Let $q(\mathbf{x}, \omega)$ be constructed so that [H2] holds and let $q_\varepsilon(\mathbf{x}, \omega)$ be as defined in (16). Let u_ε be the solution to (8) and $u_0 = \mathcal{G}f$. We assume that u_0 is continuous on D . Then we have the following strong convergence result:*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}\left\{ \left\| \frac{u_\varepsilon - u_0}{\varepsilon^{\frac{d}{2}}} + \frac{1}{\varepsilon^{\frac{d}{2}}} \mathcal{G}q\left(\frac{\cdot}{\varepsilon}, \omega\right) u_0 \right\| \right\} = 0. \quad (27)$$

Proof. Thanks to (18), we may replace $q_\varepsilon(\mathbf{x}, \omega)$ by $\tilde{q}_\varepsilon(\mathbf{x}, \omega) = q\left(\frac{\mathbf{x}}{\varepsilon}, \omega\right)$ in (26) up to a small error compared to $\varepsilon^{\frac{d}{2}}$. Indeed,

$$\begin{aligned} \mathbb{E}\left\{ \left\| \frac{1}{\varepsilon^{\frac{d}{2}}} \mathcal{G}\left(q\left(\frac{\cdot}{\varepsilon}, \omega\right) - q_\varepsilon(\cdot, \omega)\right) u_0 \right\| \right\} &= \mathbb{E}\left\{ \chi_{\Omega^\varepsilon}(\omega) \left\| \frac{1}{\varepsilon^{\frac{d}{2}}} \mathcal{G}q\left(\frac{\cdot}{\varepsilon}, \omega\right) u_0 \right\| \right\} \\ &\leq \|\mathcal{G}\| \|u_0\|_{L^\infty(D)} \mathbb{E}\left\{ \chi_{\Omega^\varepsilon}(\omega) \left\| \frac{1}{\varepsilon^{\frac{d}{2}}} q\left(\frac{\cdot}{\varepsilon}, \omega\right) \right\| \right\} \ll 1. \end{aligned}$$

\square

The rescaled corrector $\varepsilon^{-\frac{d}{2}} \mathcal{G}q\left(\frac{\cdot}{\varepsilon}, \omega\right) u_0$ does not converge strongly to its limit. Rather, it should be interpreted as a stochastic oscillatory integral whose limiting distribution is governed by the central limit theorem [15, 23]. We consider such limits first in the one-dimensional case and second for arbitrary space dimensions.

2.2 Oscillatory integral in one space dimension

In dimension $d = 1$, the leading term of the corrector $\varepsilon^{-\frac{1}{2}}(u_\varepsilon - u_0)$ is thus given by:

$$u_{1\varepsilon}(x, \omega) = \int_D -G(x, y) \frac{1}{\sqrt{\varepsilon}} q\left(\frac{y}{\varepsilon}, \omega\right) u_0(y) dy, \quad (28)$$

where D is an interval (a, b) . The convergence is more precise in dimension $d = 1$ than in higher space dimensions. For the Helmholtz equation, the Green function in $d = 1$ is

Lipschitz continuous and we will assume this regularity for the rest of the section; see the next section for less regular Green's functions. Then $u_{1\varepsilon}(x, \omega)$ is of class $\mathcal{C}(D)$ \mathbb{P} -a.s. and we can seek convergence in that functional class. Since $u_0 = \mathcal{G}f$, it is continuous for $f \in L^2(D)$.

The variance of the random variable $u_{1\varepsilon}(x, \omega)$ is given by

$$\mathbb{E}\{u_{1\varepsilon}^2(x, \omega)\} = \int_{D^2} G(x, y)G(x, z) \frac{1}{\varepsilon} R\left(\frac{y-z}{\varepsilon}\right) u_0(y)u_0(z) dydz. \quad (29)$$

Because $R(x)$ is assumed to be integrable, the above integral converges, as $\varepsilon \rightarrow 0$, to the following limit:

$$\mathbb{E}\{u_1^2(x, \omega)\} = \int_D G^2(x, y) \hat{R}(0) u_0^2(y) dy, \quad (30)$$

where

$$\hat{R}(0) = \sigma^2 := \int_{-\infty}^{\infty} R(r) dr = 2 \int_0^{\infty} \mathbb{E}\{q(0)q(r)\} dr. \quad (31)$$

Because (28) is an average of random variables decorrelating sufficiently fast, we expect a central limit-type result to show that $u_{1\varepsilon}(x, \omega)$ converges to a Gaussian random variable. Combined with the variance (31), we expect the limit to be the following stochastic integral:

$$u_1(x, \omega) = -\sigma \int_D G(x, y) u_0(y) dW_y(\omega), \quad (32)$$

where $dW_y(\omega)$ is standard white noise on $(\mathcal{C}(D), \mathcal{B}(\mathcal{C}(D)), \mathbb{P})$ [8]. More precisely, we show the following result:

Theorem 2.8 *Let us assume that $G(x, y)$ is Lipschitz continuous. Then, under the conditions of Proposition 2.7, the process $u_{1\varepsilon}(x, \omega)$ converges weakly and in distribution in the space of continuous paths $\mathcal{C}(D)$ to the limit $u_1(x, \omega)$ in (32). As a consequence, the corrector to homogenization satisfies that*

$$\frac{u_\varepsilon - u_0}{\sqrt{\varepsilon}}(x) \xrightarrow[n \geq 1]{\text{dist.}} -\sigma \int_D G(x, y) u_0(y) dW_y, \quad \text{as } \varepsilon \rightarrow 0, \quad (33)$$

in the space of integrable paths $L^1(D)$.

Proof. We recall the classical result on the weak convergence of random variables with values in the space of continuous paths [8]:

Proposition 2.9 *Suppose $(Z_n; 1 \leq n \leq \infty)$ are random variables with values in the space of continuous functions $\mathcal{C}(D)$. Then Z_n converges weakly (in distribution) to Z_∞ provided that:*

- (a) *any finite-dimensional joint distribution $(Z_n(x_1), \dots, Z_n(x_k))$ converges to the joint distribution $(Z_\infty(x_1), \dots, Z_\infty(x_k))$ as $n \rightarrow \infty$.*
- (b) *(Z_n) is a tight sequence of random variables. A sufficient condition for tightness of (Z_n) is the following Kolmogorov criterion: there exist positive constants ν , β , and δ such that*

$$\begin{aligned} (i) \quad & \sup_{n \geq 1} \mathbb{E}\{|Z_n(t)|^\nu\} < \infty, \quad \text{for some } t \in D, \\ (ii) \quad & \mathbb{E}\{|Z_n(s) - Z_n(t)|^\beta\} \lesssim |t - s|^{1+\delta}, \end{aligned} \quad (34)$$

uniformly in $n \geq 1$ and $t, s \in D$.

Tightness. Tightness of $u_{1\varepsilon}(x, \omega)$ is obtained with $\nu = \beta = 2$ and $\delta = 1$. Indeed, we easily obtain that

$$\mathbb{E}\{|u_{1\varepsilon}(x, \omega)|^2\} \lesssim 1,$$

in fact uniformly in $x \in D$. Now by assumption on $G(x, y)$ we obtain that

$$\begin{aligned} \mathbb{E}\{|u_{1\varepsilon}(x, \omega) - u_{1\varepsilon}(\xi, \omega)|^2\} &= \mathbb{E}\left(\int_D [G(x, y) - G(\xi, y)] \frac{1}{\sqrt{\varepsilon}} q\left(\frac{y}{\varepsilon}\right) u_0(y) dy\right)^2 \\ &= \int_{D^2} [G(x, y) - G(\xi, y)][G(x, \zeta) - G(\xi, \zeta)] \frac{1}{\varepsilon} R\left(\frac{\zeta - y}{\varepsilon}\right) u_0(y) u_0(\zeta) dy d\zeta \\ &\lesssim |x - \xi|^2 \int_{D^2} \frac{1}{\varepsilon} |R\left(\frac{\zeta - y}{\varepsilon}\right)| u_0(y) u_0(\zeta) dy d\zeta \lesssim |x - \xi|^2, \end{aligned}$$

since the correlation function $R(r)$ is integrable and u_0 is bounded. This proves tightness of the sequence $u_{1\varepsilon}(x, \omega)$, or equivalently weak convergence of the measures \mathbb{P}_ε generated by $u_{1\varepsilon}(x, \omega)$ on $(\mathcal{C}(D), \mathcal{B}(\mathcal{C}(D)))$.

Finite dimensional distributions. Now any finite-dimensional distribution $(u_{1\varepsilon}(x_j, \omega))_{1 \leq j \leq n}$ has the characteristic function

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{i \sum_{j=1}^n k_j u_{1\varepsilon}(x_j, \omega)}\}, \quad \mathbf{k} = (k_1, \dots, k_n).$$

The above characteristic function can be recast as

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{i \int_D m(y) \frac{1}{\sqrt{\varepsilon}} q\left(\frac{y}{\varepsilon}\right) dy}\}, \quad m(y) = \sum_{j=1}^n k_j G(x_j, y) u_0(y).$$

As a consequence, convergence of the finite dimensional distributions will be proved if we can show convergence of:

$$I_{m\varepsilon} := \int_D m(y) \frac{1}{\sqrt{\varepsilon}} q\left(\frac{y}{\varepsilon}\right) dy \xrightarrow{\text{dist.}} I_m := \int_D m(y) \sigma dW_y, \quad \varepsilon \rightarrow 0, \quad (35)$$

for arbitrary continuous moments $m(y)$. Such integrals have been extensively analyzed in the literature, see e.g. [9, 33], where the above integral, for $D = (a, b)$ may be seen as the solution $x_\varepsilon(b)$ of the following ordinary differential equation with random coefficients:

$$\dot{x}_\varepsilon = \frac{1}{\sqrt{\varepsilon}} q\left(\frac{t}{\varepsilon}\right) m(t), \quad x_\varepsilon(a) = 0.$$

Since we will use the same methodology in higher space dimensions, we give a short proof of (35) using the central limit theorem for correlated discrete random variables as stated e.g. in [10, 14].

Approximation by piecewise constant integrand. Note that if we replace $m(y)$ by $m_h(y)$, then

$$\mathbb{E}\{(I_{m\varepsilon} - I_{m_h\varepsilon})^2\} \lesssim \|m - m_h\|_\infty^2, \quad (36)$$

where $\|\cdot\|_\infty$ is the uniform norm on D . It is therefore sufficient to consider (35) for a sequence of functions m_h converging to m in the uniform sense. Since m is (uniformly)

continuous, we can approximate it by piecewise constant functions m_h that are constant on M intervals of size $h = \frac{b-a}{M}$. Let m_{hj} be the value of m_h on the j^{th} interval and define the random variables

$$M_{\varepsilon j} = m_{hj} \int_{(j-1)h}^{jh} \frac{1}{\sqrt{\varepsilon}} q\left(\frac{y}{\varepsilon}\right) dy.$$

Independence of random variables. We want to show that the variables $M_{\varepsilon j}$ become independent in the limit $\varepsilon \rightarrow 0$. This is done by showing that

$$\mathcal{E}(\mathbf{k}) = \left| \mathbb{E}\{e^{i \sum_{j=1}^M k_j M_{\varepsilon j}}\} - \prod_{j=1}^M \mathbb{E}\{e^{ik_j M_{\varepsilon j}}\} \right| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

for all $\mathbf{k} = \{k_j\}_{1 \leq j \leq M} \in \mathbb{R}^M$. Let $\mathbf{k} \in \mathbb{R}^M$ fixed, $0 < \eta < \frac{h}{2}$ and define

$$P_{\varepsilon j}^{\eta} = m_{hj} \int_{(j-1)h+\eta}^{jh-\eta} \frac{1}{\sqrt{\varepsilon}} q\left(\frac{y}{\varepsilon}\right) dy, \quad Q_{\varepsilon j}^{\eta} = M_{\varepsilon j} - P_{\varepsilon j}^{\eta}.$$

Now we write

$$\mathbb{E}\{e^{i \sum_{j=1}^M k_j M_{\varepsilon j}}\} = \mathbb{E}\{[e^{ik_1 Q_{\varepsilon 1}^{\eta}} - 1] e^{ik_1 P_{\varepsilon 1}^{\eta} + i \sum_{j=2}^M k_j M_{\varepsilon j}}\} + \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^{\eta} + i \sum_{j=2}^M k_j M_{\varepsilon j}}\}.$$

