CENTRAL LIMITS AND HOMOGENIZATION IN RANDOM MEDIA *

GUILLAUME BAL †

Abstract. We consider the perturbation of elliptic pseudo-differential operators $P(\mathbf{x}, \mathbf{D})$ with more than square integrable Green's functions by random, rapidly varying, sufficiently mixing, potentials of the form $q(\frac{\mathbf{x}}{\varepsilon}, \omega)$. We analyze the source and spectral problems associated to such operators and show that the rescaled difference between the perturbed and unperturbed solutions may be written asymptotically as $\varepsilon \to 0$ as explicit Gaussian processes. Such results may be seen as central limit corrections to homogenization (law of large numbers). Similar results are derived for more general elliptic equations with random coefficients in one dimension of space. The results are based on the availability of a rapidly converging integral formulation for the perturbed solutions and on the use of classical central limit results for random processes with appropriate mixing conditions.

Key words. Homogenization, central limit, differential equations with random coefficients.

AMS subject classifications. 35R60, 35J05, 35P20, 60H05.

1. Introduction.

There are many practical applications of partial differential equations with coefficients that oscillate at a faster scale than the scale of the domain on which the equation is solved. In such settings, it is often necessary to model the coefficients as random processes, whose properties are known only at a statistical level.

Since numerical simulations of the resulting partial differential equation become a daunting task, two simplifications are typically considered. The first simplification consists in assuming that the coefficients oscillate very rapidly and replacing the equation with random coefficients by a homogenized equation with deterministic (effective medium) coefficients; for homogenization in the periodic and random settings, see e.g. [8, 39] and [12, 17, 30, 34, 36, 37, 42], respectively.

The solution to the equation with random equations may also be interpreted as a functional of an infinite number of random variables and may be expanded in polynomial chaoses [18, 46]. A second simplification consists then in discretizing the randomness in the coefficients over sufficiently low dimensional subspaces; see e.g. [3, 27, 28, 29, 38, 48] for references on this active area of research. Such problems, which are posed in domains of dimension $d+Q$, where d is spatial dimension and Q the dimension of the random space, are often computationally very intensive.

In several practical settings such as e.g. the analysis of geological basins or the manufacturing of composite materials, one may be interested in an intermediate situation, where random fluctuations are observed and yet the random environment is so rich that full solutions of the equation with random coefficients may not be feasible. In this paper, we are interested in characterizing the random fluctuations about the homogenized limit, which arise as an application of the central limit theory.

In many cases of practical interest, such as those involving the elliptic operator $\nabla \cdot a_\varepsilon(\mathbf{x}, \omega)\nabla$, with $\mathbf{x} \in D \subset \mathbb{R}^d$ and $\omega \in \Omega$ the space of random realizations, the calculation of the homogenized tensor is difficult and does not admit analytic expressions except in very simple cases [30]. The amplitude of the corrector to homogenization,

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†Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027; gb2030@columbia.edu

let alone its statistical description, remains largely open. The best estimates currently available in spatial dimension $d \geq 2$ may be found in [47]; see also [20, 21], [19] for discrete equations, and [5] for applications of such error estimates. Only in one dimension of space do we have an explicit characterization of the effective diffusion coefficient and of the corrector [14]. Unlike the case of periodic media, where the corrector is proportional to the size of the cell of periodicity ε, the random corrector to the homogenized solution is an explicitly characterized Gaussian process of order $\sqrt{\varepsilon}$ when the random coefficient has integrable correlation [14]. In the case of correlations that are non integrable and of the form $R(t) \sim t^{-\alpha}$ for some $0 < \alpha < 1$, the corrector may be shown to be still an explicitly characterized Gaussian process, but now of order $\varepsilon^{\frac{\alpha}{2}}$ [6].

The explicit characterizations of the correctors obtained in [6, 14] are based on the availability of explicit solutions to the heterogeneous elliptic equation. Correctors to homogenization have been obtained in other settings. The analysis of homogenized solutions and central limit correctors to evolution equations with time dependent randomly varying coefficients is well known; see e.g. [10, 23, 25, 32, 35, 41]. The asymptotic limit of boundary value problems requires different mathematical techniques. We refer the reader to [26, 45] for results in the setting of one-dimensional problems. Note that in the case of a much stronger potential, in dimension $d = 1$ of the form $\varepsilon^{-\frac{1}{2}} q_\varepsilon$ instead of q_ε in the above Helmholtz operator, the deterministic homogenization limit no longer holds and the solution of a corresponding evolution equation converges to another well identified limit; see [43].

In spatial dimensions two and higher, a methodology to compute the Gaussian fluctuations for boundary value problems of the form $-\Delta u_\varepsilon + F(u_\varepsilon, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) = f(\mathbf{x})$ was developed in [24]. An explicit expression for the fluctuations was obtained and proved for the linear equation $(-\Delta + \lambda + q(\frac{\mathbf{x}}{\varepsilon}))u_\varepsilon(\mathbf{x}) = f(\mathbf{x})$ in dimension $d = 3$. In this paper, we revisit the problem and generalize it to linear problems of the form $P(\mathbf{x}, \mathbf{D})u_\varepsilon + q_\varepsilon(\mathbf{x})u_\varepsilon = f(\mathbf{x})$ with an unperturbed equation $P(\mathbf{x}, \mathbf{D})u = f$, which admits a Green's function $G(\mathbf{x}, \mathbf{y})$ that is more than square integrable (see (2.4) below). The prototypical example of interest is the operator $P(\mathbf{x}, \mathbf{D}) = -\nabla \cdot a(\mathbf{x})\nabla + q_0(\mathbf{x})$ with sufficiently smooth (deterministic) coefficients $a(\mathbf{x})$ and $q_0(\mathbf{x})$ posed on a bounded domain with, say, Dirichlet boundary conditions, for which the Green's function is more than square integrable in dimensions $1 \leq d \leq 3$.

Under appropriate mixing conditions on the random process $q_\varepsilon(\mathbf{x}, \omega)$, we will show that arbitrary spatial moments of the correctors $\varepsilon^{-\frac{d}{2}}(u_\varepsilon - u_0, M)$ where u_ε and u_0 are the solutions to perturbed and unperturbed equations, respectively, and where M is a smooth function, converge in distribution to Gaussian random variables, which admit a convenient and explicit representation as a stochastic integral with respect to a standard (multi-parameter) Wiener process. If we denote by u_1 the weak limit of $w_{1\varepsilon} = \varepsilon^{-\frac{d}{2}}(u_\varepsilon - u_0)$, we observe, for $1 \leq d \leq 3$, that $\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x}, \omega)\}$ converges to $\mathbb{E}\{u_1^2(\mathbf{x}, \omega)\}$, where $v_{1\varepsilon}$ is the leading term in $w_{1\varepsilon}$ up to an error term we prove is of order $O(\varepsilon^d)$ in $L^1(\Omega \times D)$. This shows that the limiting process u_1 captures all the fluctuations of the corrector to homogenization. This result is no longer valid in $d \geq 4$ and in homogenization in periodic media in arbitrary dimension, where the weak limit of the corrector captures a fraction of the energy of that corrector. The square integrability of the Green's function thus appears as a natural condition in the framework of homogenization in random media.

We obtain similar expressions for the spectral elements of the perturbed elliptic equation. We find that the correctors to the eigenvalues and the spatial moments of the correctors to the corresponding eigenvectors converge in distribution to Gaussian variables as the correlation length ε vanishes. In the setting $d = 1$, we obtain similar result for more general elliptic operators of the form $-\frac{d}{dx}(a_\varepsilon \frac{d}{dx}) + q_0 + q_\varepsilon$ by appropriate use of harmonic coordinates [34].

As we already mentioned, the theory developed here allows us to characterize the statistical properties of the solutions to equations with random coefficients in the limit where the correlation length (the scale of the heterogeneities) is small compared to the overall size of the domain. Asymptotically explicit expressions for the correctors may also find applications in the testing of numerical algorithms. Several numerical schemes have been developed to estimate the heterogeneous solution accurately in the regime of validity of homogenization; see e.g. [1, 2, 21, 22, 40]. Explicit expressions for the correctors allow us to check whether these algorithms capture the central limit corrections as well. Another application concerns the reconstruction of the constitutive parameters of an equation from redundant measurements. In such cases, lower-variance reconstructions are available when the cross-correlations are known and used optimally in the inversion; see e.g. [44]. The correctors obtained in this paper provide asymptotic estimates for the cross-correlation of the measured data and allow us to obtain lower-variance reconstructions for the constitutive parameters; see [7].

An outline for the rest of the paper is as follows. Section 2 summarizes the results obtained in the paper. The analysis of the correctors to homogenization is undertaken in section 3 for perturbations of $P(\mathbf{x}, \mathbf{D})$ by a random potential $q_\varepsilon(\mathbf{x})$. The generalization to a more general one-dimensional elliptic source problem in detailed in section 4. The results on the correctors obtained for source problems are then extended to correctors for spectral problems in section 5. The results obtained for the spectral problems are then briefly applied to the analysis of evolution equations.

2. Main results. Let us consider an equation of the form:

$$P(\mathbf{x}, \mathbf{D}) u_\varepsilon + q_\varepsilon u_\varepsilon = f, \qquad \mathbf{x} \in D \tag{2.1}$$

with $u_\varepsilon = 0$ on ∂D, where $P(\mathbf{x}, \mathbf{D})$ is a (deterministic) self-adjoint, elliptic, pseudo-differential operator and D an open bounded domain in $\mathbb{R}^d$. We assume that $P(\mathbf{x}, \mathbf{D})$ is invertible with symmetric and “more than square integrable” Green’s function. More precisely, we assume that the equation

$$P(\mathbf{x}, \mathbf{D}) u = f, \qquad \mathbf{x} \in D \tag{2.2}$$

with $u = 0$ on ∂D admits a unique solution given by:

$$u(\mathbf{x}) = \mathcal{G} f(\mathbf{x}) := \int_D G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \tag{2.3}$$

where the real-valued, non-negative, symmetric kernel $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x})$ (these assumptions can be relaxed in Section 3) has more than square integrable singularities:

$$\mathbf{x} \mapsto \Big(\int_D |G|^{2+\eta}(\mathbf{x}, \mathbf{y}) d\mathbf{y} \Big)^{\frac{1}{2+\eta}} \quad \text{is bounded on } D \text{ for some } \eta > 0. \tag{2.4}$$

The assumption is typically satisfied for operators of the form $P(\mathbf{x}, D) = -\nabla \cdot a(\mathbf{x}) \nabla + \sigma(\mathbf{x})$ for $a(\mathbf{x})$ uniformly bounded and coercive and $\sigma(\mathbf{x}) \geq 0$ in dimension $d \leq 3$, with $\eta = +\infty$ when $d = 1$ (i.e., G is bounded), $\eta < \infty$ for $d = 2$, and $\eta < 1$ for $d = 3$.