Using the strong mixing condition (6), we find that

$$\left| \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^{\eta} + i \sum_{j=2}^M k_j M_{\varepsilon j}}\} - \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^{\eta}}\} \mathbb{E}\{e^{i \sum_{j=2}^M k_j M_{\varepsilon j}}\} \right| \lesssim \varphi\left(\frac{2\eta}{\varepsilon}\right).$$

Now we find that $\mathbb{E}\{Q_{\varepsilon j}^{\eta}\} = 0$ and $\mathbb{E}\{[Q_{\varepsilon j}^{\eta}]^2\} \lesssim \eta$. The latter result comes from integrating $\varepsilon^{-1} R\left(\frac{t-s}{\varepsilon}\right) ds dt$ over a cube of size $O(\eta^2)$. Since $|e^{ix} - 1| \lesssim |x|$, we deduce that

$$\left| \mathbb{E}\{[e^{ik_1 Q_{\varepsilon 1}^{\eta}} - 1] e^{ik_1 P_{\varepsilon 1}^{\eta} + iZ}\} \right| \leq \mathbb{E}\{[e^{ik_1 Q_{\varepsilon 1}^{\eta}} - 1]^2\}^{\frac{1}{2}} \lesssim \eta^{\frac{1}{2}},$$

for an arbitrary random variable Z (equal to 0 or to $\sum_{j=2}^M k_j M_{\varepsilon j}$ here). Thus,

$$\left| \mathbb{E}\{e^{ik_1 M_{\varepsilon 1} + i \sum_{j=2}^M k_j M_{\varepsilon j}}\} - \mathbb{E}\{e^{ik_1 M_{\varepsilon 1}}\} \mathbb{E}\{e^{i \sum_{j=2}^M k_j M_{\varepsilon j}}\} \right| \lesssim \varphi\left(\frac{2\eta}{\varepsilon}\right) + \eta^{\frac{1}{2}}.$$

By induction, we thus find that for all $0 < \eta < \frac{h}{2}$,

$$\mathcal{E} \lesssim M \varphi\left(\frac{2\eta}{\varepsilon}\right) + \eta^{\frac{1}{2}}.$$

This expression tends to 0 say for $\eta = \varepsilon^{\frac{1}{2}}$. This shows that the random variables $M_{\varepsilon j}$ become independent as $\varepsilon \rightarrow 0$. We show below that each $M_{\varepsilon j}$ converges to a centered Gaussian variable as $\varepsilon \rightarrow 0$. The sum over j thus yields in the limit a centered Gaussian variable with variance the sum of the M individual variances.

Central Limit Theorem for discrete random variables. By stationarity of the process $q(x, \omega)$, we are thus led to showing that

$$\int_0^h \frac{1}{\sqrt{\varepsilon}} q\left(\frac{y}{\varepsilon}\right) dy \xrightarrow{\text{dist.}} \int_0^h \sigma dW_y = \sigma W_h = \sigma \mathcal{N}(0, h), \quad \varepsilon \rightarrow 0,$$

where $\mathcal{N}(0, h)$ is the centered Gaussian variable with variance h . We break up h into $N = h/\varepsilon$ (which we assume is an integer) intervals and call

$$q_j = \int_{(j-1)\varepsilon}^{j\varepsilon} \frac{1}{\varepsilon} q\left(\frac{y}{\varepsilon}\right) dy = \int_{j-1}^j q(y) dy, \quad j \in \mathbb{Z}.$$

The q_j are stationary mixing random variables and we are interested in the limit

$$\sqrt{\varepsilon} \sum_{j=1}^N q_j = \frac{\sqrt{h}}{\sqrt{N}} \sum_{j=1}^N q_j. \quad (37)$$

Following remark 3 in [10], we introduce \mathcal{A}_m and \mathcal{A}^m as the σ -algebras generated by $(q_j)_{j \leq m}$ and $(q_j)_{j \geq m}$, respectively. Let then

$$\rho(n) = \sup \left\{ \frac{\mathbb{E}\{(\eta - \mathbb{E}\{\eta\})(\xi - \mathbb{E}\{\xi\})\}}{(\mathbb{E}\{\eta^2\} \mathbb{E}\{\xi^2\})^{\frac{1}{2}}}; \eta \in L^2(\mathcal{A}_0), \quad \xi \in L^2(\mathcal{A}^n) \right\}. \quad (38)$$

Then provided that $\sum_{n \geq 1} \rho(n) < \infty$, we obtain the following central limit theorem

$$\frac{\sqrt{h}}{\sqrt{N}} \sum_{j=1}^N q_j \xrightarrow{\text{dist.}} \sqrt{h} \sigma \mathcal{N}(0, 1) \equiv \sigma \mathcal{N}(0, h), \quad (39)$$

where $\mathcal{N}(0, 1)$ is the standard normal variable, where \equiv is used to mean equality in distribution, and where $\sigma^2 = \sum_{n \in \mathbb{Z}} \mathbb{E}\{q_0 q_n\}$. It remains to verify that the two definitions of σ above and in (31) agree and that $\sum_{n \geq 1} \rho(n) < \infty$. Note that

$$\sum_{n \in \mathbb{Z}} \mathbb{E}\{q_0 q_n\} = \int_0^1 \int_{-\infty}^{\infty} \mathbb{E}\{q(y)q(z)\} dy dz = \int_0^1 \int_{-\infty}^{\infty} \mathbb{E}\{q(y)q(y+z)\} dy dz = \int_0^1 \hat{R}(0) dy = \hat{R}(0),$$

thanks to (31). Now we observe that $\rho(n) \leq \varphi(n-1)$ so that summability of $\rho(n)$ is implied by the integrability of $\varphi(r)$ on \mathbb{R}^+ . This concludes the proof of the convergence in distribution of $u_{1\varepsilon}$ in the space of continuous paths $\mathcal{C}(D)$.

It now remains to recall the convergence result (27) to obtain (33) in the space of integrable paths. \square

2.3 Oscillatory integral in arbitrary space dimensions

In dimension $1 \leq d \leq 3$, the leading term in the corrector $\varepsilon^{-\frac{d}{2}}(u_\varepsilon - u_0)$ is given by:

$$u_{1\varepsilon}(\mathbf{x}, \omega) = \int_D -G(\mathbf{x}, \mathbf{y}) \frac{1}{\varepsilon^{\frac{d}{2}}} q_\varepsilon(\mathbf{y}, \omega) u_0(\mathbf{y}) d\mathbf{y}. \quad (40)$$

The variance of $u_{1\varepsilon}(\mathbf{x}, \omega)$ is given by

$$\mathbb{E}\{u_{1\varepsilon}^2(\mathbf{x}, \omega)\} = \int_{D^2} G(\mathbf{x}, \mathbf{y})G(\mathbf{x}, \mathbf{z})\frac{1}{\varepsilon^d}R\left(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\right)u_0(\mathbf{y})u_0(\mathbf{z})d\mathbf{y}d\mathbf{z}.$$

As in the one-dimensional case, it converges as $\varepsilon \rightarrow 0$ to the limit

$$\mathbb{E}\{u_1^2(\mathbf{x}, \omega)\} = \sigma^2 \int_D G^2(\mathbf{x}, \mathbf{y})u_0^2(\mathbf{y})d\mathbf{y}, \quad \sigma^2 = \int_{\mathbb{R}^d} \mathbb{E}\{q(\mathbf{0})q(\mathbf{y})\}d\mathbf{y}. \quad (41)$$

Because of the singularities of the Green's function $G(\mathbf{x}, \mathbf{y})$ in dimension $d \geq 2$, we prove here less accurate results than those obtained in dimension $d = 1$ in the preceding section.

We want to obtain convergence of the above corrector in distribution on $(\Omega, \mathcal{F}, \mathbb{P})$ and weakly in D . More precisely, let $M_k(\mathbf{x})$ for $1 \leq k \leq K$ be sufficiently smooth functions such that

$$m_k(\mathbf{y}) = - \int_D M_k(\mathbf{x})G(\mathbf{x}, \mathbf{y})u_0(\mathbf{y})d\mathbf{x} = -\mathcal{G}M_k(\mathbf{y})u_0(\mathbf{y}), \quad 1 \leq k \leq K, \quad (42)$$

are continuous functions (we thus assume that $u_0(\mathbf{x})$ is continuous as well). Let us introduce the random variables

$$I_{k\varepsilon}(\omega) = \int_D m_k(\mathbf{y})\frac{1}{\varepsilon^{\frac{d}{2}}}q\left(\frac{\mathbf{y}}{\varepsilon}, \omega\right)d\mathbf{y}. \quad (43)$$

Because of (18), the accumulation points of the integrals $I_{k\varepsilon}(\omega)$ are not modified if $q\left(\frac{\mathbf{y}}{\varepsilon}, \omega\right)$ is replaced by $q_\varepsilon(\mathbf{y}, \omega)$. The main result of this section is the following:

Theorem 2.10 *Under the above conditions and the hypotheses of Proposition 2.7, the random variables $I_{k\varepsilon}(\omega)$ converge in distribution to the mean zero Gaussian random variables $I_k(\omega)$ as $\varepsilon \rightarrow 0$, where the correlation matrix is given by*

$$\Sigma_{jk} = \mathbb{E}\{I_j I_k\} = \sigma^2 \int_D m_j(\mathbf{y})m_k(\mathbf{y})d\mathbf{y}, \quad (44)$$

where σ is given by

$$\sigma^2 = \int_{\mathbb{R}^d} \mathbb{E}\{q(\mathbf{0})q(\mathbf{y})\}d\mathbf{y}. \quad (45)$$

Moreover, we have the stochastic representation

$$I_k(\omega) = \int_D m_k(\mathbf{y})\sigma dW_{\mathbf{y}}, \quad (46)$$

where $dW_{\mathbf{y}}$ is standard multi-parameter Wiener process [34].

As a result, for $M(\mathbf{x})$ sufficiently smooth, we obtain that

$$\left(\frac{u_\varepsilon - u_0}{\varepsilon^{\frac{d}{2}}}, M\right) \xrightarrow{\text{dist.}} -\sigma \int_D \mathcal{G}M(\mathbf{y})\mathcal{G}f(\mathbf{y})dW_{\mathbf{y}}. \quad (47)$$

Proof. The convergence in (47) is a direct consequence of (46) since

$$\int_{D^2} M(\mathbf{x})G(\mathbf{x}, \mathbf{y})u_0(\mathbf{y})dW_{\mathbf{y}}d\mathbf{x} = \int_D \mathcal{G}M(\mathbf{y})\mathcal{G}f(\mathbf{y})dW_{\mathbf{y}},$$

and of the strong convergence (27) in Proposition 2.7. The equality (46) is directly deduced from (44) since $I_k(\omega)$ is a (multivariate) Gaussian variable. In order to prove (44), we use a methodology similar to that in the proof of Theorem 2.8.

The characteristic function of the random variables $I_{k\varepsilon}(\omega)$ is given by

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{i\sum_{k=1}^K k_j I_{j\varepsilon}(\omega)}\}, \quad \mathbf{k} = (k_1, \dots, k_K),$$

and may be recast as

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{i\int_D m(\mathbf{y})\varepsilon^{-\frac{d}{2}}q(\frac{\mathbf{y}}{\varepsilon}, \omega)d\mathbf{y}}\}, \quad m(\mathbf{y}) = \sum_{j=1}^K k_j m_j(\mathbf{y}).$$

So (44) follows from showing that

$$I_\varepsilon(\omega) = \int_D m(\mathbf{y})\frac{1}{\varepsilon^{\frac{d}{2}}}q\left(\frac{\mathbf{y}}{\varepsilon}, \omega\right)d\mathbf{y} \xrightarrow{\text{dist.}} \int_D m(\mathbf{y})\sigma dW_{\mathbf{y}}, \quad (48)$$

for an arbitrary continuous function $m(\mathbf{y})$. As in the one-dimensional case and for the same reasons, we replace $m(\mathbf{y})$ by $m_h(\mathbf{y})$, which is constant on small hyper-cubes \mathcal{C}_j of size h (and volume h^d) and that there are $M \approx h^{-d}$ of them. Because ∂D is assumed to be sufficiently smooth, it can be covered by $M_S \approx h^{-d+1}$ cubes and we set $m_h(\mathbf{x}) = 0$ on those cubes. The contribution to $I_\varepsilon(\omega)$ is seen to converge to 0 as $h \rightarrow 0$ in the mean-square sense as in (36).