In order to avoid resonances from occurring in (2.1), the process $q_\varepsilon(\mathbf{x}, \omega)$ is a modification -see Section 3 for the details- of $\tilde{q}_\varepsilon(\mathbf{x}, \omega) = q(\frac{\mathbf{x}}{\varepsilon}, \omega)$, a mean zero, (strictly) stationary, process defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ [16]. We assume that $q(\mathbf{x}, \omega)$ has an integrable correlation function:

$$R(\mathbf{x}) = \mathbb{E}\{q(\mathbf{y}, \omega) q(\mathbf{y} + \mathbf{x}, \omega)\}, \tag{2.5}$$

where $\mathbb{E}$ is mathematical expectation associated to $\mathbb{P}$, and that it is strongly mixing in the sense given in (3.1) below. We define the following variance:

$$\hat{R}(\mathbf{0}) = \sigma^2 := \int_{\mathbb{R}^d} R(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^d} \mathbb{E}\{q(0) q(\mathbf{x})\} d\mathbf{x}. \tag{2.6}$$

We formally recast (2.1) as $u_\varepsilon = \mathcal{G}(f - q_\varepsilon u_\varepsilon)$, where $\mathcal{G} = P(\mathbf{x}, D)^{-1}$, and thus:

$$u_\varepsilon = \mathcal{G} f - \mathcal{G} q_\varepsilon \mathcal{G} f + \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u_\varepsilon. \tag{2.7}$$

The process $\tilde{q}_\varepsilon$ is modified on a set of measure $O(\varepsilon)$ to ensure that the above equation admits a unique solution $\mathbb{P}-$a.s. One of the main results of this paper is that:

$$\frac{u_\varepsilon - u_0}{\varepsilon^{\frac{d}{2}}} \xrightarrow{\text{dist.}} -\sigma \int_D G(\mathbf{x}, \mathbf{y}) u_0(\mathbf{y}) dW_{\mathbf{y}}, \quad \text{as } \varepsilon \to 0, \tag{2.8}$$

weakly in space (i.e., after integration against a sufficiently smooth deterministic function $M(\mathbf{x})$), where $u_0 = \mathcal{G} f$ and $dW_{\mathbf{y}}$ is standard multi-parameter Wiener process [33]; see Theorem 3.7 when G is bounded and Theorem 3.8 when G verifies (2.4). The right-hand side in (2.8) is the limit of $-\varepsilon^{-\frac{d}{2}} \mathcal{G} q_\varepsilon \mathcal{G} f$. We observe that the variance of the latter term converges to the variance of the right-hand side in (2.8).

When the Green's function is no longer square integrable, the variance of the corrector $-\varepsilon^{-\frac{d}{2}} \mathcal{G} q_\varepsilon \mathcal{G} f(\mathbf{x})$ is much larger than that of $-\sigma \int_D G(\mathbf{x}, \mathbf{y}) u_0(\mathbf{y}) dW_{\mathbf{y}}$, which implies that energy is lost while passing to the weak limit in (2.8). This is the case for the elliptic operator $P(\mathbf{x}, \mathbf{D}) = -\nabla \cdot a \nabla + q_0$ in dimension $d \geq 4$; see section 3.3.

We then extend the previous results to the general one-dimensional equation:

$$-\frac{d}{dx} a_\varepsilon(x, \omega) \frac{d}{dx} u_\varepsilon + (q_0 + q_\varepsilon(x, \omega)) u_\varepsilon = \rho_\varepsilon(x, \omega) f(x), \qquad x \in D = (0, 1), \tag{2.9}$$

with Dirichlet conditions $u_\varepsilon(0) = u_\varepsilon(1) = 0$ to simplify the presentation. The random coefficients $a_\varepsilon(x, \omega)$ and $\rho_\varepsilon(x, \omega)$ are uniformly bounded from above and below: $0 < a_0 \leq a_\varepsilon(x, \omega), \rho_\varepsilon(x, \omega) \leq a_0^{-1}$. The (deterministic) absorption term q_0 is assumed to be a non-negative constant to simplify the presentation.

We assume that $a_\varepsilon(x, \omega) = a(\frac{x}{\varepsilon}, \omega)$, $q_\varepsilon(x, \omega) = q(\frac{x}{\varepsilon}, \omega)$, and $\rho_\varepsilon(x, \omega) = \rho(\frac{x}{\varepsilon}, \omega)$, where $a(x, \omega)$, $q(x, \omega)$, and $\rho(x, \omega)$ are strictly stationary processes on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume appropriate joint mixing conditions on the random processes (see Section 4) and integrability of the cross-correlation functions $R_{fg}(\mathbf{x})$ for $\{f, g\} \in \{a, q, \rho\}$, where $R_{fg}(\mathbf{x}) = \mathbb{E}\{f(\mathbf{y}, \omega) g(\mathbf{y} + \mathbf{x}, \omega)\}$. Let us define:

$$b_\varepsilon(x) = \frac{a^*}{a_\varepsilon(x, \omega)} - 1, \qquad \tilde{q}_\varepsilon(x, \omega) = q_\varepsilon(x, \omega) - q_0 b_\varepsilon(x), \quad \delta\rho_\varepsilon = \rho_\varepsilon - \mathbb{E}\{\rho\}, \tag{2.10}$$

where $(a^*)^{-1} = \mathbb{E}\{a^{-1}\}$. The process $\tilde{q}_\varepsilon(x, \omega)$ will be modified on a set of measure $O(\varepsilon)$ to ensure existence of a solution to (2.9). We denote by $G(x, y)$ the Green's function of the homogenized equation (2.9), where a_ε is replaced by a^*, q_ε by 0 and ρ_ε by $\bar{\rho} = \mathbb{E}\{\rho\}$. Then we have that

$$\frac{u_\varepsilon - u_0}{\sqrt{\varepsilon}}(x) \xrightarrow{\text{dist.}} \int_0^1 \sigma(x, t) dW_t, \tag{2.11}$$

where W_t is standard Brownian motion and

$$\begin{aligned} \sigma^2(x,t) &= 2\int_0^\infty \mathbb{E}\{F(x,t,0)F(x,t,\tau)\}d\tau, \\ F(x,t,\tau) &= H_b(x,t)b(\tau) + H_\rho(x,t)\delta\rho(\tau) - H_q(x,t)\tilde{q}(\tau). \end{aligned} \tag{2.12}$$

Here, we have defined:

$$\begin{aligned} &H_b(x,t) = \int_0^1 \Big[\chi_x(t)\frac{\partial}{\partial x}G(x,y;1) + \chi_y(t)\frac{\partial}{\partial y}G(x,y;1) + \frac{\partial}{\partial L}G(x,y;1)\Big]\bar{\rho}f(y)dy \\ &H_\rho(x,t) = G(x,t)f(t), \qquad H_q(x,t) = G(x,t)\int_0^1 G(t,z)f(z)dz, \end{aligned} \tag{2.13}$$

where $\chi_x(t) = 1$ if $0 < t < x$ and $\chi_x(t) = 0$ otherwise. The homogeneous Green's function $G(x,y) = G(x,y;1)$, where $G(x,y;L)$ is defined in (4.2) below and the homogenized solution is defined as $u_0(x) = \bar{\rho}\int_0^1 G(x,t)f(t)dt$. The proof of the result is based on a change of variables to harmonic coordinates and the techniques used to prove the convergence result in (2.8).

The above two problems may be recast as $u_\eta(\omega) = A_\eta(\omega)f$ for a deterministic source term f, where $A_\eta(\omega)$ is the solution operator and $\eta = \varepsilon^{\frac{d}{2}}$. Let A be the formal limit of A_η as $\eta \to 0$. For the above problems, we show that

$$\mathbb{E}\{\|A_\eta - A\|^2\} \lesssim \eta^2, \tag{2.14}$$

which is sufficient to adapt results in [31] and obtain Gaussian fluctuations for the leading eigenvalues and eigenvectors of the compact, self-adjoint, operator A_η. More precisely, let $(\lambda_n^\eta, u_n^\eta)$ and (λ_n, u_n) be the spectral elements of A_η and A, respectively, where the eigenvalues are ordered in decreasing order (assuming they are non-negative to simplify). The results obtained on the convergence of the source problems allow us to assume that

$$\frac{A_\eta - A}{\eta}u_n(\mathbf{x}) \xrightarrow{\text{dist.}} \int_D \sigma_n(\mathbf{x},\mathbf{y})dW_{\mathbf{y}}, \tag{2.15}$$

weakly in space for some known kernel $\sigma_n(\mathbf{x},\mathbf{y})$. Then we find that

$$\frac{\lambda_n^\eta - \lambda_n}{\eta} \xrightarrow{\text{dist.}} \int_{D^2} u_n(\mathbf{x})\sigma_n(\mathbf{x},\mathbf{y})dW_{\mathbf{y}}d\mathbf{x} := \int_D \Lambda_n(\mathbf{y})dW_{\mathbf{y}} \qquad \text{as } \eta \to 0. \tag{2.16}$$

The eigenvalue correctors are therefore Gaussian variables, which may conveniently be written as a stochastic integral that is quadratic in the eigenvectors since $\sigma_n(\mathbf{x},\mathbf{y})$ is a linear functional of u_n. The correlations between different correctors may also be obtained as

$$\mathbb{E}\Big\{\frac{\lambda_n^\eta - \lambda_n}{\eta}\frac{\lambda_m^\eta - \lambda_m}{\eta}\Big\} \xrightarrow{\eta\to 0} \int_D \Lambda_n(\mathbf{x})\Lambda_m(\mathbf{x})d\mathbf{x}. \tag{2.17}$$

By proper normalization, (u_n, u_n^η) is equal to 1 plus an error term of order $O(\eta^2)$ on average. We find the limiting behavior of the Fourier coefficients of $u_n^\eta - u_n$ and obtain that for $m \neq n$:

$$\Big(\frac{u_n^\eta - u_n}{\eta}, u_m\Big) \xrightarrow{\text{dist.}} \frac{1}{\lambda_n - \lambda_m}\int_{D^2} u_m(\mathbf{x})\sigma_n(\mathbf{x},\mathbf{y})dW_{\mathbf{y}}d\mathbf{x}. \tag{2.18}$$

Note that the convergence holds for fixed values of n as $\eta \to 0$. We do not have convergence of the eigenelements for values of, say, $n = \varepsilon^{-\gamma}$ for $\gamma > 0$. The results obtained on the spectral elements allow us to address the convergence of solutions to several evolution equations; see Section 5.3.

3. Correctors for Helmholtz equations. In this section, we analyze the convergence properties of u_ε given by (2.7) assuming that the Green's function of the unperturbed problem (2.2) satisfies (2.4).

We define $\tilde{q}_\varepsilon(\mathbf{x},\omega) = q(\frac{\mathbf{x}}{\varepsilon},\omega)$, where $q(\mathbf{x},\omega)$ is a mean zero, strictly stationary, process defined on an abstract probability space $(\Omega,\mathcal{F},\mathbb{P})$ [16]. We assume that $q(\mathbf{x},\omega)$ has an integrable correlation function defined in (2.5). We also assume that $q(\mathbf{x},\omega)$ is strongly mixing in the following sense. For two Borel sets $A, B \subset \mathbb{R}^d$, we denote by $\mathcal{F}_A$ and $\mathcal{F}_B$ the sub-σ algebras of $\mathcal{F}$ generated by the field $q(\mathbf{x},\omega)$ for $\mathbf{x} \in A$ and $\mathbf{x} \in B$, respectively. Then we assume the existence of a $(\rho-)$ mixing coefficient $\varphi(r)$ such that

$$\Big|\frac{\mathbb{E}\big\{(\eta-\mathbb{E}\{\eta\})(\xi-\mathbb{E}\{\xi\})\big\}}{\big(\mathbb{E}\{\eta^2\}\mathbb{E}\{\xi^2\}\big)^{\frac{1}{2}}}\Big| \leq \varphi\big(2\,d(A,B)\big) \tag{3.1}$$

for all (real-valued) square integrable random variables η on $(\Omega,\mathcal{F}_A,\mathbb{P})$ and ξ on $(\Omega,\mathcal{F}_B,\mathbb{P})$. Here, $d(A,B)$ is the Euclidean distance between the Borel sets A and B. The multiplicative factor 2 in (3.1) is here only for convenience. Moreover, we assume that $\varphi(r)$ is bounded and decreasing. We will impose additional restrictions on the process to ensure that the equation (2.7) admits a unique solution.

3.1. Existence of solutions and error estimates. In order for the above equation to admit a unique solution, we need to ensure that $(I-\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon)$ is invertible $\mathbb{P}-$a.s. We modify the process $\tilde{q}_\varepsilon(\mathbf{x},\omega)$ defined above on a set of measure of order ε^d so that $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon$ has spectral radius bounded by $\rho<1$ $\mathbb{P}-$a.s. To do so and to estimate the source term $\mathcal{G}f-\mathcal{G}q_\varepsilon\mathcal{G}f$ in (2.7), we need a few lemmas.

LEMMA 3.1. *Let $q(\mathbf{x},\omega)$ be strongly mixing so that* (3.1) *holds and such that $\mathbb{E}\{q^6\}<\infty$. Then, we have:*

$$\big|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\big| \lesssim \sup_{\{\mathbf{y}_k\}_k=\{\mathbf{x}_k\}_k} \varphi^{\frac{1}{2}}(|\mathbf{y}_1-\mathbf{y}_3|)\varphi^{\frac{1}{2}}(|\mathbf{y}_2-\mathbf{y}_4|)\mathbb{E}\{q^6\}^{\frac{2}{3}}. \tag{3.2}$$

We use the notation $a \lesssim b$ when there is a positive constant C such that $a \leq Cb$.

Proof. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two points in $\{\mathbf{x}_k\}_{1\leq k\leq 4}$ such that $d(\mathbf{y}_1,\mathbf{y}_2) \geq d(\mathbf{x}_i,\mathbf{x}_j)$ for all $1\leq i,j\leq 4$ and such that $d(\mathbf{y}_1,\{\mathbf{z}_3,\mathbf{z}_4\}) \leq d(\mathbf{y}_2,\{\mathbf{z}_3,\mathbf{z}_4\})$, where $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{z}_3,\mathbf{z}_4\}$ $=\{\mathbf{x}_k\}_{1\leq k\leq 4}$. Let us call $\mathbf{y}_3$ a point in $\{\mathbf{z}_3,\mathbf{z}_4\}$ closest to $\mathbf{y}_1$. We call $\mathbf{y}_4$ the remaining point in $\{\mathbf{x}_k\}_{1\leq k\leq 4}$. We have, using (3.1) and $\mathbb{E}\{q\}=0$, that:

$$\mathcal{E} := \big|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\big| \lesssim \varphi(2|\mathbf{y}_1-\mathbf{y}_3|)(\mathbb{E}\{q^2\})^{\frac{1}{2}}\big(\mathbb{E}\{(q(\mathbf{y}_2)q(\mathbf{y}_3)q(\mathbf{y}_4))^2\}\big)^{\frac{1}{2}}.$$

The last two terms are bounded by $\mathbb{E}\{q^6\}^{\frac{1}{6}}$ and $\mathbb{E}\{q^6\}^{\frac{1}{2}}$, respectively, using Hölder's inequality. Because $\varphi(r)$ is assumed to be decreasing, we deduce that:

$$\big|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\big| \lesssim \varphi(|\mathbf{y}_1-\mathbf{y}_3|)\mathbb{E}\{q^6\}^{\frac{2}{3}}. \tag{3.3}$$

If $\mathbf{y}_4$ is (one of) the closest point(s) to $\mathbf{y}_2$, then the same arguments show that

$$\big|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\big| \lesssim \varphi(|\mathbf{y}_2-\mathbf{y}_4|)\mathbb{E}\{q^6\}^{\frac{2}{3}}. \tag{3.4}$$

Otherwise, $\mathbf{y}_3$ is the closest point to $\mathbf{y}_2$, and we find that: $\mathcal{E} \lesssim \varphi(2|\mathbf{y}_2-\mathbf{y}_3|)\mathbb{E}\{q^6\}^{\frac{2}{3}}$. However, by construction, $|\mathbf{y}_2-\mathbf{y}_4| \leq |\mathbf{y}_1-\mathbf{y}_2| \leq |\mathbf{y}_1-\mathbf{y}_3|+|\mathbf{y}_3-\mathbf{y}_2| \leq 2|\mathbf{y}_2-\mathbf{y}_3|$, so (3.4) is still valid (this is where the factor 2 in (3.1) is used). Combining (3.3) and (3.4), the result follows from $a\wedge b \leq (ab)^{\frac{1}{2}}$ for $a,b\geq 0$, where $a\wedge b=\min(a,b)$. ∎

LEMMA 3.2. *Let q_ε be a stationary process $q_\varepsilon(\mathbf{x},\omega) = q(\frac{\mathbf{x}}{\varepsilon},\omega)$ with integrable correlation function in (2.5) and $f \in L^2(D)$ a deterministic function. Then we have:*

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2_{L^2(D)}\} \lesssim \varepsilon^d \|f\|^2_{L^2(D)}. \tag{3.5}$$

Let q_ε satisfy one of the following additional hypotheses:

[H1] *There is a constant C such that $|q(\mathbf{x},\omega)| < C$ $d\mathbf{x} \times \mathbb{P}$-a.s.*

[H2] *$\mathbb{E}\{q^6\} < \infty$ and $q(\mathbf{x},\omega)$ is strongly mixing with mixing coefficient in (3.1) such that $\varphi^{\frac12}(r)$ is bounded and $r^{d-1}\varphi^{\frac12}(r)$ is integrable on $\mathbb{R}^+$.*

Then we find the following bound for the operator $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon$:

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|^2_{\mathcal{L}(L^2(D))}\} \lesssim \varepsilon^d. \tag{3.6}$$

Note that [H2] and $r \mapsto \varphi(r)$ decreasing impose that $\varphi(r) = o(r^{-2d})$; see [4].

Proof [Lemma 3.2]. Here and below, we denote $\|\cdot\| = \|\cdot\|_{L^2(D)}$ and calculate

$$\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x}) = \int_D \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)f(\mathbf{z})d\mathbf{z},$$

so that by the Cauchy-Schwarz inequality, we have

$$|\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x})|^2 \le \|f\|^2 \int_D \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)^2 d\mathbf{z}.$$

By definition of the correlation function, we thus find that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2 \int_{D^4} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\boldsymbol{\zeta})R\Big(\frac{\mathbf{y}-\boldsymbol{\zeta}}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\mathbf{z})d\mathbf{x}d\mathbf{y}d\boldsymbol{\zeta}d\mathbf{z}. \tag{3.7}$$

Extending $G(\mathbf{x},\mathbf{y})$ by 0 outside $D \times D$, we find in the Fourier domain that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2 \int_{D^2}\int_{\mathbb{R}^d} |\widehat{G(\mathbf{x},\cdot)G(}\mathbf{z},\cdot)|^2(\mathbf{p})\varepsilon^d\hat{R}(\varepsilon\mathbf{p})d\mathbf{p}d\mathbf{x}d\mathbf{z}.$$

Here $\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}f(\mathbf{x})d\mathbf{x}$ is the Fourier transform of $f(\mathbf{x})$. Since $R(\mathbf{x})$ is integrable, then $\hat{R}(\varepsilon\mathbf{p})$ (which is always non-negative by e.g. Bochner's theorem) is bounded by a constant we call R_0 so that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2\varepsilon^d R_0 \int_{D^3} G^2(\mathbf{x},\mathbf{y})G^2(\mathbf{z},\mathbf{y})d\mathbf{x}d\mathbf{y}d\mathbf{z} \lesssim \|f\|^2\varepsilon^d R_0,$$

by the square-integrability assumption on $G(\mathbf{x},\mathbf{y})$. This yields (3.5). Let us now consider (3.6). We denote by $\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}}$ the norm $\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}(L^2(D))}$ and calculate that

$$\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\phi(\mathbf{x}) = \int_D \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)q_\varepsilon(\mathbf{z})\phi(\mathbf{z})d\mathbf{z}.$$

Therefore,

$$\Big(\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\phi(\mathbf{x})\Big)^2 \le \int_D \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})q_\varepsilon(\mathbf{z})d\mathbf{y}\Big)^2 d\mathbf{z}\int_D \phi^2(\mathbf{z})d\mathbf{z},$$

by Cauchy-Schwarz. This shows that

$$\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|^2_{\mathcal{L}}(\omega) \le \int_{D^2}\Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)^2 q^2_\varepsilon(\mathbf{z})d\mathbf{z}d\mathbf{x}.$$

When $q_\varepsilon(\mathbf{z},\omega)$ is bounded $\mathbb{P}-$a.s., the above proof leading to (3.5) applies and we obtain (3.6) under hypothesis [H1]. Using Lemma 3.1, we obtain that

$$\mathbb{E}\{q_\varepsilon(\mathbf{y})q_\varepsilon(\boldsymbol{\zeta})q_\varepsilon^2(\mathbf{z})\} \lesssim \varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\Big)\varphi^{\frac{1}{2}}(0) + \varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\Big)\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{z}-\boldsymbol{\zeta}|}{\varepsilon}\Big).$$

Under hypothesis [H2], we thus obtain that

$$\begin{aligned}\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|^2_{\mathcal{L}}\} \quad \lesssim \quad & \int_{D^4} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\boldsymbol{\zeta})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\mathbf{z})d\mathbf{y}d\boldsymbol{\zeta}d\mathbf{x}d\mathbf{z} \\ & + \int_{D^2}\Big(\int_D G(\mathbf{x},\mathbf{y})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)^2 d\mathbf{x}d\mathbf{z}.\end{aligned}$$