We define the random variables

$$M_{\varepsilon j}(\omega) = m_{h_j} \int_{\mathcal{C}_j} \frac{1}{\varepsilon^{\frac{d}{2}}}q\left(\frac{\mathbf{y}}{\varepsilon}, \omega\right)d\mathbf{y}, \quad 1 \leq j \leq M,$$

where m_{h_j} is the value of m_h on \mathcal{C}_j and are interested in the limiting distribution as $\varepsilon \rightarrow 0$ of the random variable

$$I_\varepsilon^h(\omega) = \sum_{j=1}^M M_{\varepsilon j}(\omega). \quad (49)$$

We show below that these random variables are again independent in the limit $\varepsilon \rightarrow 0$ and each variable converges to a centered Gaussian variable. As a consequence, $I_\varepsilon^h(\omega)$ converges in distribution to a centered Gaussian variable whose variance is the sum of the variances of the variables $M_{\varepsilon j}(\omega)$ in the limit $\varepsilon \rightarrow 0$.

That the random variables $M_{\varepsilon j}$ are independent in the limit $\varepsilon \rightarrow 0$ is shown using a similar method to that of the one-dimensional case. We want to obtain that

$$\mathcal{E}(\mathbf{k}) = \left| \mathbb{E}\{e^{i\sum_{j=1}^M k_j M_{\varepsilon j}}\} - \prod_{j=1}^M \mathbb{E}\{e^{ik_j M_{\varepsilon j}}\} \right| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad \text{for all } \mathbf{k} = \{k_j\}_j \in \mathbb{R}^M.$$

Let $0 < \eta < \frac{h}{2}$ and $\mathcal{D}_j^\eta = \{\mathbf{x} \in \mathcal{C}_j; d(\mathbf{x}, \partial\mathcal{C}_j) > \eta\}$. We define

$$P_{\varepsilon j}^\eta = m_{hj} \int_{\mathcal{D}_j^\eta} \frac{1}{\varepsilon^{\frac{d}{2}}} q\left(\frac{\mathbf{y}}{\varepsilon}, \omega\right) d\mathbf{y}, \quad Q_{\varepsilon j}^\eta = M_{\varepsilon j} - P_{\varepsilon j}^\eta.$$

We write again:

$$\mathbb{E}\{e^{i \sum_{j=1}^M k_j M_{\varepsilon j}}\} = \mathbb{E}\{[e^{ik_1 Q_{\varepsilon 1}^\eta} - 1] e^{ik_1 P_{\varepsilon 1}^\eta + i \sum_{j=2}^M k_j M_{\varepsilon j}}\} + \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^\eta + i \sum_{j=2}^M k_j M_{\varepsilon j}}\}.$$

Using the strong mixing condition (6), we find that

$$\left| \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^\eta + i \sum_{j=2}^M k_j M_{\varepsilon j}}\} - \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^\eta}\} \mathbb{E}\{e^{i \sum_{j=2}^M k_j M_{\varepsilon j}}\} \right| \lesssim \varphi\left(\frac{2\eta}{\varepsilon}\right).$$

We find as in the one-dimensional case that $\mathbb{E}\{Q_{\varepsilon j}^\eta\} = 0$ and $\mathbb{E}\{[Q_{\varepsilon j}^\eta]^2\} \lesssim \eta h^{(d-1)} \lesssim \eta$ with a bound independent of ε . This comes from integrating $\varepsilon^{-d} R(\frac{\mathbf{x}-\mathbf{y}}{\varepsilon}) d\mathbf{x} d\mathbf{y}$ on a domain of size $O([\eta h^{d-1}]^2)$. The rest of the proof follows as in the one-dimensional case.

It remains to address the convergence of $M_{\varepsilon j}$ as $\varepsilon \rightarrow 0$. By invariance of $q(\mathbf{x})$, it is sufficient to consider integrals on the cube $[\mathbf{0}, \mathbf{h}]$, with $\mathbf{h} = (h, \dots, h)$. It now remains to show that

$$\int_{[\mathbf{0}, \mathbf{h}]} \frac{1}{\varepsilon^{\frac{d}{2}}} q\left(\frac{\mathbf{y}}{\varepsilon}, \omega\right) d\mathbf{y} \xrightarrow{\text{dist.}} \sigma \int_{[\mathbf{0}, \mathbf{h}]} dW_{\mathbf{y}} = \sigma \mathcal{N}(0, h^d). \quad (50)$$

For a multi-index $\mathbf{j} \in \mathbb{Z}^d$, we define

$$q_{\mathbf{j}}(\omega) = \int_{\mathbf{j} + [\mathbf{0}, \mathbf{1}]} q(\mathbf{y}, \omega) d\mathbf{y}.$$

Then (50) will follow by homogeneity if we can show that

$$\frac{1}{\sigma n^{\frac{d}{2}}} \sum_{\mathbf{j} \in [\mathbf{0}, \mathbf{n}]} q_{\mathbf{j}} \xrightarrow{\text{dist.}} \mathcal{N}(0, 1). \quad (51)$$

The latter result is proved in e.g. [10, 20]. The results in these references are stated in terms of α -mixing coefficients. Since α coefficients are bounded by ρ coefficients [20, p.4], we state the results in terms of less optimal ρ -mixing coefficients; see also [14].

Let A and B be subsets of \mathbb{Z}^d and let \mathcal{A} and \mathcal{B} be the σ algebras generated by $q_{\mathbf{j}}$ on A and B , respectively. Then we define

$$\rho(n) = \sup \left\{ \frac{\mathbb{E}\{(\eta - \mathbb{E}\{\eta\})(\xi - \mathbb{E}\{\xi\})\}}{(\mathbb{E}\{\eta^2\} \mathbb{E}\{\xi^2\})^{\frac{1}{2}}}; \eta \in L^2(\mathcal{A}), \quad \xi \in L^2(\mathcal{B}), \quad d(A, B) \geq n \right\}.$$

We then assume that $\mathbb{E}\{q_{\mathbf{j}}^6\} < \infty$ as in hypothesis [H2] and that $\rho(n) = o(n^{-d})$ and that

$$\sum_{n=0}^{\infty} n^{d-1} \rho^{\frac{1}{2}}(n) < \infty. \quad (52)$$

Then we verify that the hypotheses in [10] (see also [20, p.48]) are satisfied so that (51) holds with

$$\sigma^2 = \sum_{\mathbf{j} \in \mathbb{Z}^d} \mathbb{E}\{q_{\mathbf{0}} q_{\mathbf{j}}\}.$$

We verify as in the one-dimensional case that the above σ agrees with that in definition (45). Now we verify that (52) is a consequence of the integrability of $r^{d-1} \varphi^{\frac{1}{2}}(r)$. The decay $\rho(n) = o(n^{-d})$ is obtained when $\varphi(r)$ decays faster than $r^{-d-\eta}$ for some $\eta > 0$; see Remark 2.3. \square

2.4 Larger fluctuations, random and periodic homogenization

We now consider several generalizations of the results presented in earlier sections and compare homogenization in periodic and random media. The results stated in the preceding sections, corresponding to the case $\alpha = 0$ below, generalize to larger fluctuations of the form:

$$\tilde{q}_\varepsilon(\mathbf{x}, \omega) = \frac{1}{\varepsilon^{\alpha d}} q\left(\frac{\mathbf{x}}{\varepsilon}, \omega\right). \quad (53)$$

The corrector $-\mathcal{G}q_\varepsilon\mathcal{G}f$ is now of order $\varepsilon^{d(\frac{1}{2}-\alpha)}$ for $0 \leq \alpha < \frac{1}{2}$. The next-order corrector, given by $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f$ in (22), is bounded in $L^1(\Omega \times D)$ by $\varepsilon^{d(\frac{1+\eta}{2+\eta}-2\alpha)}$ according to Lemma 2.6. The order of this term is smaller than the order of the leading corrector $\varepsilon^{d(\frac{1}{2}-\alpha)}$ again provided that $0 \leq \alpha < \frac{\eta}{2(2+\eta)}$, which converges to $\frac{1}{2}$ for $d = 1, 2$ as $\eta \rightarrow \infty$ and converges to $\frac{1}{6}$ for $d = 3$ as $\eta \rightarrow 1$.

In dimensions $d = 1, 2$, we can infer from these results that $\varepsilon^{-d(\frac{1}{2}-\alpha)}(u_\varepsilon - u_0)$ converges in distribution to the limits obtained in the preceding sections as $\varepsilon \rightarrow 0$ provided that $0 \leq \alpha < \frac{1}{2}$. The proof presented in this paper extends to the values $0 \leq \alpha < \frac{1}{4}$. Indeed, the proof is based on imposing that the spectral radius of $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon$ is sufficiently small using (13) in Lemma 2.2, which for (53), translates into $\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}(L^2(D))}^2\} \lesssim \varepsilon^{d(1-4\alpha)}$. We then verify that all results leading to Proposition 2.7 generalize when $0 < \alpha < \frac{1}{4}$ to yield (27) with $\varepsilon^{\frac{d}{2}}$ replaced by $\varepsilon^{d(\frac{1}{2}-\alpha)}$. A proof of convergence for $0 \leq \alpha < \frac{1}{2}$ would presumably require us to analyze all the terms in the formal expansion

$$u_\varepsilon = \sum_{k=0}^{\infty} (-\mathcal{G}q_\varepsilon)^k \mathcal{G}f, \quad (54)$$

something we do not address here. In the limiting case $\alpha = \frac{1}{2}$, the above theory breaks down and u_ε no longer converges to the deterministic solution u_0 as is shown in the temporal one-dimensional case in [44].

The results on the corrector $u_\varepsilon - u_0$ obtained in the preceding sections, namely Theorems 2.8 and 2.10 are valid for $1 \leq d \leq 3$. If we admit that the expansion (54) involves a first term u_0 , a second term $-\mathcal{G}q_\varepsilon u_0$, and smaller order terms, then the results obtained in Theorem 2.10 show that $u_\varepsilon - u_0$ converges weakly in space and in distribution to a process of order $O(\varepsilon^{\frac{d}{2}})$. The critical case $d = 4$ yields a correction of order ε^2 , whereas $\varepsilon^{\frac{d}{2}}$ would be even smaller for $d \geq 5$.

The theory presented in this paper does not allow us to justify (54) when $d \geq 4$ because the corresponding Green's function are no longer square integrable. Another argument shows that corrections of order ε^2 correspond to a transition and that we should not expect quite the same results for $d \leq 3$ and $d \geq 4$. Indeed, let us consider the problem in the periodic case:

$$\begin{aligned} -\Delta u_\varepsilon + q\left(\frac{\mathbf{x}}{\varepsilon}\right)u_\varepsilon &= f & D \\ u_\varepsilon &= 0 & \partial D, \end{aligned} \quad (55)$$

on a smooth open, bounded, domain $D \subset \mathbb{R}^d$, where $q(\mathbf{y})$ is $[0, 1]^d$ -periodic. Then following [7], we introduce the fast scale $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$ and introduce a function $u_\varepsilon = u_\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$.

Gradients $\nabla_{\mathbf{x}}$ become $\frac{1}{\varepsilon}\nabla_{\mathbf{y}} + \nabla_{\mathbf{x}}$ and (55) becomes formally

$$\left(-\frac{1}{\varepsilon^2}\Delta_{\mathbf{y}} - \frac{2}{\varepsilon}\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}} - \Delta_{\mathbf{x}} + q(\mathbf{y})\right)u_{\varepsilon}(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}).$$

Plugging the expansion $u_{\varepsilon} = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$ into the above equality and equating like powers of ε yields three equations. The first equation shows that $u_0 = u_0(\mathbf{x})$. The second equation shows that $u_1 = u_1(\mathbf{x})$, which we can choose as $u_1 \equiv 0$. The third equation $-\Delta_{\mathbf{y}}u_2 - \Delta_{\mathbf{x}}u_0 + q(\mathbf{y})u_0 = f(\mathbf{x})$, admits a solution provided that

$$-\Delta_{\mathbf{x}}u_0 + \langle q \rangle u_0 = f(\mathbf{x}), \quad D$$

with $u_0 = 0$ on ∂D . Here, $\langle q \rangle$ is the average of q on $[0, 1]^d$, which we assume is sufficiently large that the above equation admits a unique solution. We recast the above equation as $u_0 = \mathcal{G}_D f$. The corrector u_2 thus solves

$$-\Delta_{\mathbf{y}}u_2 = \left(\langle q \rangle - q(\mathbf{y})\right)u_0(\mathbf{x}),$$

and is uniquely defined along with the constraint $\langle u_2 \rangle = 0$. We denote the solution operator of the above cell problem as $\mathcal{G}_{\#}$ so that $u_2 = -\mathcal{G}_{\#}(q - \langle q \rangle)\mathcal{G}f$. Thus formally, we have obtained that

$$u_{\varepsilon}(\mathbf{x}) = \mathcal{G}f(\mathbf{x}) - \varepsilon^2 \mathcal{G}_{\#}(q - \langle q \rangle)\left(\frac{\mathbf{x}}{\varepsilon}\right)\mathcal{G}f(\mathbf{x}) + \text{l.o.t.} \quad (56)$$

We thus observe that the corrector $u_{2\varepsilon}(\mathbf{x}) := u_2(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ is of order $O(\varepsilon^2)$ in the L^2 sense, say. In the sense of distributions, however, the corrector may be of order $o(\varepsilon^m)$ for all integer m in the sense that $\int_D M(\mathbf{x})u_{2\varepsilon}(\mathbf{x})d\mathbf{x} \ll \varepsilon^m$ for all m when $M(\mathbf{x})u_0(\mathbf{x}) \in \mathcal{C}_0^{\infty}(D)$.