Because $r^{d-1}\varphi^{\frac{1}{2}}(r)$ is integrable, then $\mathbf{x}\mapsto\varphi^{\frac{1}{2}}(|\mathbf{x}|)$ is integrable as well and the bound of the first term above under hypothesis [H2] is done as in (3.7) by replacing $R(\mathbf{x})$ by $\varphi^{\frac{1}{2}}(|\mathbf{x}|)$. The second term is bounded, using the Cauchy-Schwarz inequality, by

$$\int_D\Big(\int_D\Big(\int_D G^2(\mathbf{x},\mathbf{y})d\mathbf{x}\Big)G^2(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)\Big(\int_D\varphi\Big(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\Big)d\mathbf{y}\Big)d\mathbf{z} \lesssim \varepsilon^d,$$

since $\mathbf{x}\mapsto\varphi(|\mathbf{x}|)$ is integrable, D is bounded, and (2.4) holds. □

Applying the previous result to the process $\tilde{q}_\varepsilon(\mathbf{x},\omega) = q(\frac{\mathbf{x}}{\varepsilon},\omega)$, we obtain from the Chebyshev inequality that

$$\mathbb{P}(\omega;\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\tilde{q}_\varepsilon\|_{\mathcal{L}} > \rho) \lesssim \frac{\mathbb{E}\{\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\tilde{q}_\varepsilon\|^2_{\mathcal{L}}\}}{\rho^2} \lesssim \varepsilon^d. \tag{3.8}$$

On the set $\Omega_\varepsilon\subset\Omega$ of measure $\mathbb{P}(\Omega_\varepsilon)\lesssim\varepsilon^d$ where $\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\tilde{q}_\varepsilon\|_{\mathcal{L}} > \rho$, we modify the potential $\tilde{q}_\varepsilon$ and set it to e.g. 0. We thus construct

$$q_\varepsilon(\mathbf{x},\omega) = \left\{\begin{array}{ll}\tilde{q}_\varepsilon(\mathbf{x},\omega) & \omega\in\Omega\backslash\Omega_\varepsilon, \\ 0 & \omega\in\Omega_\varepsilon.\end{array}\right. \tag{3.9}$$

Lemma 3.3. *The results obtained for* $\tilde{q}_\varepsilon(\mathbf{x},\omega) = q(\frac{\mathbf{x}}{\varepsilon},\omega)$ *in Lemma 3.2 hold for* $q_\varepsilon(\mathbf{x},\omega)$ *constructed in* (3.9).

Proof. For instance,

$$\begin{aligned}&\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} = \mathbb{E}\{\chi_{\Omega^\varepsilon}(\omega)\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} + \mathbb{E}\{\chi_{\Omega\backslash\Omega^\varepsilon}(\omega)\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \\ &= \mathbb{E}\{\chi_{\Omega\backslash\Omega^\varepsilon}(\omega)\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}f\|^2\} \le \mathbb{E}\{\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}f\|^2\} \lesssim \varepsilon^d\|f\|^2.\end{aligned}$$

The same proof holds for the second bound (3.6). □

We need to ensure that the oscillatory integrals studied in subsequent sections are not significantly modified when $q(\frac{\mathbf{x}}{\varepsilon},\omega)$ is replaced by the new $q_\varepsilon(\mathbf{x},\omega)$. Let $I_\varepsilon = \varepsilon^{-\frac{d}{2}}\|q(\frac{\mathbf{x}}{\varepsilon},\omega) - q_\varepsilon(\mathbf{x},\omega)\|$. Then, under [H1] or [H2], we have

$$\lim_{\varepsilon\to0}\mathbb{E}\big\{I_\varepsilon\big\} \equiv \lim_{\varepsilon\to0}\mathbb{E}\Big\{\chi_{\Omega^\varepsilon}(\omega)\Big\|\frac{1}{\varepsilon^{\frac{d}{2}}}q\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\Big\|\Big\} = 0. \tag{3.10}$$

Indeed, by Hölder's inequality, boundedness of D and stationarity of q, we have $\mathbb{E}\{I_\varepsilon\} \lesssim \mathbb{E}\{\chi^{p'}_{\Omega^\varepsilon}(\omega))\}^{\frac{1}{p'}}\varepsilon^{-\frac{d}{2}}\mathbb{E}\{|q|^p\}^{\frac{1}{p}} \lesssim \varepsilon^{d(\frac{1}{2}-\frac{1}{p})}(\mathbb{E}|q|^p)^{\frac{1}{p}}$, for $\frac{1}{p}+\frac{1}{p'}=1$. The result follows when $p>2$.

With the modified potential, (2.7) admits a unique solution $\mathbb{P}$-a.s. and we find that $\|u_\varepsilon\|(\omega) \lesssim \|\mathcal{G}f\| + \|\mathcal{G}q_\varepsilon\mathcal{G}f\|$ $\mathbb{P}-$ a.s., where $\|\cdot\|$ denotes $L^2(D)$ norm. Using the first result of Lemma 3.2, we find that

$$\mathbb{E}\{\|u_\varepsilon\|^2\} \lesssim \|f\|^2. \tag{3.11}$$

Now we can address the behavior of the correctors. We find that

$$(I - \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon)(u_\varepsilon - u_0) = -\mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f. \tag{3.12}$$

Using the results of Lemma 3.2, we obtain that

LEMMA 3.4. *Let u_ε be the solution to the heterogeneous problem* (2.1) *and u_0 the solution to the corresponding unperturbed problem. Then we have that*

$$\Big(\mathbb{E}\{\|u_\varepsilon - u_0\|^2\}\Big)^{\frac{1}{2}} \lesssim \varepsilon^{\frac{d}{2}}\|f\|. \tag{3.13}$$

Note that writing $u_\varepsilon = A_\varepsilon f$ and $u_0 = A_0 f$, with A_ε and A_0 the solution operators of the heterogeneous and homogeneous equations, respectively, we have just shown that

$$\mathbb{E}\{\|A_\varepsilon - A_0\|^2\} \lesssim \varepsilon^d. \tag{3.14}$$

Now $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)$ is bounded by ε^d in $L^1(\Omega; L^2(D))$ by Cauchy-Schwarz:

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)\|\} \le \Big(\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|^2_{\mathcal{L}}\}\Big)^{\frac{1}{2}}\Big(\mathbb{E}\{\|u_\varepsilon - u_0\|^2\}\Big)^{\frac{1}{2}} \lesssim \varepsilon^d.$$

LEMMA 3.5. *Under hypothesis* [H2] *of Lemma 3.2, we find that*

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \varepsilon^{2d\frac{1+\eta}{2+\eta}}\|f\|^2 \ll \varepsilon^d\|f\|^2, \tag{3.15}$$

where η is such that $\mathbf{y} \mapsto \Big(\int_D |G|^{2+\eta}(\mathbf{x},\mathbf{y})d\mathbf{x}\Big)^{\frac{1}{2+\eta}}$ is uniformly bounded on D.

Proof. By Cauchy-Schwarz,

$$|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x})|^2 \le \|f\|^2 \int_D \Big(\int_{D^2} G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})q_\varepsilon(\mathbf{z})G(\mathbf{z},\mathbf{t})d\mathbf{y}d\mathbf{z}\Big)^2 d\mathbf{t}.$$

So we want to estimate

$$A = \mathbb{E}\{\int_{D^6} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\boldsymbol{\zeta})q_\varepsilon(\mathbf{y})q_\varepsilon(\boldsymbol{\zeta})G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\boldsymbol{\xi})q_\varepsilon(\mathbf{z})q_\varepsilon(\boldsymbol{\xi})G(\mathbf{z},\mathbf{t})G(\boldsymbol{\xi},\mathbf{t})d[\boldsymbol{\xi}\boldsymbol{\zeta}\mathbf{yzxt}]\},$$

where $d[\mathbf{x}_1\ldots\mathbf{x}_n] \equiv d\mathbf{x}_1\ldots d\mathbf{x}_n$. We use (3.2) to obtain that $A \lesssim A_1 + A_2 + A_3$:

$$\begin{aligned} A_1 &= \int_{D^6} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\boldsymbol{\zeta})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\boldsymbol{\xi})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{z}-\boldsymbol{\xi}|}{\varepsilon}\Big)G(\mathbf{z},\mathbf{t})G(\boldsymbol{\xi},\mathbf{t})d[\boldsymbol{\xi}\boldsymbol{\zeta}\mathbf{yzxt}],\\ A_2 &= \int_{D^2}\Big(\int_{D^2} G(\mathbf{x},\mathbf{y})G(\mathbf{y},\mathbf{z})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\Big)G(\mathbf{z},\mathbf{t})d\mathbf{y}d\mathbf{z}\Big)^2 d\mathbf{t}d\mathbf{x},\\ A_3 &= \int_{D^6} G(\mathbf{x},\mathbf{y})G(\boldsymbol{\xi},\mathbf{t})G(\mathbf{x},\boldsymbol{\zeta})G(\mathbf{z},\mathbf{t})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\xi}|}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\boldsymbol{\xi})\varphi^{\frac{1}{2}}\Big(\frac{|\boldsymbol{\zeta}-\mathbf{z}|}{\varepsilon}\Big)d[\boldsymbol{\xi}\boldsymbol{\zeta}\mathbf{yzxt}]. \end{aligned}$$

Denote $F_{\mathbf{x},\mathbf{t}}(\mathbf{y},\mathbf{z}) = G(\mathbf{x},\mathbf{y})G(\mathbf{y},\mathbf{z})G(\mathbf{z},\mathbf{t})$. Then in the Fourier domain, we find that

$$A_1 \lesssim \int_{D^2}\int_{\mathbb{R}^{2d}} \varepsilon^{2d}\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{p})\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{q})|\hat{F}_{\mathbf{x},\mathbf{t}}(\mathbf{p},\mathbf{q})|^2 d\mathbf{p}d\mathbf{q}d\mathbf{x}d\mathbf{t}.$$

Here $\widehat{\varphi^{\frac{1}{2}}}(\mathbf{p})$ is the Fourier transform of $\mathbf{x} \mapsto \varphi^{\frac{1}{2}}(|\mathbf{x}|)$. Since $\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{p})$ is bounded because $r^{d-1}\varphi^{\frac{1}{2}}(r)$ is integrable on $\mathbb{R}^+$, we deduce that

$$A_1 \lesssim \varepsilon^{2d} \int_{D^4} G^2(\mathbf{x},\mathbf{y})G^2(\mathbf{y},\mathbf{z})G^2(\mathbf{z},\mathbf{t})d\mathbf{x}d\mathbf{y}d\mathbf{z}d\mathbf{t} \lesssim \varepsilon^{2d},$$

using the integrability condition imposed on $G(\mathbf{x},\mathbf{y})$.

Using $2ab \leq a^2+b^2$ for $(a,b) = (G(\mathbf{x},\mathbf{y}), G(\mathbf{x},\boldsymbol{\zeta}))$ and $(a,b) = (G(\boldsymbol{\xi},\mathbf{t}), G(\mathbf{z},\mathbf{t}))$ successively, and integrating in $\mathbf{t}$ and $\mathbf{x}$, we find that

$$A_3 \lesssim \int_{D^4} G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\boldsymbol{\xi})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\xi}|}{\varepsilon}\Big)\varphi^{\frac{1}{2}}\Big(\frac{|\boldsymbol{\zeta}-\mathbf{z}|}{\varepsilon}\Big)d[\mathbf{y}\boldsymbol{\zeta}\mathbf{z}\boldsymbol{\xi}],$$

thanks to (2.4). Now with $(a,b) = (G(\mathbf{y},\mathbf{z}), G(\boldsymbol{\zeta},\boldsymbol{\xi}))$, we find that

$$A_3 \lesssim \int_{D^4} G^2(\mathbf{y},\mathbf{z})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\xi}|}{\varepsilon}\Big)\varphi^{\frac{1}{2}}\Big(\frac{|\boldsymbol{\zeta}-\mathbf{z}|}{\varepsilon}\Big)d[\mathbf{y}\boldsymbol{\zeta}\mathbf{z}\boldsymbol{\xi}] \lesssim \varepsilon^{2d},$$

since $\varphi^{\frac{1}{2}}$ is integrable and G is square integrable on the bounded domain D.

Let us now consider the contribution A_2. We write the squared integral as a double integral over the variables $(\mathbf{y},\boldsymbol{\zeta},\mathbf{z},\boldsymbol{\xi})$ and dealing with the integration in $\mathbf{x}$ and $\mathbf{t}$ using $2ab \leq a^2+b^2$ as in the A_3 contribution, obtain that

$$A_2 \lesssim \int_{D^4} G(\mathbf{y},\boldsymbol{\zeta})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\Big)G(\mathbf{z},\boldsymbol{\xi})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{z}-\boldsymbol{\xi}|}{\varepsilon}\Big)d[\mathbf{y}\boldsymbol{\zeta}\mathbf{z}\boldsymbol{\xi}].$$

Using Hölder's inequality, we obtain that

$$A_2 \lesssim \Big(\Big(\int_0^\infty \varphi^{\frac{p'}{2}}\Big(\frac{r}{\varepsilon}\Big)r^{d-1}dr\Big)^{\frac{1}{p'}}\Big(\int_{D^2} G^p(\mathbf{y},\mathbf{z})d\mathbf{y}d\mathbf{z}\Big)^{\frac{1}{p}}\Big)^2 \lesssim \varepsilon^{2d\frac{1+\eta}{2+\eta}},$$

with $p = 2+\eta$ and $p' = \frac{2+\eta}{1+\eta}$ since $\varphi^{\frac{1}{2}}(r)r^{d-1}$, whence $\varphi^{\frac{p'}{2}}(r)r^{d-1}$, is integrable. ∎

The above lemma applies to the stationary process $\tilde{q}_\varepsilon(\mathbf{x},\omega)$, and using the same proof as in Lemma 3.3, for the modified process $q_\varepsilon(\mathbf{x},\omega)$ in (3.9). We have therefore obtained that

$$\mathbb{E}\{\|u_\varepsilon - u + \mathcal{G}q_\varepsilon\mathcal{G}f\|\} \lesssim \varepsilon^{d\frac{1+\eta}{2+\eta}}. \tag{3.16}$$

For what follows, it is useful to recast the above result as:

PROPOSITION 3.6. *Let $q(\mathbf{x},\omega)$ be constructed so that [H2] holds and let $q_\varepsilon(\mathbf{x},\omega)$ be as defined in* (3.9)*. Let u_ε be the solution to* (2.7) *and $u_0 = \mathcal{G}f$. We assume that u_0 is continuous on D. Then we have the following result:*

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big\{\Big\|\frac{u_\varepsilon - u_0}{\varepsilon^{\frac{d}{2}}} + \frac{1}{\varepsilon^{\frac{d}{2}}}\mathcal{G}q\Big(\frac{\cdot}{\varepsilon},\omega\Big)u_0\Big\|\Big\} = 0. \tag{3.17}$$

Proof. Thanks to (3.10), we may replace $q_\varepsilon(\mathbf{x},\omega)$ by $\tilde{q}_\varepsilon(\mathbf{x},\omega) = q(\frac{\mathbf{x}}{\varepsilon},\omega)$ in (3.16) up to a small error compared to $\varepsilon^{\frac{d}{2}}$. Indeed,

$$\begin{aligned}&\mathbb{E}\Big\{\Big\|\frac{1}{\varepsilon^{\frac{d}{2}}}\mathcal{G}\Big(q\Big(\frac{\cdot}{\varepsilon},\omega\Big) - q_\varepsilon(\cdot,\omega)\Big)u_0\Big\|\Big\} = \mathbb{E}\Big\{\chi_{\Omega^\varepsilon}(\omega)\Big\|\frac{1}{\varepsilon^{\frac{d}{2}}}\mathcal{G}q\Big(\frac{\cdot}{\varepsilon},\omega\Big)u_0\Big\|\Big\}\\ &\leq \|\mathcal{G}\|\|u_0\|_{L^\infty(D)}\mathbb{E}\Big\{\chi_{\Omega^\varepsilon}(\omega)\Big\|\frac{1}{\varepsilon^{\frac{d}{2}}}q\Big(\frac{\cdot}{\varepsilon},\omega\Big)\Big\|\Big\} \ll 1.\end{aligned}$$

∎

The rescaled corrector $\varepsilon^{-\frac{d}{2}}\mathcal{G}q(\frac{\cdot}{\varepsilon},\omega)u_0$ does not converge strongly to its limit. Rather, it should be interpreted as a stochastic oscillatory integral whose limiting distribution is governed by the central limit theorem [16, 23].

3.2. Oscillatory integrals and central limits. The convergence of oscillatory integrals $\varepsilon^{-\frac{d}{2}}\mathcal{G}q(\frac{\cdot}{\varepsilon},\omega)\mathcal{G}f$ to Gaussian processes is an application of the central limit theorem. It is well known in one dimension of space and can be generalized in several dimensions of space using the central limit theorem for discrete random variables as it appears in [11]. The details of the convergence are presented in [4]; we merely summarize here the main steps of the derivation.

We consider such limits first in the one-dimensional case and second for arbitrary space dimensions. In one dimension of space, the Green's function $G(x,y)$ is typically Lipschitz continuous and we will assume this regularity for the first part of this section. Then, the leading term of the corrector $\varepsilon^{-\frac{1}{2}}(u_\varepsilon - u_0)$, given by:

$$u_{1\varepsilon}(x,\omega) = \int_D -G(x,y)\frac{1}{\sqrt{\varepsilon}}q(\frac{y}{\varepsilon},\omega)u_0(y)dy, \tag{3.18}$$

is of class $\mathcal{C}(D)$ $\mathbb{P}$-a.s. and we can seek convergence in that functional class. Since $u_0 = \mathcal{G}f$, it is continuous for $f \in L^2(D)$. Then we have:

THEOREM 3.7. *Let us assume that $G(x,y)$ is Lipschitz continuous. Then, under the conditions of Proposition 3.6,*

$$u_{1\varepsilon}(x,\omega) \xrightarrow{\text{dist.}} -\sigma\int_D G(x,y)u_0(y)dW_y, \qquad \text{as } \varepsilon \to 0, \tag{3.19}$$

in the space of continuous paths $\mathcal{C}(D)$, where $dW_y(\omega)$ is the standard Wiener measure on $(\mathcal{C}(D),\mathcal{B}(\mathcal{C}(D)),\mathbb{P})$. As a consequence, the corrector to homogenization satisfies:

$$\frac{u_\varepsilon - u_0}{\sqrt{\varepsilon}}(x) \xrightarrow{\text{dist.}} -\sigma\int_D G(x,y)u_0(y)dW_y, \qquad \text{as } \varepsilon \to 0, \tag{3.20}$$

in the space of integrable paths $L^1(D)$.

Proof. To prove (3.19), we need to show tightness and convergence of the finite dimensional distributions of $u_{1\varepsilon}$. Tightness of $u_{1\varepsilon}$ follows from the fact that $\mathbb{E}\{|u_{1\varepsilon}(x,\omega)|^2\} \lesssim 1$, and that

$$\begin{aligned}
&\mathbb{E}\{|u_{1\varepsilon}(x,\omega) - u_{1\varepsilon}(\xi,\omega)|^2\} = \mathbb{E}\Big(\int_D [G(x,y)-G(\xi,y)]\frac{1}{\sqrt{\varepsilon}}q(\frac{y}{\varepsilon})u_0(y)dy\Big)^2 \\
&= \int_{D^2}[G(x,y)-G(\xi,y)][G(x,\zeta)-G(\xi,\zeta)]\frac{1}{\varepsilon}R(\frac{\zeta-y}{\varepsilon})u_0(y)u_0(\zeta)dyd\zeta \\
&\lesssim |x-\xi|^2\int_{D^2}\frac{1}{\varepsilon}|R(\frac{\zeta-y}{\varepsilon})|u_0(y)u_0(\zeta)dyd\zeta \lesssim |x-\xi|^2,
\end{aligned}$$

since the correlation function $R(r)$ is integrable and u_0 is bounded.

The convergence of the finite dimensional distributions is addressed as follows; see [4] for more details. The finite-dimensional distribution $(u_{1\varepsilon}(x_j,\omega))_{1\le j\le n}$ has the characteristic function for $\mathbf{k} = (k_1,\ldots,k_n)$:

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{i\sum_{j=1}^n k_j u_{1\varepsilon}(x_j,\omega)}\} = \mathbb{E}\{e^{i\int_D m(y)\frac{1}{\sqrt{\varepsilon}}q_\varepsilon(y)dy}\}, \quad m(y) = \sum_{j=1}^n k_j G(x_j,y)u_0(y).$$

It thus remains to show that

$$I_{m\varepsilon} := \int_D m(y)\frac{1}{\sqrt{\varepsilon}}q(\frac{y}{\varepsilon})dy \xrightarrow{\text{dist.}} I_m := \int_D m(y)\sigma dW_y, \qquad \varepsilon \to 0, \tag{3.21}$$

for arbitrary continuous moments $m(y)$. This is done by approximating $m(y)$ by $m_h(y)$ constant on intervals of size of order h so that we have to analyze random variables of the form $M_{\varepsilon j} = m_{hj} \int_{(j-1)h}^{jh} \frac{1}{\sqrt{\varepsilon}} q(\frac{y}{\varepsilon}) dy$. We show that the variables $M_{\varepsilon j}$ become independent in the limit $\varepsilon \to 0$ and converge in distribution to $m_{hj}\sigma\mathcal{N}(0,h)$, where $\mathcal{N}(0,h)$ is the centered Gaussian variable with variance h. Our assumptions on φ allow us to verify the mixing properties required in [11] to apply the central limit theorem to the discrete random variables $q_j = \int_j^{j+1} q(y)dy$ appearing in $M_{\varepsilon j}$.