A similar behavior occurs for the random corrector

$$v_{1\varepsilon}(\mathbf{x}, \omega) = \int_D -G(\mathbf{x}, \mathbf{y})q\left(\frac{\mathbf{y}}{\varepsilon}, \omega\right)u_0(\mathbf{y})d\mathbf{y}. \quad (57)$$

Theorem 2.10 shows that $(v_{1\varepsilon}, M(\mathbf{x}))$ is of order $O(\varepsilon^{\frac{d}{2}})$ for $M(\mathbf{x})$ and $u_0(\mathbf{x})$ sufficiently smooth and that $\varepsilon^{-\frac{d}{2}}(v_{1\varepsilon}, M(\mathbf{x}))$ converges in distribution to a Gaussian random variable. This result, however, does not hold in the $L^2(D)$ -sense for $d \geq 4$ when $G(\mathbf{x}, \mathbf{y})$ is the fundamental solution of the Helmholtz equation $-\Delta + q_0(\mathbf{x})$ on D . Indeed, we can prove that

Proposition 2.11 *Provided that $u_0(\mathbf{x})$ and $\hat{R}(\boldsymbol{\xi})$ are sufficiently smooth, we obtain that:*

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x}, \omega)\} \sim \begin{cases} \varepsilon^d \hat{R}(\mathbf{0}) \int_D G^2(\mathbf{x}, \mathbf{y})u_0^2(\mathbf{y})d\mathbf{y} & 1 \leq d \leq 3 \\ \varepsilon^4 |\ln \varepsilon| \frac{(2\pi)^4 \hat{R}(\mathbf{0})}{c_4} u_0^2(\mathbf{x}) & d = 4 \\ \varepsilon^4 u_0^2(\mathbf{x})(2\pi)^d \int_{\mathbb{R}^d} \frac{\hat{R}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^4} d\boldsymbol{\xi} & d \geq 5. \end{cases} \quad (58)$$

Here $a_{\varepsilon} \sim b_{\varepsilon}$ means $a_{\varepsilon} = b_{\varepsilon}(1 + o(1))$.

Proof. We calculate:

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x}, \omega)\} = \int_{D^2} G(\mathbf{x}, \mathbf{y})G(\mathbf{x}, \mathbf{z})R\left(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\right)u_0(\mathbf{y})u_0(\mathbf{z})d\mathbf{y}d\mathbf{z}. \quad (59)$$

Extending $G(\mathbf{x}, \cdot)$ by 0 outside of D , by the Parseval equality this is equal to

$$(2\pi)^d \int_{\mathbb{R}^{2d}} |\mathcal{F}_{\mathbf{y} \rightarrow \boldsymbol{\xi}}(G(\mathbf{x}, \mathbf{y})u_0(\mathbf{y}))|^2(\boldsymbol{\xi})\varepsilon^d \hat{R}(\varepsilon\boldsymbol{\xi})d\boldsymbol{\xi},$$

where $\mathcal{F}_{\mathbf{x} \rightarrow \boldsymbol{\xi}}$ is the Fourier transform from \mathbf{x} to $\boldsymbol{\xi}$. In dimension $1 \leq d \leq 3$, since $\hat{R}(\varepsilon\boldsymbol{\xi}) \rightarrow \hat{R}(\mathbf{0})$ pointwise, the Lebesgue dominated convergence theorem yields the result. In dimension $d \geq 4$, however, the Green function is no longer square integrable and the integral is larger than ε^d .

Let us consider the cases $d \geq 4$. We first replace $G(\mathbf{x}, \mathbf{y})$ by $c_d|\mathbf{x}-\mathbf{y}|^{2-d}$ where c_d is the measure of the unit sphere S^{d-1} . The difference $G(\mathbf{x}, \mathbf{y}) - c_d|\mathbf{x}-\mathbf{y}|^{2-d}$ is a function bounded by $C|\mathbf{x}-\mathbf{y}|^{3-d}$, which yields a smaller contribution to $\mathbb{E}\{v_{1\varepsilon}^2\}$. We leave the details to the reader. We also replace $u_0(\mathbf{y})$ by $u_0(\mathbf{x})$, up to an error bounded by $|\mathbf{x}-\mathbf{y}|^\alpha$ as soon as $u_0(\mathbf{x})$ is of class $C^{0,\alpha}(D)$. This contribution again provides a lower order term to $\mathbb{E}\{v_{1\varepsilon}^2\}$. Similarly, we replace $u_0(\mathbf{z})$ by $u_0(\mathbf{x})$ and thus obtain that

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x}, \omega)\} \sim u_0^2(\mathbf{x}) \int_{D^2} \frac{1}{c_d|\mathbf{x}-\mathbf{y}|^{d-2}} \frac{1}{c_d|\mathbf{x}-\mathbf{z}|^{d-2}} R\left(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\right) d\mathbf{y}d\mathbf{z}.$$

Let $\alpha > 0$ and $B(\mathbf{x}, \alpha)$ the ball of center \mathbf{x} and radius α so that $B(\mathbf{x}, \alpha) \subset D$. Because all singularities occur when \mathbf{y} and \mathbf{z} are in the vicinity of \mathbf{x} , we use the proof of the case $1 \leq d \leq 3$ to show that up to a term of order ε^d , we can replace D by $B(\mathbf{x}, \alpha)$ so that

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x}, \omega)\} \sim u_0^2(\mathbf{x}) \int_{B^2(\mathbf{0}, \alpha)} \frac{1}{c_d|\mathbf{y}|^{d-2}} \frac{1}{c_d|\mathbf{z}|^{d-2}} R\left(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\right) d\mathbf{y}d\mathbf{z}. \quad (60)$$

Now for $d \geq 5$, using the dominated convergence theorem, we can replace $B(\mathbf{0}, \alpha)$ by \mathbb{R}^d because the Green function is square integrable at infinity, whence

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x}, \omega)\} \sim u_0^2(\mathbf{x}) \int_{\mathbb{R}^{2d}} \frac{1}{c_d|\mathbf{y}|^{d-2}} \frac{1}{c_d|\mathbf{z}|^{d-2}} R\left(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\right) d\mathbf{y}d\mathbf{z}.$$

This, however, by the Parseval equality, is equal to

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x}, \omega)\} \sim u_0^2(\mathbf{x})(2\pi)^d \int_{\mathbb{R}^d} \frac{1}{|\boldsymbol{\xi}|^4} \varepsilon^d \hat{R}(\varepsilon\boldsymbol{\xi})d\boldsymbol{\xi} = u_0^2(\mathbf{x})(2\pi)^d \int_{\mathbb{R}^d} \frac{1}{|\boldsymbol{\xi}|^4} \varepsilon^4 \hat{R}(\boldsymbol{\xi})d\boldsymbol{\xi},$$

since the Fourier transform of the fundamental solution of the Laplacian is $|\boldsymbol{\xi}|^{-2}$.

When $d = 4$, we come back to (60), and replace one of the integrals (in \mathbf{z}) on $B(\mathbf{0}, \alpha)$ by an integral on \mathbb{R}^d using again the dominated convergence theorem and the other integral by an integration on $B_\varepsilon^\alpha = B(\mathbf{0}, \alpha) \cap B(\mathbf{0}, \varepsilon)$, with an error that we can verify is of order $O(\varepsilon^4)$. This yields the term

$$\begin{aligned} & \int_{B(\mathbf{0}, \alpha) \times \mathbb{R}^d} \frac{1}{c_4^2|\mathbf{y}|^2|\mathbf{z}|^2} R\left(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\right) d\mathbf{y}d\mathbf{z} = \int_{B_\varepsilon^\alpha \times \mathbb{R}^d} \frac{(2\pi)^4 \varepsilon^2}{c_4|\mathbf{y}|^2|\boldsymbol{\xi}|^2} \hat{R}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \mathbf{y}} d\boldsymbol{\xi}d\mathbf{y} + O(\varepsilon^4) \\ & = \int_{B_1^\alpha \times \mathbb{R}^d} \frac{(2\pi\varepsilon)^4}{c_4|\mathbf{y}|^2|\boldsymbol{\xi}|^2} \hat{R}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \mathbf{y}} d\boldsymbol{\xi}d\mathbf{y} + O(\varepsilon^4) = \hat{R}(\mathbf{0})(2\pi\varepsilon)^4 \int_{B_1^\alpha} \frac{1}{c_4^2|\mathbf{y}|^4} d\mathbf{y} + O(\varepsilon^4) \\ & = \frac{\hat{R}(\mathbf{0})(2\pi\varepsilon)^4}{c_4} \int_1^{\frac{\alpha}{\varepsilon}} \frac{|\mathbf{y}|^3}{|\mathbf{y}|^4} d|\mathbf{y}| + O(\varepsilon^4) = \frac{\hat{R}(\mathbf{0})(2\pi\varepsilon)^4}{c_4} |\ln \varepsilon| + O(\varepsilon^4). \end{aligned}$$

Here, we have assumed that $|\hat{R}(\boldsymbol{\xi}) - \hat{R}(\mathbf{0})|$ was bounded by $C|\boldsymbol{\xi}|^\beta$ for some $\beta > 0$. \square

In all dimensions, we thus obtain that $\varepsilon^{-\frac{d}{2}}v_{1\varepsilon}(\mathbf{x}, \omega)$ converges (weakly and in distribution) to a limit $u_1(\mathbf{x}, \omega) = -\int_D G(\mathbf{x}, \mathbf{y})u_0(\mathbf{y})dW_{\mathbf{y}}$. In dimensions $1 \leq d \leq 3$, we have proved that u_1 was the limit of the corrector to homogenization $\varepsilon^{-\frac{d}{2}}(u_\varepsilon - u_0)$. The above calculation shows that the limit u_1 captures all the energy in the oscillations of the homogenization corrector $v_{1\varepsilon}$ in the sense that $\varepsilon^{-d}\mathbb{E}\{\|v_{1\varepsilon}\|_{L^2(D)}^2\}$ converges to $\mathbb{E}\{\|u_1\|_{L^2(D)}^2\}$.

In higher dimensions $d \geq 4$, as in the case of homogenization in periodic media, some energy is lost while passing to the (weak) limit. The corrector $u_{1\varepsilon} = \varepsilon^{-\frac{d}{2}}v_{1\varepsilon}$ converges weakly and in distribution to the limit u_1 . However, while the energy of the asymptotic corrector is $\varepsilon^{\frac{d}{2}}(\mathbb{E}\{\|u_1\|_{L^2(D)}^2\})^{\frac{1}{2}}$, the energy of the true corrector $(\mathbb{E}\{\|v_{1\varepsilon}\|_{L^2(D)}^2\})^{\frac{1}{2}}$ is of order $O(\varepsilon^2)$ for $d \geq 5$ and of order $O(\varepsilon^2|\ln \varepsilon|^{\frac{1}{2}})$ for $d = 4$. Most of the energy of the correctors is lost in passing from $u_{1\varepsilon}$ to its weak limit u_1 .