This concludes the proof of the convergence in distribution of $u_{1\varepsilon}$ in the space of continuous paths $\mathcal{C}(D)$. It now remains to recall the convergence result (3.17) to obtain (3.20) in the space of integrable paths. ∎

In arbitrary dimension, the leading term in $\varepsilon^{-\frac{d}{2}}(u_\varepsilon - u_0)$ is given by:

$$u_{1\varepsilon}(\mathbf{x},\omega) = \int_D -G(\mathbf{x},\mathbf{y}) \frac{1}{\varepsilon^{\frac{d}{2}}} q_\varepsilon(\mathbf{y},\omega) u_0(\mathbf{y}) d\mathbf{y}. \tag{3.22}$$

Because of the singularities of the Green's function $G(\mathbf{x},\mathbf{y})$ in dimension $d \geq 2$, we obtain convergence of the above corrector in distribution on $(\Omega, \mathcal{F}, \mathbb{P})$ and weakly in D. More precisely, let $M_k(\mathbf{x})$, $1 \leq k \leq K$, be sufficiently smooth functions such that

$$m_k(\mathbf{y}) = -\int_D M_k(\mathbf{x}) G(\mathbf{x},\mathbf{y}) u_0(\mathbf{y}) d\mathbf{x} = -\mathcal{G} M_k(\mathbf{y}) u_0(\mathbf{y}), \quad 1 \leq k \leq K, \tag{3.23}$$

are continuous functions (we thus assume that $u_0(\mathbf{x})$ is continuous as well). Let us introduce the random variables

$$I_{k\varepsilon}(\omega) = \int_D m_k(\mathbf{y}) \frac{1}{\varepsilon^{\frac{d}{2}}} q\Big(\frac{\mathbf{y}}{\varepsilon},\omega\Big) d\mathbf{y}. \tag{3.24}$$

Because of (3.10), the accumulation points of the integrals $I_{k\varepsilon}(\omega)$ are not modified if $q(\frac{\mathbf{y}}{\varepsilon},\omega)$ is replaced by $q_\varepsilon(\mathbf{y},\omega)$. Then we have:

THEOREM 3.8. *Under the above conditions and the hypotheses of Proposition 3.6, the random variables $I_{k\varepsilon}(\omega)$ converge in distribution to the mean zero Gaussian random variables $I_k(\omega)$ as $\varepsilon \to 0$, where the correlation matrix is given by*

$$\Sigma_{jk} = \mathbb{E}\{I_j I_k\} = \sigma^2 \int_D m_j(\mathbf{y}) m_k(\mathbf{y}) d\mathbf{y}. \tag{3.25}$$

Here, we have defined:

$$\sigma^2 = \int_{\mathbb{R}^d} \mathbb{E}\{q(\mathbf{0})q(\mathbf{y})\} d\mathbf{y}, \qquad I_k(\omega) = \int_D m_k(\mathbf{y}) \sigma dW_{\mathbf{y}}, \tag{3.26}$$

where $dW_{\mathbf{y}}$ is standard multi-parameter Wiener process [33]. As a result, for $M(\mathbf{x})$ sufficiently smooth, we obtain that

$$\Big(\frac{u_\varepsilon - u_0}{\varepsilon^{\frac{d}{2}}}, M\Big) \xrightarrow{\text{dist.}} -\sigma \int_D \mathcal{G}M(\mathbf{y}) \mathcal{G}f(\mathbf{y}) dW_{\mathbf{y}}. \tag{3.27}$$

Proof. The convergence in (3.27) is a direct consequence of the second equality in (3.26) and the strong convergence (3.17) in Proposition 3.6. The second equality in (3.26) is directly obtained from (3.25) since $I_k(\omega)$ is a (multivariate) Gaussian variable. In order to prove (3.25), we use a methodology similar to that in the proof of Theorem 3.7.

The convergence in (3.21) is now multi-dimensional and we approximate $m(\mathbf{y})$ by $m_h(\mathbf{y})$, which is constant and equal to m_{hj} on small hyper-cubes $\mathcal{C}_j$ of size h (and volume h^d); there are $M \approx h^{-d}$ of them. Because ∂D is assumed to be sufficiently smooth, it can be covered by $M_S \approx h^{-d+1}$ cubes and we set $m_h(\mathbf{x}) = 0$ on those cubes. We then define the random variables $M_{\varepsilon j}(\omega) = m_{hj} \int_{\mathcal{C}_j} \frac{1}{\varepsilon^{\frac{d}{2}}} q(\frac{\mathbf{y}}{\varepsilon}, \omega) d\mathbf{y}$, show that they become asymptotically independent as $\varepsilon \to 0$, and apply the central limit theorem for the discrete random variables $q_{\mathbf{j}}(\omega) = \int_{\mathbf{j}+[\mathbf{0},\mathbf{1}]} q(\mathbf{y}, \omega) d\mathbf{y}$ with $\mathbf{j} \in \mathbb{Z}^d$ and $[\mathbf{0}, \mathbf{1}] = [0, 1]^d$. Our assumptions on $\varphi(r)$ allow us to show that the variables $q_{\mathbf{j}}(\omega)$ are sufficiently mixing so that the central limit theorem in [11, 15] applies; see [4]. □

3.3. Larger fluctuations, random and periodic homogenization. The results stated in the preceding section generalize to larger fluctuations of the form:

$$\tilde{q}_\varepsilon(\mathbf{x}, \omega) = \frac{1}{\varepsilon^{\alpha d}} q\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big), \tag{3.28}$$

with q_ε the same modification of $\tilde{q}_\varepsilon$ as before. The corrector $-\mathcal{G} q_\varepsilon \mathcal{G} f$ is now of order $\varepsilon^{d(\frac{1}{2}-\alpha)}$ for $0 \le \alpha < \frac{1}{2}$. The next-order corrector, given by $\mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon \mathcal{G} f$ in (3.12), is bounded in $L^1(\Omega \times D)$ by $\varepsilon^{d(\frac{1+\eta}{2+\eta} - 2\alpha)}$ according to Lemma 3.5. The order of this term is smaller than the order of the leading corrector $\varepsilon^{d(\frac{1}{2}-\alpha)}$ again provided that $0 \le \alpha < \frac{\eta}{2(2+\eta)}$, which converges to $\frac{1}{2}$ for $d = 1, 2$ as $\eta \to \infty$ and converges to $\frac{1}{6}$ for $d = 3$ as $\eta \to 1$. In dimension $d = 1, 2$, we can infer from these results that $\varepsilon^{-d(\frac{1}{2}-\alpha)}(u_\varepsilon - u_0)$ converges in distribution to the limits obtained in the preceding sections as $\varepsilon \to 0$ provided that $0 \le \alpha < \frac{1}{2}$. The proof presented in this paper extends only to the values $0 \le \alpha < \frac{1}{4}$ since it is based on imposing that the spectral radius of $\mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon$ be sufficiently small using (3.6) in Lemma 3.2. For (3.28), this translates into $\mathbb{E}\{\|\mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon\|^2_{\mathcal{L}(L^2(D))}\} \lesssim \varepsilon^{d(1-4\alpha)}$. We then verify that all results leading to Proposition 3.6 generalize when $0 < \alpha < \frac{1}{4}$ to yield (3.17) with $\varepsilon^{\frac{d}{2}}$ replaced by $\varepsilon^{d(\frac{1}{2}-\alpha)}$. Note that in the limiting case $\alpha = \frac{1}{2}$, u_ε does not converge to the deterministic solution u_0 as is shown in the temporal one-dimensional case in [43].

Let $P(\mathbf{x}, \mathbf{D}) = -\nabla \cdot a(\mathbf{x}) \nabla + q_0(\mathbf{x})$. The results on the corrector $u_\varepsilon - u_0$ obtained in Theorems 3.7 and 3.8 are valid for $1 \le d \le 3$. If we admit the expansion in (2.7) that $u_\varepsilon - u_0 = -\mathcal{G} q_\varepsilon u_0$ plus smaller order terms, then the results obtained in Theorem 3.8 show that $u_\varepsilon - u_0$ converges weakly in space and in distribution to a process of order $O(\varepsilon^{\frac{d}{2}})$ for all dimensions. The theory of this paper does not allow us to justify (2.7) when $d \ge 4$ because the Green's functions are no longer square integrable. Corrections of order ε^2 thus correspond to a transition that we also find in the periodic case:

$$-\Delta u_\varepsilon + q\Big(\frac{\mathbf{x}}{\varepsilon}\Big) u_\varepsilon = f \quad \text{ in } D, \tag{3.29}$$

with $u_\varepsilon = 0$ on ∂D, defined on a smooth open, bounded, domain $D \subset \mathbb{R}^d$, where $q(\mathbf{y})$ is $[0, 1]^d$-periodic. Following [8], we introduce the fast scale $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$ and introduce a decomposition $u_\varepsilon = u_\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$. Replacing $\nabla_{\mathbf{x}}$ by $\frac{1}{\varepsilon}\nabla_{\mathbf{y}} + \nabla_{\mathbf{x}}$ in (3.29) and equating like powers of ε yields three equations; see [4] for additional details. The first equation shows that $u_0 = u_0(\mathbf{x})$. We can choose $u_1(\mathbf{x}) = 0$ in the second equation. The third equation $-\Delta_{\mathbf{y}} u_2 - \Delta_{\mathbf{x}} u_0 + q(\mathbf{y}) u_0 = f(\mathbf{x})$, admits a solution provided that $-\Delta_{\mathbf{x}} u_0 + \langle q \rangle u_0 = f(\mathbf{x})$, in D with $u_0 = 0$ on ∂D. Here, $\langle q \rangle$ is the average of q on $[0, 1]^d$, which we assume is sufficiently large so that the above equation admits a unique solution. We recast the above equation as $u_0 = \mathcal{G}_D f$. The corrector u_2

thus solves $-\Delta_{\mathbf{y}} u_2 = \big(\langle q\rangle - q(\mathbf{y})\big)u_0(\mathbf{x})$, and by the Fredholm alternative is uniquely defined along with the constraint $\langle u_2\rangle = 0$. We denote the solution operator of the above cell problem as $\mathcal{G}_\#$ so that $u_2(\mathbf{x},\mathbf{y}) = -\mathcal{G}_\#(q-\langle q\rangle)(\mathbf{y})\mathcal{G}f(\mathbf{x})$. Thus formally:

$$u_\varepsilon(\mathbf{x}) = \mathcal{G}f(\mathbf{x}) - \varepsilon^2\mathcal{G}_\#(q-\langle q\rangle)\Big(\frac{\mathbf{x}}{\varepsilon}\Big)\mathcal{G}f(\mathbf{x}) + \text{ l.o.t.} \tag{3.30}$$

We thus observe that the corrector $u_{2\varepsilon}(\mathbf{x}) := u_2(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})$ is of order $O(\varepsilon^2)$ in the L^2 sense, say. In the sense of distributions, however, integrations by parts show that the corrector may be of order $o(\varepsilon^m)$ for all integer m in the sense that the oscillatory integral $\int_D M(\mathbf{x})u_{2\varepsilon}(\mathbf{x})d\mathbf{x} \ll \varepsilon^m$ for all m when $M(\mathbf{x})u_0(\mathbf{x}) \in \mathcal{C}_0^\infty(D)$.

A similar behavior occurs for the random integral

$$v_{1\varepsilon}(\mathbf{x},\omega) = \int_D -G(\mathbf{x},\mathbf{y})q\Big(\frac{\mathbf{y}}{\varepsilon},\omega\Big)u_0(\mathbf{y})d\mathbf{y}, \tag{3.31}$$

which behaves like $\varepsilon^{\frac{d}{2}}u_{1\varepsilon}$ defined in (3.22) thanks to (3.10). Theorem 3.8 shows that $(v_{1\varepsilon}, M(\mathbf{x}))$ is of order $O(\varepsilon^{\frac{d}{2}})$ for $M(\mathbf{x})$ and $u_0(\mathbf{x})$ sufficiently smooth and that $\varepsilon^{-\frac{d}{2}}(v_{1\varepsilon}, M(\mathbf{x}))$ converges in distribution to a Gaussian random variable. This result, however, does not hold in the $L^2(D)-$sense for $d \geq 4$ when $G(\mathbf{x},\mathbf{y})$ is the fundamental solution of the Helmholtz equation $-\Delta + q_0(\mathbf{x})$ on D. Indeed, we prove that:

PROPOSITION 3.9. *For $u_0(\mathbf{x})$ and $\hat{R}(\boldsymbol{\xi})$ Hölder continuous, we have:*

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x},\omega)\} \sim \begin{cases} \varepsilon^d \hat{R}(\mathbf{0})\displaystyle\int_D G^2(\mathbf{x},\mathbf{y})u_0^2(\mathbf{y})d\mathbf{y} & 1 \leq d \leq 3 \\ \varepsilon^4 |\ln\varepsilon| \dfrac{(2\pi)^4\hat{R}(\mathbf{0})}{c_4}u_0^2(\mathbf{x}) & d = 4 \\ \varepsilon^4 u_0^2(\mathbf{x})(2\pi)^d \displaystyle\int_{\mathbb{R}^d}\frac{\hat{R}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^4}d\boldsymbol{\xi} & d \geq 5. \end{cases} \tag{3.32}$$

Here $a_\varepsilon \sim b_\varepsilon$ means $a_\varepsilon = b_\varepsilon(1+o(1))$.

Proof. We calculate:

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x},\omega)\} = \int_{D^2} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\mathbf{z})R\Big(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\Big)u_0(\mathbf{y})u_0(\mathbf{z})d\mathbf{y}d\mathbf{z}. \tag{3.33}$$

Extending $G(\mathbf{x},\cdot)$ by 0 outside of D, by the Parseval equality the above term is equal to $(2\pi)^d\int_{\mathbb{R}^{2d}} |\mathcal{F}_{\mathbf{y}\to\boldsymbol{\xi}}(G(\mathbf{x},\mathbf{y})u_0(\mathbf{y}))|^2(\boldsymbol{\xi})\varepsilon^d\hat{R}(\varepsilon\boldsymbol{\xi})d\boldsymbol{\xi}$, where $\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}}$ is the Fourier transform from $\mathbf{x}$ to $\boldsymbol{\xi}$. In dimension $1 \leq d \leq 3$, since $\hat{R}(\varepsilon\boldsymbol{\xi}) \to \hat{R}(\mathbf{0})$ pointwise, the Lebesgue dominated convergence theorem yields the result. In dimension $d \geq 4$, however, the Green function is no longer square integrable and the integral is larger than ε^d.

For $d \geq 4$, we replace $G(\mathbf{x},\mathbf{y})$ by $c_d|\mathbf{x}-\mathbf{y}|^{2-d}$ where c_d is the measure of the unit sphere S^{d-1}. The difference $G(\mathbf{x},\mathbf{y}) - c_d|\mathbf{x}-\mathbf{y}|^{2-d}$ is a function bounded by $C|\mathbf{x}-\mathbf{y}|^{3-d}$, which yields a smaller contribution to $\mathbb{E}\{v_{1\varepsilon}^2\}$. We also replace $u_0(\mathbf{y})$ by $u_0(\mathbf{x})$, up to an error bounded by $|\mathbf{x}-\mathbf{y}|^\alpha$ as soon as $u_0(\mathbf{x})$ is of class $\mathcal{C}^{0,\alpha}(D)$. Similarly, we replace $u_0(\mathbf{z})$ by $u_0(\mathbf{x})$ and thus obtain that

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x},\omega)\} \sim u_0^2(\mathbf{x})\int_{D^2}\frac{1}{c_d|\mathbf{x}-\mathbf{y}|^{d-2}}\frac{1}{c_d|\mathbf{x}-\mathbf{z}|^{d-2}}R\Big(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\Big)d\mathbf{y}d\mathbf{z}.$$

Let $\alpha > 0$ and $B(\mathbf{x},\alpha)$ the ball of center $\mathbf{x}$ and radius α so that $B(\mathbf{x},\alpha) \subset D$. Because all singularities occur when $\mathbf{y}$ and $\mathbf{z}$ are in the vicinity of $\mathbf{x}$, we use the proof of the case $1 \le d \le 3$ to show that up to a term of order ε^d, we can replace D by $B(\mathbf{x},\alpha)$:

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x},\omega)\} \sim u_0^2(\mathbf{x}) \int_{B^2(\mathbf{0},\alpha)} \frac{1}{c_d|\mathbf{y}|^{d-2}} \frac{1}{c_d|\mathbf{z}|^{d-2}} R\Big(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\Big) d\mathbf{y} d\mathbf{z}. \tag{3.34}$$

Now for $d \ge 5$, using the dominated convergence theorem, we can replace $B(\mathbf{0},\alpha)$ by $\mathbb{R}^d$ because the Green function is square integrable at infinity, whence

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x},\omega)\} \sim u_0^2(\mathbf{x}) \int_{\mathbb{R}^{2d}} \frac{1}{c_d|\mathbf{y}|^{d-2}} \frac{1}{c_d|\mathbf{z}|^{d-2}} R\Big(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\Big) d\mathbf{y} d\mathbf{z}.$$

This, however, by the Parseval equality, is equal to

$$\mathbb{E}\{v_{1\varepsilon}^2(\mathbf{x},\omega)\} \sim u_0^2(\mathbf{x})(2\pi)^d \int_{\mathbb{R}^d} \frac{1}{|\boldsymbol{\xi}|^4} \varepsilon^d \hat{R}(\varepsilon\boldsymbol{\xi}) d\boldsymbol{\xi} = u_0^2(\mathbf{x})(2\pi)^d \int_{\mathbb{R}^d} \frac{1}{|\boldsymbol{\xi}|^4} \varepsilon^4 \hat{R}(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

since the Fourier transform of the fundamental solution of the Laplacian is $|\boldsymbol{\xi}|^{-2}$.

When $d = 4$, we come back to (3.34), and replace one of the integrals (in $\mathbf{z}$) on $B(\mathbf{0},\alpha)$ by an integral on $\mathbb{R}^d$ using again the dominated convergence theorem and the other integral by an integration on $B_\varepsilon^\alpha = B(\mathbf{0},\alpha) \cap B(\mathbf{0},\varepsilon)$, with an error that we can verify is of order $O(\varepsilon^4)$. This yields the term

$$\begin{aligned}
&\int_{B(\mathbf{0},\alpha)\times\mathbb{R}^d} \frac{1}{c_4^2|\mathbf{y}|^2|\mathbf{z}|^2} R\Big(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\Big) d\mathbf{y} d\mathbf{z} = \int_{B_\varepsilon^\alpha\times\mathbb{R}^d} \frac{(2\pi)^4\varepsilon^2}{c_4|\mathbf{y}|^2|\boldsymbol{\xi}|^2} \hat{R}(\boldsymbol{\xi}) e^{i\frac{\boldsymbol{\xi}\cdot\mathbf{y}}{\varepsilon}} d\boldsymbol{\xi} d\mathbf{y} + O(\varepsilon^4) \\
&= \int_{B_1^{\frac{\alpha}{\varepsilon}}\times\mathbb{R}^d} \frac{(2\pi\varepsilon)^4}{c_4|\mathbf{y}|^2|\boldsymbol{\xi}|^2} \hat{R}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\mathbf{y}} d\boldsymbol{\xi} d\mathbf{y} + O(\varepsilon^4) = \hat{R}(\mathbf{0})(2\pi\varepsilon)^4 \int_{B_1^{\frac{\alpha}{\varepsilon}}} \frac{1}{c_4^2|\mathbf{y}|^4} d\mathbf{y} + O(\varepsilon^4) \\
&= \frac{\hat{R}(\mathbf{0})(2\pi\varepsilon)^4}{c_4} \int_1^{\frac{\alpha}{\varepsilon}} \frac{|\mathbf{y}|^3}{|\mathbf{y}|^4} d|\mathbf{y}| + O(\varepsilon^4) = \frac{\hat{R}(\mathbf{0})(2\pi\varepsilon)^4}{c_4} |\ln\varepsilon| + O(\varepsilon^4).
\end{aligned}$$

Here, we have assumed that $|\hat{R}(\boldsymbol{\xi}) - \hat{R}(\mathbf{0})|$ was bounded by $C|\boldsymbol{\xi}|^\beta$ for some $\beta > 0$. ◻

In all dimensions, $\varepsilon^{-\frac{d}{2}} v_{1\varepsilon}(\mathbf{x},\omega)$ converges (weakly and in distribution) to a limit $u_1(\mathbf{x},\omega) = -\int_D G(\mathbf{x},\mathbf{y}) u_0(\mathbf{y}) dW_{\mathbf{y}}$. In dimension $1 \le d \le 3$, we have proved that u_1 was the limit of $\varepsilon^{-\frac{d}{2}}(u_\varepsilon - u_0)$. The above calculation shows that the limit u_1 captures all the energy in the oscillations of the homogenization corrector $\varepsilon^{-\frac{d}{2}} v_{1\varepsilon}$ in the sense that $\varepsilon^{-d}\mathbb{E}\{\|v_{1\varepsilon}\|^2_{L^2(D)}\}$ converges to $\mathbb{E}\{\|u_1\|^2_{L^2(D)}\}$.