3 Correctors for one-dimensional elliptic problems

In this section, we consider the homogenization of the following one-dimensional elliptic problems:

$$\begin{aligned} -\frac{d}{dx}a_\varepsilon(x, \omega)\frac{d}{dx}u_\varepsilon + (q_0 + q_\varepsilon(x, \omega))u_\varepsilon &= \rho_\varepsilon(x, \omega)f(x), & x \in D = (0, 1), \\ u_\varepsilon(0) = u_\varepsilon(1) &= 0. \end{aligned} \quad (61)$$

We consider homogeneous Dirichlet conditions to simplify the presentation. The coefficients $a_\varepsilon(x, \omega)$ and $\rho_\varepsilon(x, \omega)$ are uniformly bounded from above and below: $0 < a_0 \leq a_\varepsilon(x, \omega), \rho_\varepsilon(x, \omega) \leq a_0^{-1}$. The (deterministic) absorption term q_0 is assumed to be a non-negative constant. The generalization to a non-negative smooth function $q_0(x)$ can be done.

We assume that $a_\varepsilon(x, \omega) = a(\frac{x}{\varepsilon}, \omega)$, $q_\varepsilon(x, \omega) = q(\frac{x}{\varepsilon}, \omega)$, and $\rho_\varepsilon(x, \omega) = \rho(\frac{x}{\varepsilon}, \omega)$, where $a(x, \omega)$, $q(x, \omega)$, and $\rho(x, \omega)$ are strictly stationary processes on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will modify the mean-zero process $q_\varepsilon(x, \omega)$ as in the preceding section and assume here to simplify that $q(x, \omega)$ is bounded \mathbb{P} -a.s. We also assume that the cross-correlations $R_{fg}(\mathbf{x})$ are integrable for $\{f, g\} \in \{a, q, \rho\}$, where

$$R_{fg}(\mathbf{x}) = \mathbb{E}\{f(\mathbf{y}, \omega)g(\mathbf{y} + \mathbf{x}, \omega)\}. \quad (62)$$

We also assume that the coefficients are jointly strongly mixing in the sense of (6), where for two Borel sets A and B in \mathbb{R}^d , we denote by \mathcal{F}_A and \mathcal{F}_B the σ -algebras generated by the random fields $a(\mathbf{x}, \omega)$, $q(\mathbf{x}, \omega)$, and $\rho(\mathbf{x}, \omega)$. We still assume that the ρ -mixing coefficient $\varphi(r)$ is integrable and such that $\varphi^{\frac{1}{2}}$ is also integrable.

In the case where $q_\varepsilon = 0$ and $\rho_\varepsilon = 0$, the corrector to the homogenization limit u_0 has been considered in [12]. For general sufficiently mixing coefficients a_ε with positive variance $\sigma^2 = 2\int_0^\infty \mathbb{E}\{a(0)a(t)\}dt > 0$, we obtain that $u_\varepsilon - u_0$ is of order $\sqrt{\varepsilon}$ and converges in distribution to a Gaussian process. This section aims at generalizing the result to (61) using the results of the preceding section and a change of variables based on harmonic coordinates [35].

Let us introduce the change of variables

$$z_\varepsilon(x) = a^* \int_0^x \frac{1}{a_\varepsilon(t)} dt, \quad \frac{dz_\varepsilon}{dx} = \frac{a^*}{a_\varepsilon(x)}, \quad a^* = \frac{1}{\mathbb{E}\{a^{-1}\}}, \quad (63)$$

and $\tilde{u}_\varepsilon(z) = u_\varepsilon(x)$. Note that $\mathbb{E}\{z_\varepsilon(x)\} = x$. Then we find, with $x = x(z_\varepsilon)$ that

$$\begin{aligned} -(a^*)^2 \frac{d^2}{dz^2} \tilde{u}_\varepsilon + a^* q_0 \tilde{u}_\varepsilon + a_\varepsilon [(1 - a_\varepsilon^{-1} a^*) q_0 + q_\varepsilon] \tilde{u}_\varepsilon &= a_\varepsilon \rho_\varepsilon f, & 0 < z < z_\varepsilon(1) \\ \tilde{u}_\varepsilon(0) = \tilde{u}_\varepsilon(z_\varepsilon(1)) &= 0. \end{aligned} \quad (64)$$

Let us introduce the following Green's function

$$\begin{aligned} -a^* \frac{d^2}{dx^2} G(x, y; L) + q_0 G(x, y; L) &= \delta(x - y) \\ G(0, y; L) = G(L, y; L) &= 0, \end{aligned} \quad (65)$$

and set $G(x, y) = G(x, y; 1)$. Then, defining

$$\tilde{q}_\varepsilon(x, \omega) = (1 - a_\varepsilon^{-1}(x, \omega) a^*) q_0 + q_\varepsilon(x, \omega), \quad (66)$$

we find that

$$\begin{aligned} \tilde{u}_\varepsilon(z) &= \int_0^{z_\varepsilon(1)} G(z, y; z_\varepsilon(1)) (\rho_\varepsilon f - \tilde{q}_\varepsilon \tilde{u}_\varepsilon)(x(y)) \frac{a_\varepsilon}{a^*}(x(y)) dy, \\ u_\varepsilon(x) &= \int_0^1 G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1)) (\rho_\varepsilon f - \tilde{q}_\varepsilon u_\varepsilon)(y) dy. \end{aligned}$$

We recast the above equation as

$$u_\varepsilon(x, \omega) = \mathcal{G}_\varepsilon(\rho_\varepsilon f - \tilde{q}_\varepsilon u_\varepsilon), \quad \mathcal{G}_\varepsilon u(x) = \int_0^1 G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1)) u(y) dy. \quad (67)$$

After one more iteration, we obtain the following integral equation:

$$u_\varepsilon = \mathcal{G}_\varepsilon \rho_\varepsilon f - \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \rho_\varepsilon f + \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon u_\varepsilon. \quad (68)$$

Since $a_0 a^* x \leq z_\varepsilon(x, \omega) \leq a^* a_0^{-1} x$ \mathbb{P} -a.s., the Green's operator \mathcal{G}_ε is bounded \mathbb{P} -a.s. and the results of Lemma 2.2 generalize to the case where the operator \mathcal{G}_ε replaces \mathcal{G} . As in (15), we thus modify \tilde{q}_ε (i.e. we modify a_ε and q_ε) on a set of measure ε so that $\|\mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon\|_{\mathcal{L}} \leq r < 1$ and obtain that (18) holds.

Let us introduce the notation

$$\rho_\varepsilon = \bar{\rho} + \delta \rho_\varepsilon, \quad \bar{\rho} = \mathbb{E}\{\rho\}, \quad \mathcal{G}_\varepsilon = \mathcal{G} + \delta \mathcal{G}_\varepsilon, \quad \mathcal{G} = \mathbb{E}\{\mathcal{G}_\varepsilon\}, \quad u_0 = \mathcal{G} \bar{\rho} f. \quad (69)$$

We also define

$$\delta z_\varepsilon(x) = z_\varepsilon(x) - x = \int_0^x b\left(\frac{t}{\varepsilon}\right) dt, \quad b(t, \omega) = \frac{a^*}{a(t, \omega)} - 1. \quad (70)$$

We first obtain the

Lemma 3.1 *We have that*

$$\mathbb{E}\{|\delta z_\varepsilon(x)\delta z_\varepsilon(y)|\} \lesssim \varepsilon, \quad 0 \leq x, y \leq 1. \quad (71)$$

The operator \mathcal{G}_ε may be decomposed as

$$\mathcal{G}_\varepsilon = \mathcal{G} + \mathcal{G}_{1\varepsilon} + \mathcal{R}_\varepsilon, \quad (72)$$

where

$$\mathcal{G}_{1\varepsilon}f(x) = \int_0^1 \left(\delta z_\varepsilon(x) \frac{\partial}{\partial x} + \delta z_\varepsilon(y) \frac{\partial}{\partial y} + \delta z_\varepsilon(1) \frac{\partial}{\partial L} \right) G(x, y; 1) f(y) dy. \quad (73)$$

We also have the following estimates

$$\mathbb{E}\{\|\mathcal{G}_{1\varepsilon}\|^2\} \lesssim \varepsilon, \quad \mathbb{E}\{\|\mathcal{R}_\varepsilon\|\} \lesssim \varepsilon. \quad (74)$$

Proof. We first use the fact that

$$\mathbb{E}\{|\delta z_\varepsilon(x)\delta z_\varepsilon(y)|\} \leq \left(\mathbb{E}\{(\delta z_\varepsilon(x)\delta z_\varepsilon(y))^2\} \right)^{\frac{1}{2}}.$$

Denoting by $b_\varepsilon(x, \omega) = b(\frac{x}{\varepsilon}, \omega)$, we have to show that

$$\mathbb{E}\left\{ \int_0^x \int_0^x \int_0^y \int_0^y b_\varepsilon(z_1)b_\varepsilon(z_2)b_\varepsilon(z_3)b_\varepsilon(z_4) d[z_1 z_2 z_3 z_4] \right\} \lesssim \varepsilon^2.$$

Now using the mixing property of the mean-zero field b_ε and the integrability of $\varphi^{\frac{1}{2}}(r)$, we obtain the result using (9) as in the proof of Lemma 2.6.

The integral defining \mathcal{G}_ε is split into two contributions, according as $y < x$ or $y > x$. On these two intervals, $G(x, y; L)$ is twice differentiable, and we thus have the expansion

$$G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1)) = G(x, y; 1) + \left(\delta z_\varepsilon(x) \frac{\partial}{\partial x} + \delta z_\varepsilon(y) \frac{\partial}{\partial y} + \delta z_\varepsilon(1) \frac{\partial}{\partial L} \right) G(x, y; 1) + r_\varepsilon,$$

where the Lagrange remainder $r_\varepsilon = r_\varepsilon(x, z_\varepsilon(x), y, z_\varepsilon(y), z_\varepsilon(1))$ is quadratic in the variables $(\delta z_\varepsilon(x), \delta z_\varepsilon(y), \delta z_\varepsilon(1))$ and involves second-order derivatives of $G(x, y; 1)$ at points (ξ, ζ, L) between $(x, y; 1)$ and $(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1))$.

From (71) and the fact that second-order derivatives of G are \mathbb{P} -a.s. uniformly bounded on each interval $y < x$ and $y > x$ (we use here again the fact that $a_0 a^* x \leq z_\varepsilon(x, \omega) \leq a^* a_0^{-1} x$ \mathbb{P} -a.s.), we thus obtain that $\mathbb{E}\{|r_\varepsilon(\cdot)|\} \lesssim \varepsilon$. This also shows the bound for $\mathbb{E}\{\|\mathcal{R}_\varepsilon\|\}$ in (74). The bound for $\mathbb{E}\{\|\mathcal{G}_{1\varepsilon}\|^2\}$ is obtained similarly. \square

Because we have assumed that \tilde{q}_ε and ρ_ε were bounded \mathbb{P} -a.s., we can replace \mathcal{G}_ε by $\mathcal{G} + \mathcal{G}_{1\varepsilon}$ in (68) up to an error of order ε in $L^1(\Omega; L^2(D))$. The case of q_ε and ρ_ε bounded on average would require to address their correlation with r_ε defined in the proof of the preceding lemma. This is not considered here.

We recast (68) as

$$u_\varepsilon - u_0 = (\mathcal{G}_\varepsilon \rho_\varepsilon - \mathcal{G} \bar{\rho})f - \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \rho_\varepsilon f + \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon (u_\varepsilon - u_0) + \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G} f. \quad (75)$$

Because $G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1))$ and ρ_ε are uniformly bounded \mathbb{P} -a.s., the proof of Lemma 2.2 generalizes to give us that

$$\mathbb{E}\{\|\mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon\|^2\} + \mathbb{E}\{\|\mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \rho_\varepsilon f\|^2\} + \mathbb{E}\{\|(\mathcal{G}_\varepsilon \rho_\varepsilon - \mathcal{G} \bar{\rho})f\|^2\} \lesssim \varepsilon. \quad (76)$$

So far, since moreover $\|\mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon\|_{\mathcal{L}} \leq r < 1$, we have thus obtained the following result:

Lemma 3.2 *Let u_ε be the solution to the heterogeneous problem (61) and u_0 the solution to the corresponding homogenized problem. Then we have that*

$$(\mathbb{E}\{\|u_\varepsilon - u_0\|^2\})^{\frac{1}{2}} \lesssim \sqrt{\varepsilon}\|f\|. \quad (77)$$

The estimate (24) with $d = 1$ is thus verified in the context of the elliptic equation (61). As a consequence, we find that $\mathbb{E}\{\|u_\varepsilon - u_0\|^2\} \lesssim \varepsilon$ so that by Cauchy-Schwarz and (76),

$$\mathbb{E}\{\|\mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon (u_\varepsilon - u_0)\|\} \lesssim \varepsilon.$$

It remains to exhibit the term of order $\sqrt{\varepsilon}$ in $u_\varepsilon - u_0$. Let us introduce the decomposition

$$\begin{aligned} u_\varepsilon - u_0 &= \left[\mathcal{G}_{1\varepsilon} \bar{\rho} + \mathcal{G} \delta \rho_\varepsilon - \mathcal{G} \tilde{q}_\varepsilon \mathcal{G} \bar{\rho} \right] f + s_\varepsilon, \\ s_\varepsilon &= (\delta \mathcal{G}_\varepsilon \delta \rho_\varepsilon + \mathcal{R}_\varepsilon \bar{\rho}) f - (\mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \rho_\varepsilon - \mathcal{G} \tilde{q}_\varepsilon \mathcal{G} \bar{\rho}) f + \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon (u_\varepsilon - u_0) + \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G} f. \end{aligned} \quad (78)$$

Lemma 3.3 *Let $f \in L^2(D)$. We have*

$$\mathbb{E}\{\|s_\varepsilon\|\} \lesssim \varepsilon \|f\|. \quad (79)$$

Proof. Because $G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1))$ is uniformly bounded \mathbb{P} -a.s., the proof of Lemma 2.6 generalizes to show that $\mathbb{E}\{\|\mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G} f\|^2\} \lesssim \varepsilon^2 \|f\|^2$. We already know that $\mathbb{E}\{\|\mathcal{R}_\varepsilon\|\} \lesssim \varepsilon$. It remains to address the terms $I_1 = \mathcal{G}_{1\varepsilon} \delta \rho_\varepsilon f$, $I_2 = \mathcal{G} \tilde{q}_\varepsilon \mathcal{G}_{1\varepsilon} \rho_\varepsilon f$, $I_3 = \mathcal{G}_{1\varepsilon} \tilde{q}_\varepsilon \mathcal{G} \rho_\varepsilon f$, and $I_4 = \mathcal{G} \tilde{q}_\varepsilon \mathcal{G} \delta \rho_\varepsilon f$.

Because ρ_ε is uniformly bounded \mathbb{P} -a.s., the first three terms are handled in a similar way. Let us consider $\mathbb{E}\{I_1^2\}$, which is bounded by the sum of three terms of the form

$$\mathbb{E}\left\{ \int_{D^3} \delta z_\varepsilon(v_1(x, y)) H(x, y) \delta z_\varepsilon(v_2(x, \zeta)) H(x, \zeta) \delta \rho_\varepsilon(y) \delta \rho_\varepsilon(\zeta) f(y) f(\zeta) dx dy d\zeta \right\},$$

where $v_k(x, y)$ is either x , y , or 1 for $k = 1, 2$, and $H(x, y)$ is a uniformly bounded function. Using the definition of δz_ε , we recast the above integral as

$$\int_{D^3} \int_0^{v_1} \int_0^{v_2} \mathbb{E}\{b_\varepsilon(t_1) b_\varepsilon(t_2) \delta \rho_\varepsilon(y) \delta \rho_\varepsilon(\zeta)\} dt_1 dt_2 H(x, y) H(x, \zeta) f(y) f(\zeta) dx dy d\zeta.$$

Using (9), we see that the above integral is bounded by terms of the form

$$\int_{D^3} \int_0^{v_1} \int_0^{v_2} \varphi^{\frac{1}{2}}\left(\frac{u_1 - u_2}{\varepsilon}\right) \varphi^{\frac{1}{2}}\left(\frac{u_3 - u_4}{\varepsilon}\right) dt_1 dt_2 |H(x, y) H(x, \zeta)| |f(y)| |f(\zeta)| dx dy d\zeta,$$

where $(u_1, u_2, u_3, u_4) = (u_1, u_2, u_3, u_4)(t_1, t_2, y, \zeta)$ is an arbitrary (fixed) permutation of (t_1, t_2, y, ζ) . Because $\varphi(r)$ is integrable, the Cauchy-Schwarz inequality shows that the above term is $\lesssim \varepsilon^2 \|f\|^2$. The term $\mathbb{E}\{I_4^2\}$ is given by

$$\mathbb{E}\left\{ \int_{D^4} G(x, y) G(x, \zeta) \tilde{q}_\varepsilon(y) \tilde{q}_\varepsilon(\zeta) G(y, z) G(\zeta, \xi) \delta \rho_\varepsilon(z) \delta \rho_\varepsilon(\xi) f(z) f(\xi) d[xyz\xi\zeta] \right\}.$$

Since $G(x, y)$ is uniformly bounded on D , we again use (9) as above to obtain a bound of the form $\varepsilon^2 \|f\|^2$. \square

It remains to analyze the convergence of the contribution $[\mathcal{G}_{1\varepsilon} \bar{\rho} + \mathcal{G} \delta \rho_\varepsilon - \mathcal{G} \tilde{q}_\varepsilon \mathcal{G} \bar{\rho}] f$.

As in (28), we define

$$u_{1\varepsilon}(x, \omega) = \frac{1}{\sqrt{\varepsilon}} \left[\mathcal{G}_{1\varepsilon} \bar{\rho} + \mathcal{G} \delta \rho_\varepsilon - \mathcal{G} \tilde{q}_\varepsilon \mathcal{G} \bar{\rho} \right] f(x). \quad (80)$$

We recast the above term as

$$u_{1\varepsilon}(x, \omega) = \frac{1}{\sqrt{\varepsilon}} \int_0^1 \left[b\left(\frac{t}{\varepsilon}\right) H_b(x, t) + \delta \rho\left(\frac{t}{\varepsilon}\right) H_\rho(x, t) - \tilde{q}\left(\frac{t}{\varepsilon}\right) H_q(x, t) \right] dt, \quad (81)$$

with

$$\begin{aligned} H_b(x, t) &= \int_0^1 \left[\chi_x(t) \frac{\partial}{\partial x} G(x, y; 1) + \chi_y(t) \frac{\partial}{\partial y} G(x, y; 1) + \frac{\partial}{\partial L} G(x, y; 1) \right] \bar{\rho} f(y) dy \\ H_\rho(x, t) &= G(x, t) f(t) \\ H_q(x, t) &= G(x, t) \int_0^1 G(t, z) f(z) dz. \end{aligned} \quad (82)$$

where $\chi_x(t) = 1$ if $0 < t < x$ and vanishes otherwise. We have the following result.

Theorem 3.4 *Let $f \in L^\infty(0, 1)$. The process $u_{1\varepsilon}(x, \omega)$ converges weakly and in distribution in the space of continuous paths $\mathcal{C}(D)$ to the limit $u_1(x, \omega)$ given by*

$$u_1(x, \omega) = \int_0^1 \sigma(x, t) dW_t, \quad (83)$$

where W_t is standard Brownian motion and

$$\begin{aligned} \sigma^2(x, t) &= 2 \int_0^\infty \mathbb{E} \{ F(x, t, 0) F(x, t, \tau) \} d\tau, \\ F(x, t, \tau) &= H_b(x, t) b(\tau) + H_\rho(x, t) \delta \rho(\tau) - H_q(x, t) \tilde{q}(\tau). \end{aligned} \quad (84)$$

As a consequence, the corrector to homogenization thus satisfies that:

$$\frac{u_\varepsilon - u_0}{\sqrt{\varepsilon}}(x) \xrightarrow{\text{dist.}} u_1(x, \omega), \quad \text{as } \varepsilon \rightarrow 0, \quad (85)$$

in the space of integrable paths $L^1(D)$.

We may recast $u_{1\varepsilon}(x, \omega)$ as

$$u_{1\varepsilon}(x, \omega) = \sum_{k=1}^3 \frac{1}{\sqrt{\varepsilon}} \int_D p_k\left(\frac{t}{\varepsilon}\right) H_k(x, t) dt, \quad (86)$$

where the p_k are mean-zero processes and the kernels $H_k(x, t)$ are given in (82). The corrector in (83) may then be rewritten as

$$u_1(x) = \sum_{k=1}^3 \int_D \sigma_k(x, t) dW_t^j, \quad (87)$$

with three correlated standard Brownian motions such that

$$dW_t^j dW_t^k = \rho_{jk} dt, \quad (88)$$

where we have defined

$$\begin{aligned} \sigma_k(x, t) &= H_k(x, t) \sqrt{2} \left(\int_0^\infty \mathbb{E}\{p_k(0)p_k(\tau)\} d\tau \right)^{\frac{1}{2}} \\ \rho_{jk} &= \frac{\int_0^\infty \mathbb{E}\{p_j(0)p_k(\tau) + p_k(0)p_j(\tau)\} d\tau}{2 \left(\int_0^\infty \mathbb{E}\{p_j(0)p_j(\tau)\} d\tau \int_0^\infty \mathbb{E}\{p_k(0)p_k(\tau)\} d\tau \right)^{\frac{1}{2}}}. \end{aligned} \quad (89)$$

That (83) and (87) are equivalent comes from the straightforward calculation that both processes are mean zero Gaussian processes that have the same correlation function. The new equation (87) shows more clearly the linearity of the σ_k , whence $u_1(x)$, with respect to the source term $f(x)$.

Proof. We recast $u_{1\varepsilon}(x, \omega)$ as

$$u_{1\varepsilon}(x, \omega) = \sum_k \frac{1}{\sqrt{\varepsilon}} \int_D q_k\left(\frac{t}{\varepsilon}\right) H_k(x, t) dt,$$

with a decomposition similar to but slightly different from (86), where the q_k are mean-zero processes. We verify that we can choose the terms $H_k(x, t)$ in the above decomposition so that all of them are uniformly (in t) Lipschitz in x , except for one term, say $H_1(x, t)$, which is of the form

$$H_1(x, t) = \chi_x(t) L_1(x, t), \quad L_1(x, t) = \int_0^1 \frac{\partial}{\partial x} G(x, y; 1) \bar{\rho} f(y) dy,$$

where $L_1(x, t)$ is uniformly (in t) Lipschitz in x . This results from the fact that $G(x, y; 1)$ is Lipschitz continuous and that its partial derivatives are bounded and piecewise Lipschitz continuous; we leave the tedious details to the reader.

Because of the presence of the term $H_1(x, t)$ in the above expression, it is not sufficient to consider second-order moments of $u_{1\varepsilon}$ as in the proof of Thm. 2.8. Rather, we consider fourth-order moments as follows:

$$\begin{aligned} \mathbb{E}\{|u_{1\varepsilon}(x, \omega) - u_{1\varepsilon}(\xi, \omega)|^4\} &= \frac{1}{\varepsilon^2} \sum_{k_1, k_2, k_3, k_4} \int_{D^4} \mathbb{E}\{q_{k_1}\left(\frac{t_1}{\varepsilon}\right) q_{k_2}\left(\frac{t_2}{\varepsilon}\right) q_{k_3}\left(\frac{t_3}{\varepsilon}\right) q_{k_4}\left(\frac{t_4}{\varepsilon}\right)\} \times \\ &\quad \prod_{m=1}^4 (H_{k_m}(x, t_m) - H_{k_m}(\xi, t_m)) dt_1 dt_2 dt_3 dt_4. \end{aligned}$$

Using the mixing condition of the processes q_k and Lemma 2.1 (where each q in (9) may be replaced by q_k without any change in the result), we obtain that $\mathbb{E}\{|u_{1\varepsilon}(x, \omega) - u_{1\varepsilon}(\xi, \omega)|^4\}$ is bounded by a sum of terms of the form

$$\frac{1}{\varepsilon^2} \int_{D^4} \varphi^{\frac{1}{2}}\left(\frac{t_2 - t_1}{\varepsilon}\right) \varphi^{\frac{1}{2}}\left(\frac{t_4 - t_3}{\varepsilon}\right) \prod_{m=1}^4 (H_{k_m}(x, t_m) - H_{k_m}(\xi, t_m)) dt_1 dt_2 dt_3 dt_4,$$

whence is bounded by terms of the form

$$\left(\frac{1}{\varepsilon} \int_{D^2} \varphi^{\frac{1}{2}}\left(\frac{t_2 - t_1}{\varepsilon}\right) \prod_{m=1}^2 (H_{k_m}(x, t_m) - H_{k_m}(\xi, t_m)) dt_1 dt_2\right)^2.$$

When all the kernels H_{k_m} are Lipschitz continuous, then the above term is of order $|x - \xi|^4$. The largest contribution is obtained when $k_1 = k_2 = 1$ because $H_1(x, t)$ is not uniformly Lipschitz continuous. We now concentrate on that contribution. We recast

$$H_1(x, t) - H_1(\xi, t) = (\chi_x(t) - \chi_\xi(t))L_1(x, t) + \chi_\xi(t)(L_1(x, t) - L_1(\xi, t)).$$

Again, the largest contribution to the fourth moment of $u_{1\varepsilon}$ comes from the term $(\chi_x(t) - \chi_\xi(t))L_1(x, t)$ since $L_1(x, t)$ is Lipschitz continuous. Assuming that $x \geq \xi$ without loss of generality, we calculate that

$$\begin{aligned} & \int_{D^2} (\chi_x(t) - \chi_\xi(t))L_1(x, t)(\chi_\xi(s) - \chi_\xi(s))L_1(\xi, s) \frac{1}{\varepsilon} \varphi^{\frac{1}{2}}\left(\frac{t-s}{\varepsilon}\right) dt ds \\ &= \int_{\xi}^x \int_{\xi}^x L_1(x, t)L_1(\xi, s) \frac{1}{\varepsilon} \varphi^{\frac{1}{2}}\left(\frac{t-s}{\varepsilon}\right) dt ds \lesssim (x - \xi), \end{aligned}$$

since $\varphi^{\frac{1}{2}}$ is integrable. Note that this term is not of order $|\xi - x|^2$. Nonetheless, we have shown that

$$\mathbb{E}\{|u_{1\varepsilon}(x, \omega) - u_{1\varepsilon}(\xi, \omega)|^4\} \lesssim |\xi - x|^2,$$

so that we can apply the Kolmogorov criterion in Prop. 2.9 with $\nu = 2$, $\beta = 4$, and $\delta = 1$. This concludes the proof of tightness of $u_{1\varepsilon}(x, \omega)$ as a process with values in the space of continuous functions $\mathcal{C}(D)$.

It remains to verify step (a) of Prop. 2.9. The finite-dimensional distributions are treated as in the proof of Thm. 2.8 and are replaced by the analysis of random integrals of the form:

$$\frac{1}{\sqrt{\varepsilon}} \int_0^1 \left[b\left(\frac{t}{\varepsilon}\right) m_b(t) + \delta \rho\left(\frac{t}{\varepsilon}\right) m_\rho(t) + \tilde{q}\left(\frac{t}{\varepsilon}\right) m_q(t) \right] dt.$$

The functions m are continuous and can be approximated by m_h constant on intervals of size h so that we end up with M independent (in the limit $\varepsilon \rightarrow 0$) variables of the form:

$$\frac{\sqrt{h}}{\sqrt{N}} \sum_{j=1}^N m_{bh} b_j + m_{\rho h} \delta \rho_j + m_{qh} \tilde{q}_j.$$

It remains to apply the central limit theorem as in the proof of Thm. 2.8. The above random variable converges in distribution to

$$\mathcal{N}(0, h\sigma^2), \quad \sigma^2 = 2 \int_0^\infty \mathbb{E}\{(m_{bh}b + m_{\rho h}\delta\rho + m_{qh}\tilde{q})(0)(m_{bh}b + m_{\rho h}\delta\rho + m_{qh}\tilde{q})(t)\} dt.$$

This concludes our analysis of the convergence in distribution of $u_{1\varepsilon}$ to its limit in the space of continuous paths $\mathcal{C}(D)$. The convergence of $u_\varepsilon - u_0$ follows from the bound (79). \square

4 Correctors for spectral problems

4.1 Abstract convergence result

For $\omega \in \Omega$, let $A_\eta(\omega)$ be a sequence of bounded (uniformly in ω \mathbb{P} -a.s. and in $\eta > 0$), compact, self-adjoint operators, converging to a deterministic, compact, self-adjoint, operator A as $\eta \rightarrow 0$ in the sense that the following error estimate holds:

$$\mathbb{E}\|A_\eta(\omega) - A\|^p \lesssim \eta^p, \quad \text{for some } 1 \leq p < \infty, \quad (90)$$

where $\|A_\eta(\omega) - A\|$ is the $L^2(D)$ norm and D is an open subset of \mathbb{R}^d .

The operators A and \mathbb{P} -a.s. $A_\eta(\omega)$ admit the spectral decompositions (λ_n, u_n) and $(\lambda_n^\eta, u_n^\eta)$, where the real-valued eigenvalues are ordered in decreasing order of their absolute values and counted m_n times, where m_n is their multiplicity.

For λ_n , let μ_n be (one of) the closest eigenvalue of A that is different from λ_n . Let us then define the distance:

$$d_n = \frac{|\lambda_n - \mu_n|}{2}. \quad (91)$$

Following [32], we analyze the spectrum of A_η in the vicinity of λ_n . Let Γ be the circle of center λ_n and radius d_n in the complex plane and let $R(\zeta, A) = (A - \zeta)^{-1}$ be the resolvent of A defined for all complex numbers $\zeta \notin \sigma(A)$, the spectrum of A . The projection operator onto the spectral components of B inside the curve Γ is defined by

$$P_n[B] = -\frac{1}{2\pi i} \int_{\Gamma} R(\zeta, B) d\zeta. \quad (92)$$

Note that for all $\zeta \in \Gamma$, we have that $R(\zeta, A)P_n[A] = (\lambda_n - \zeta)^{-1}$. We can then prove the following result:

Proposition 4.1 *Let A_η and A be the operators described above and let λ_n be fixed. Then, for η sufficiently small with respect to d_n , we can choose m_n eigenvalues λ_n^η of A_η in the vicinity of λ_n so that the following estimate holds:*

$$\mathbb{E}\{|\lambda_n - \lambda_n^\eta|^p\} + \mathbb{E}\{\|u_n^\eta - u_n\|^p\} \lesssim \frac{\eta^p}{d_n^p} \wedge 1, \quad (93)$$

for a suitable labeling of the eigenvectors u_n^η of A^η associated to the eigenvalues λ_n^η .

Proof. It follows from [32, Theorem IV.3.18] that for those realizations ω such that $\|A_\eta(\omega) - A\| < d_n$, then there are exactly m_n eigenvalues of A_η in the d_n -vicinity of Γ . Since this also holds for every λ_m such that $d_m > d_n$, we can index the eigenvalues of A_η as the eigenvalues of A . Moreover,

$$|\lambda_n^\eta(\omega) - \lambda_n| \leq \|A_\eta(\omega) - A\|.$$

For those realizations ω such that $\|A_\eta(\omega) - A\| \geq d_n$, we choose m_n eigenvalues of $A_\eta(\omega)$ arbitrarily among the eigenvalues that have not been chosen in the d_m -vicinity of λ_m for $|\lambda_m| > |\lambda_n|$.

For all realizations, we thus obtain that

$$|\lambda_n^\eta(\omega) - \lambda_n| \lesssim \frac{\|A_\eta(\omega) - A\|}{d_n}.$$

It remains to take the p th power and average the above expression to obtain the first inequality of the proposition.

In order for the eigenvectors u_n^η and u_n to be close, we need to restrict the size of η further. To make sure the eigenvectors are sufficiently close, we need to ensure that

$$P_n[A_\eta] - P_n[A] = \frac{-1}{2\pi i} \int_{\Gamma} [R(\zeta, A_\eta) - R(\zeta, A)] d\zeta = \frac{1}{2\pi i} \int_{\Gamma} R(\zeta, A_\eta)(A_\eta - A)R(\zeta, A) d\zeta,$$

is sufficiently small. On the circle Γ and for $\|A - A_\eta\| < d_n$, we verify that

$$\sup_{\zeta \in \Gamma} \|R(\zeta, A)\| = \frac{1}{d_n}, \quad \sup_{\zeta \in \Gamma} \|R(\zeta, A_\eta)\| \leq \frac{1}{d_n - \|A - A_\eta\|},$$

by construction of d_n and by using $R^{-1}(\zeta, A_\eta) = R^{-1}(\zeta, A) + (A_\eta - A)$ and the triangle inequality

$$\|R^{-1}(\zeta, A_\eta)\| \geq \|R^{-1}(\zeta, A)\| - \|A_\eta - A\| \geq d_n - \|A_\eta - A\|.$$

Upon integrating the expression for $P_n[A_\eta] - P_n[A]$ on Γ , we find that

$$\rho := \|P_n[A_\eta] - P_n[A]\| \leq \frac{\|A_\eta - A\|}{d_n - \|A_\eta - A\|} \leq \frac{2}{d_n} \|A_\eta - A\| < 1,$$

for $2\|A_\eta - A\| < d_n$.

For self-adjoint operators A and A_η , the above bound on the distance ρ between the eigenspaces is sufficient to characterize the distance between the corresponding eigenvectors. We follow [32, I.4.6 & II.4.2] and construct the unitary operator

$$U_n^\eta = \left(I - (P_n[A_\eta] - P_n[A])^2 \right)^{-\frac{1}{2}} \left(P_n[A_\eta]P_n[A] + (I - P_n[A_\eta])(I - P_n[A]) \right). \quad (94)$$

Let $u_{n,k}$, $1 \leq k \leq m_n$ be all the eigenvectors associated to an eigenvalue λ_n , $n \geq 1$. Then the eigenspace associated to λ_n^η admits for an orthonormal basis the eigenvectors defined by [32]

$$u_{n,k}^\eta = U_n^\eta u_{n,k}, \quad 1 \leq k \leq m_n. \quad (95)$$

The relation (94) may be recast as

$$U_n^\eta = (I - R_n^\eta)(I + P_n[A_\eta](P_n[A_\eta] - P_n[A]) + (P_n[A_\eta] - P_n[A])P_n[A_\eta]),$$

where $\|R_n^\eta\| \lesssim \rho^2$. This shows that

$$\|U_n^\eta - I\| \lesssim \rho \quad \text{and} \quad \|u_{n,k}^\eta - u_{n,k}\| \lesssim \rho \lesssim \frac{1}{d_n} \|A_\eta - A\|, \quad 1 \leq k \leq m_n,$$

whenever $d_n^{-1}\|A_\eta - A\| < \mu$ for μ sufficiently small. When $d_n^{-1}\|A_\eta - A\| \geq \mu$, we find that $\|u_{n,k} - u_{n,k}^\eta\| \lesssim 2\mu\|A_\eta(\omega) - A\|/d_n$, where the vectors $u_{n,k}^\eta$ are constructed as an arbitrary orthonormal basis of the eigenspace associated to λ_n^η . Upon taking p th power and ensemble averaging, we obtain (93). \square

4.2 Correctors for eigenvalues and eigenvectors

Let (λ_n, u_n) be a solution of $Au_n = \lambda_n u_n$ and let λ_n^η and u_n^η be the solution of $A_\eta u_n^\eta = \lambda_n^\eta u_n^\eta$ defined in Proposition 4.1. We assume that (90) holds with $p = 2$.

We calculate that

$$\frac{\lambda_n^\eta - \lambda_n}{\eta} = \left(u_n, \frac{A_\eta - A}{\eta} u_n \right) + \frac{1}{\eta} \left(u_n^\eta - u_n, ((A_\eta - \lambda_n^\eta) - (A - \lambda_n)) u_n \right).$$

The last term, which we denote by $r_n^\eta(\omega)$ is bounded by $O(\eta)$ in $L^1(\Omega)$ using the results of Proposition 4.1 with $p = 2$ and the Cauchy-Schwarz inequality. Thus, $r_n^\eta(\omega)$ converges to 0 in probability.

Let us assume that the eigenvectors are defined on a domain $D \subset \mathbb{R}^d$ and that for a smooth function $M(\mathbf{x})$, we have:

$$\left(M(\mathbf{x}), \frac{A_\eta - A}{\eta} u_n(\mathbf{x}) \right) \xrightarrow{\text{dist.}} \int_{D^2} M(\mathbf{x}) \sigma_n(\mathbf{x}, \mathbf{y}) dW_{\mathbf{y}} d\mathbf{x} \quad \text{as } \eta \rightarrow 0. \quad (96)$$

Using this result, and provided that the eigenvectors $u_n(\mathbf{x})$ are sufficiently smooth, we obtain that

$$\frac{\lambda_n^\eta - \lambda_n}{\eta} \xrightarrow{\text{dist.}} \int_{D^2} u_n(\mathbf{x}) \sigma_n(\mathbf{x}, \mathbf{y}) dW_{\mathbf{y}} d\mathbf{x} := \int_D \Lambda_n(\mathbf{y}) dW_{\mathbf{y}} \quad \text{as } \eta \rightarrow 0. \quad (97)$$

The eigenvalue correctors are therefore Gaussian variables, which may conveniently be written as a stochastic integral that is quadratic in the eigenvectors since $\sigma_n(\mathbf{x}, \mathbf{y})$ is a linear functional of u_n . The correlations between different correctors may also obviously be obtained as

$$\mathbb{E} \left\{ \frac{\lambda_n^\eta - \lambda_n}{\eta} \frac{\lambda_m^\eta - \lambda_m}{\eta} \right\} \xrightarrow{\eta \rightarrow 0} \int_D \Lambda_n(\mathbf{x}) \Lambda_m(\mathbf{x}) d\mathbf{x}. \quad (98)$$

Let us now turn to the corrector for the eigenvectors. Note that

$$\|u_n - u_n^\eta\|^2 = 2(1 - (u_n, u_n^\eta)),$$

so that (u_n, u_n^η) is equal to 1 plus an error term of order $O(\eta^2)$ on average. The construction of the eigenvectors in (95) shows that $u_n - u_n^\eta$ is of order $O(\eta^2)$ in the whole eigenspace associated to the eigenvalue λ_n . It thus remains to analyze the convergence properties of $(u_n - u_n^\eta, u_m)$ for all $m \neq n$. A straightforward calculation similar to the one obtained for the eigenvalue corrector shows that

$$\left(\frac{u_n^\eta - u_n}{\eta}, (A - \lambda_n) u_m \right) = - \left(\frac{(A_\eta - \lambda_n^\eta) - (A - \lambda_n)}{\eta} u_n, u_m \right) - \frac{1}{\eta} \left((A_\eta - A)(u_n^\eta - u_n), u_m \right).$$

The last term converges to 0 in probability (and is in fact of order $O(\eta)$ in $L^1(\Omega)$ as above). We thus find that

$$\left(\frac{u_n^\eta - u_n}{\eta}, u_m \right) \xrightarrow{\text{dist.}} \frac{1}{\lambda_n - \lambda_m} \int_{D^2} u_m(\mathbf{x}) \sigma_n(\mathbf{x}, \mathbf{y}) dW_{\mathbf{y}} d\mathbf{x}. \quad (99)$$

The Fourier coefficients of the eigenvector correctors converge to Gaussian random variables. As in the case of eigenvalues, it is straightforward to estimate the cross-correlations of the Fourier coefficients corresponding to (possibly) different eigenvectors.

4.3 Applications to some specific problems

The first application pertains to the following problem:

$$A_\varepsilon = (P(\mathbf{x}, D) + q_\varepsilon)^{-1}, \quad A = P(\mathbf{x}, D)^{-1}. \quad (100)$$

Lemma 2.5 and its corollary (24) show that (90) holds with $p = 2$ and $\eta = \varepsilon^{\frac{d}{2}}$. The operators A_ε and A are also compact and self-adjoint for a large class of operators $P(\mathbf{x}, D)$ which includes the Helmholtz operator $P(\mathbf{x}, D) = -\Delta + q_0(\mathbf{x})$.

Let $(\lambda_n^\varepsilon, u_n^\varepsilon)$ be the solutions of $\lambda^\varepsilon P_\varepsilon u_\varepsilon = u_\varepsilon$ and (λ_n, u_n) the solutions of $\lambda P u = u$. Then we find that

$$\frac{\lambda_n^\varepsilon - \lambda_n}{\varepsilon^{\frac{d}{2}}} \xrightarrow{\text{dist.}} -\lambda_n \sigma \int_{D^2} u_n(\mathbf{x}) G(\mathbf{x}, \mathbf{y}) u_n(\mathbf{y}) dW_{\mathbf{y}} d\mathbf{x} = -\lambda_n^2 \sigma \int_D u_n^2(\mathbf{y}) dW_{\mathbf{y}}, \quad (101)$$

or equivalently, that for the eigenvalues of P_ε and P , we have:

$$\frac{(\lambda_n^\varepsilon)^{-1} - \lambda_n^{-1}}{\varepsilon^{\frac{d}{2}}} \xrightarrow{\text{dist.}} \sigma \int_D u_n^2(\mathbf{y}) dW_{\mathbf{y}}. \quad (102)$$

The Fourier coefficients of the eigenvectors satisfy similar expressions.

The second example is the one-dimensional elliptic equation (61). Still setting $\eta = \varepsilon^{\frac{1}{2}}$, we find that

$$\frac{A_\varepsilon - A}{\sqrt{\varepsilon}} u_n \xrightarrow{\text{dist.}} \int_D \sigma_n(x, t) dW_t,$$

where $\sigma_n(x, t)$ is defined in (84) with the source term f in (82) being replaced by $u_n(x)$. The operators A_ε and A satisfy (90) with $p = 2$ thanks to Lemma 3.2 and its corollary (24). The expressions for the eigenvalue and eigenvector correctors are thus directly given by (97) and (99), respectively.

4.4 Correctors for time dependent problems

As an application of the preceding theory, let us now consider an evolution problem of the form

$$u_t + \varepsilon P u = 0, \quad t > 0, \quad u(0) = u_0, \quad (103)$$

where ε is a constant, typically $\varepsilon = 1$ or $\varepsilon = i$, and P is a symmetric pseudodifferential operator with domain $\mathcal{D}(P) \subset L^2(D)$ for some subset $D \subset \mathbb{R}^d$ and with a compact inverse $A = P^{-1}$, which we assume without loss of generality, has positive eigenvalues.

We then consider the randomly perturbed problem

$$u_t^\eta + \varepsilon P_\eta u_\eta = 0, \quad t > 0, \quad u_\eta(0) = u_0, \quad (104)$$

where $P_\eta(\omega)$ verifies the same hypotheses as P with compact inverse $A_\eta = P_\eta^{-1}$.

We assume that A_η and A are sufficiently close so that (90) holds. Following the notation of the preceding section, we denote by λ_n and λ_n^η the eigenvalues of A and A_η and by u_n and u_n^η the corresponding eigenvectors.

We then verify that

$$u(t) = e^{-\varepsilon t P} u_0 = \sum_n e^{-\varepsilon \lambda_n t} (u_n, u_0) u_n := \sum_n \alpha_n(t) u_n, \quad \alpha_n(t) = e^{-\varepsilon \lambda_n t} (u_n, u_0).$$

and

$$u_\eta(t) = \sum_n \alpha_n^\eta(t) u_n^\eta, \quad \alpha_n^\eta(t) = e^{-\epsilon \lambda_n^\eta t} (u_n^\eta, u_0).$$

We can now compare the Fourier coefficients as follows:

$$\frac{\alpha_n^\eta - \alpha_n}{\eta} = \frac{e^{-\epsilon \lambda_n^\eta t} - e^{-\epsilon \lambda_n t}}{\eta} (u_n, u_0) + e^{-\epsilon \lambda_n t} \left(\frac{u_n^\eta - u_n}{\eta}, u_0 \right) + r_\eta, \quad (105)$$

where $|r_\eta| \rightarrow 0$ strongly in $L^p(\Omega)$ as $\eta \rightarrow 0$. This may be recast as

$$\frac{\alpha_n^\eta - \alpha_n}{\eta} = e^{-\epsilon \lambda_n t} \epsilon t \frac{\lambda_n - \lambda_n^\eta}{\eta} (u_n, u_0) + e^{-\epsilon \lambda_n t} \left(\frac{u_n^\eta - u_n}{\eta}, u_0 \right) + s_\eta, \quad (106)$$

where $|s_\eta| \rightarrow 0$ strongly in $L^p(\Omega)$ as $\eta \rightarrow 0$.

The above difference thus converges to a mean zero Gaussian random variable whose variance may easily be estimated from the results obtained in the preceding section.

Since we do not control the convergence of the eigenvectors for arbitrary values of n (because we do not control in this study the speed of convergence in distribution of the random correctors), we cannot obtain the law of the full corrector $u_\eta(t) - u(t)$. We can, however, obtain a corrector for the low frequency parts $u_N^\eta(t)$ and $u^N(t)$ of $u_\eta(t)$ and $u(t)$, respectively, where only the N first terms are kept in the sum in the index n . We may easily estimate the corrector for $(u_N^\eta(t) - u^N(t), u_m)$ using the above expansion for the Fourier coefficients and the results obtained in the preceding section. We again obtain that the corrector is a mean zero Gaussian variable whose variance may be calculated explicitly.

Other time-dependent equations may be treated in a similar way. For instance, the wave equation

$$u_{tt} + Pu = 0, \quad u(0) = u_0, \quad u_t(0) = g_0, \quad (107)$$

where P is a symmetric operator with compact and positive definite inverse, may be recast as

$$w_t - Aw = 0, \quad w(0) = w_0, \quad w = \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ P & 0 \end{pmatrix}. \quad (108)$$

We verify that the eigenvalues λ_n of A are purely imaginary and equal to $\pm i\sqrt{\lambda_P}$, where λ_P are the positive eigenvalues of P . The orthogonal projector onto the n th eigenspace of A is found to be

$$\Pi_{A,\lambda} = \begin{pmatrix} \Pi_{P,-\lambda^2} & 0 \\ 0 & \lambda \Pi_{P,-\lambda^2} \end{pmatrix},$$

so that

$$\begin{pmatrix} u \\ u_t \end{pmatrix} (t) = \sum_\lambda e^{-\lambda t} \begin{pmatrix} \Pi_{P,-\lambda^2} & 0 \\ 0 & \lambda \Pi_{P,-\lambda^2} \end{pmatrix} \begin{pmatrix} u_0 \\ g_0 \end{pmatrix}. \quad (109)$$

A similar expression may be used for the perturbed problem $u_\eta(t)$, where P is replaced by P_η . The results presented earlier in this section easily generalize to provide an estimate for the low frequency component of $u(t) - u_\eta(t)$. We leave the details to the reader.

When the Green's function associated to the operator $\partial_t + \epsilon P$ is sufficiently regular, for instance when $\epsilon P = -\Delta$, more refined results may be obtained by considering expansions similar to the expansion (8) considered for steady-state problems. We do not consider such developments here.

5 Conclusions

We have considered the corrector to the homogenization of source and spectral problems for the Helmholtz equation with highly oscillatory random potential. The method works because the operator $\mathcal{G}q_\varepsilon$ appearing in (8) may be seen as lower-order, in the sense that it converges rapidly to 0 with ε . This requires that the homogenized solution $\mathcal{G}f$ be a good approximation to the source problem (8). The method was then generalized to the one-dimensional elliptic problem (61), which after a change of variables to harmonic coordinates, may also be recast as an integral equation (68) with a term $\mathcal{G}_\varepsilon\tilde{q}_\varepsilon$ that may also be seen as lower-order.

Such expansions are not currently available for more challenging problems of the form $-\nabla \cdot a_\varepsilon(\mathbf{x}, \omega)\nabla u_\varepsilon = f$, augmented with appropriate boundary conditions. The use of the Green's function to the homogenized elliptic equation does not allow for a rapidly converging expansion of the form (8) or (68). The analysis of correctors for such equations, for which current state of the art estimations are given in [49], remains an open problem; see [18] for a related discretized elliptic equation.

The correctors were analyzed here in the setting where the random coefficients have integrable correlation function $R(\mathbf{x})$ in (5) (and additional mixing properties). The expansions in (8) and (68) may be generalized to random coefficients with correlation functions $R(\mathbf{x})$ which decay as $|\mathbf{x}|^{-\alpha d}$ for some $0 < \alpha < 1$ as $|\mathbf{x}| \rightarrow \infty$. In such frameworks, following the expansions obtained in [5], we expect random correctors with Gaussian statistics and amplitudes of order $\varepsilon^{\alpha\frac{d}{2}}$ rather than $\varepsilon^{\frac{d}{2}}$, at least for dimensions $1 \leq d \leq 3$ for the Helmholtz problem. These long-range effects will be analyzed elsewhere.

Acknowledgment

The author would like to thank Josselin Garnier, Wenjia Jing, Tomasz Komorowski, and George Papanicolaou for stimulating discussions on the subject of equations with random coefficients and central limit theorems. This work was supported in part by NSF Grant DMS-0239097 and an Alfred P. Sloan Fellowship.

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