In higher dimension $d \ge 4$, as in the case of homogenization in periodic media, most of the energy is lost while passing to the (weak) limit. While the energy of the asymptotic corrector $\varepsilon^{\frac{d}{2}} u_1$ is $\varepsilon^{\frac{d}{2}}(\mathbb{E}\{\|u_1\|^2_{L^2(D)}\}^{\frac{1}{2}} = O(\varepsilon^{\frac{d}{2}})$, the energy of the random corrector $v_{1\varepsilon}$ (and that of $\varepsilon^{\frac{d}{2}} u_{1\varepsilon}$) is $(\mathbb{E}\{\|v_{1\varepsilon}\|^2_{L^2(D)}\}^{\frac{1}{2}}$, which is of order $O(\varepsilon^2)$ for $d \ge 5$ and of order $O(\varepsilon^2|\ln\varepsilon|^{\frac{1}{2}})$ for $d = 4$. Most of the energy of the random correctors to homogenization is lost when passing from $u_{1\varepsilon}$ or $\varepsilon^{-\frac{d}{2}} v_{1\varepsilon}$ to its weak limit u_1 because $u_{1\varepsilon}$ remains highly oscillatory in dimension $d \ge 4$.

4. Correctors for one-dimensional elliptic problems. In this section, we consider the homogenization of the one-dimensional elliptic problem (2.9) presented in Section 2. We assume that the random coefficients are jointly strongly mixing in the sense of (3.1), where for two Borel sets A and B in $\mathbb{R}^d$, we denote by $\mathcal{F}_A$ and $\mathcal{F}_B$ the σ-algebras generated by the random fields $a(\mathbf{x},\omega)$, $q(\mathbf{x},\omega)$, and $\rho(\mathbf{x},\omega)$ for $\mathbf{x} \in A$ and $\mathbf{x} \in B$, respectively. We still assume that the ρ-mixing coefficient $\varphi(r)$ is integrable and such that $\varphi^{\frac{1}{2}}$ is also integrable.

In the case where $q_\varepsilon = 0$ and $\rho_\varepsilon = 0$, the corrector to the homogenization limit u_0 has been considered in [13]. For general sufficiently mixing coefficients a_ε with positive variance $\sigma^2 = 2\int_0^\infty \mathbb{E}\{a(0)a(t)\}dt > 0$, we obtain that $u_\varepsilon - u_0$ is of order $\sqrt{\varepsilon}$ and converges in distribution to a Gaussian process. This section aims at generalizing the result to (2.9) using the results of the preceding section and the following change of variables in harmonic coordinates [34]:

$$z_\varepsilon(x) = a^* \int_0^x \frac{1}{a_\varepsilon(t)}dt, \qquad \frac{dz_\varepsilon}{dx} = \frac{a^*}{a_\varepsilon(x)}, \qquad a^* = \frac{1}{\mathbb{E}\{a^{-1}\}}, \tag{4.1}$$

and $\tilde{u}_\varepsilon(z) = u_\varepsilon(x)$. Note that $\mathbb{E}\{z_\varepsilon(x)\} = x$. Then we find, with $x = x(z_\varepsilon)$, that

$$-(a^*)^2 \frac{d^2}{dz^2}\tilde{u}_\varepsilon + a^* q_0 \tilde{u}_\varepsilon + a_\varepsilon[(1 - a_\varepsilon^{-1}a^*)q_0 + q_\varepsilon]\tilde{u}_\varepsilon = a_\varepsilon \rho_\varepsilon f, \qquad 0 < z < z_\varepsilon(1)$$

with $\tilde{u}_\varepsilon(0) = \tilde{u}_\varepsilon(z_\varepsilon(1)) = 0$. Let us introduce the following Green's function

$$-a^* \frac{d^2}{dx^2} G(x,y;L) + q_0 G(x,y;L) = \delta(x-y) \tag{4.2}$$

with $G(0,y;L) = G(L,y;L) = 0$ and set $G(x,y) = G(x,y;1)$. With $\tilde{q}_\varepsilon(x,\omega)$ defined in (2.10), we find that

$$\begin{aligned} \tilde{u}_\varepsilon(z) &= \int_0^{z_\varepsilon(1)} G(z,y;z_\varepsilon(1))(\rho_\varepsilon f - \tilde{q}_\varepsilon \tilde{u}_\varepsilon)(x(y))\frac{a_\varepsilon}{a^*}(x(y))dy, \\ u_\varepsilon(x) &= \int_0^1 G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1))(\rho_\varepsilon f - \tilde{q}_\varepsilon u_\varepsilon)(y)dy. \end{aligned}$$

Upon defining $\mathcal{G}_\varepsilon u(x) = \int_0^1 G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1))u(y)dy$, we obtain that:

$$u_\varepsilon = \mathcal{G}_\varepsilon \rho_\varepsilon f - \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \rho_\varepsilon f + \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon u_\varepsilon. \tag{4.3}$$

Since $a_0 a^* x \leq z_\varepsilon(x,\omega) \leq a^* a_0^{-1} x$ $\mathbb{P}-$a.s., the Green's operator $\mathcal{G}_\varepsilon$ is bounded $\mathbb{P}-$a.s. and the results of Lemma 3.2 generalize to the case where the operator $\mathcal{G}_\varepsilon$ replaces $\mathcal{G}$. As in (3.8), we thus modify $\tilde{q}_\varepsilon$ (i.e. we modify a_ε and q_ε) on a set of measure $O(\varepsilon)$ so that $\|\mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon\|_{\mathcal{L}} \leq r < 1$ and obtain that (3.10) holds. Let us introduce the notation:

$$\rho_\varepsilon = \bar{\rho} + \delta\rho_\varepsilon,\ \bar{\rho} = \mathbb{E}\{\rho\}, \qquad \mathcal{G}_\varepsilon = \mathcal{G} + \delta\mathcal{G}_\varepsilon, \quad \mathcal{G} = \mathbb{E}\{\mathcal{G}_\varepsilon\}, \quad u_0 = \mathcal{G}\bar{\rho}f. \tag{4.4}$$

$$\delta z_\varepsilon(x) = z_\varepsilon(x) - x = \int_0^x b\Big(\frac{t}{\varepsilon}\Big)dt, \qquad b(t,\omega) = \frac{a^*}{a(t,\omega)} - 1. \tag{4.5}$$

We first obtain the

Lemma 4.1. *The operator $\mathcal{G}_\varepsilon$ may be decomposed as $\mathcal{G}_\varepsilon = \mathcal{G} + \mathcal{G}_{1\varepsilon} + \mathcal{R}_\varepsilon$, with*

$$\mathcal{G}_{1\varepsilon} f(x) = \int_0^1 \Big(\delta z_\varepsilon(x)\frac{\partial}{\partial x} + \delta z_\varepsilon(y)\frac{\partial}{\partial y} + \delta z_\varepsilon(1)\frac{\partial}{\partial L}\Big)G(x,y;1)f(y)dy, \tag{4.6}$$

and $\mathcal{G}f(x) = \int_0^1 G(x,y)f(y)dy$. We have the following estimates:

$$\mathbb{E}\{\|\mathcal{G}_{1\varepsilon}\|^2\} + \mathbb{E}\{\|\mathcal{R}_\varepsilon\|\} + \mathbb{E}\{|\delta z_\varepsilon(x)\delta z_\varepsilon(y)|\} \lesssim \varepsilon, \qquad 0 \leq x,y \leq 1. \tag{4.7}$$

Proof. We first use the fact that $\mathbb{E}\{|\delta z_\varepsilon(x)\delta z_\varepsilon(y)|\} \leq \big(\mathbb{E}\{(\delta z_\varepsilon(x)\delta z_\varepsilon(y))^2\}\big)^{\frac12}$. Denoting by $b_\varepsilon(x,\omega) = b(\frac{t}{\varepsilon},\omega)$, we have to show that

$$\mathbb{E}\Big\{\int_0^x\int_0^x\int_0^y\int_0^y b_\varepsilon(z_1)b_\varepsilon(z_2)b_\varepsilon(z_3)b_\varepsilon(z_4)d[z_1z_2z_3z_4]\Big\} \lesssim \varepsilon^2.$$

Now using the mixing property of the mean zero field b_ε and the integrability of $\varphi^{\frac12}(r)$, we obtain the result using (3.2) as in the proof of Lemma 3.5. The integral defining $\mathcal{G}_\varepsilon$ is split into two contributions, according as $y<x$ or $y>x$. On these two intervals, $G(x,y;L)$ is twice differentiable, and we thus have the expansion

$$G(z_\varepsilon(x),z_\varepsilon(y);z_\varepsilon(1)) = G(x,y;1)+\Big(\delta z_\varepsilon(x)\frac{\partial}{\partial x}+\delta z_\varepsilon(y)\frac{\partial}{\partial y}+\delta z_\varepsilon(1)\frac{\partial}{\partial L}\Big)G(x,y;1)+r_\varepsilon,$$

where the Lagrange remainder $r_\varepsilon = r_\varepsilon(x,z_\varepsilon(x),y,z_\varepsilon(y),z_\varepsilon(1))$ is quadratic in the variables $(\delta z_\varepsilon(x),\delta z_\varepsilon(y),\delta z_\varepsilon(1))$ and involves second-order derivatives of $G(x,y;1)$ at points (ξ,ζ,L) between $(x,y;1)$ and $(z_\varepsilon(x),z_\varepsilon(y);z_\varepsilon(1))$.

From (4.7) and the fact that second-order derivatives of G are $\mathbb{P}-$a.s. uniformly bounded on each interval $y<x$ and $y>x$ (we use here again the fact that $a_0a^*x \leq z_\varepsilon(x,\omega) \leq a^*a_0^{-1}x$ $\mathbb{P}-$a.s.), we thus obtain that $\mathbb{E}\{|r_\varepsilon(.)|\} \lesssim \varepsilon$. This also shows the bound for $\mathbb{E}\{\|\mathcal{R}_\varepsilon\|\}$ in (4.7). The bound for $\mathbb{E}\{\|\mathcal{G}_{1\varepsilon}\|^2\}$ is obtained similarly. ◻

Because we have assumed that $\tilde q_\varepsilon$ and ρ_ε were bounded uniformly, we can replace $\mathcal{G}_\varepsilon$ by $\mathcal{G}+\mathcal{G}_{1\varepsilon}$ in (4.3) up to an error of order ε in $L^1(\Omega;L^2(D))$. The case of q_ε and ρ_ε bounded on average would require us to address their correlation with r_ε defined in the proof of the preceding lemma. This is not considered here. We recast (4.3) as

$$u_\varepsilon - u_0 = (\mathcal{G}_\varepsilon\rho_\varepsilon - \mathcal{G}\bar\rho)f - \mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\rho_\varepsilon f + \mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\tilde q_\varepsilon(u_\varepsilon-u_0) + \mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}f. \tag{4.8}$$

Because $G(z_\varepsilon(x),z_\varepsilon(y);z_\varepsilon(1))$ and ρ_ε are uniformly bounded, the proof of Lemma 3.2 generalizes to give us that

$$\mathbb{E}\{\|\mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\tilde q_\varepsilon\|^2\} + \mathbb{E}\{\|\mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\rho_\varepsilon f\|^2\} + \mathbb{E}\{\|(\mathcal{G}_\varepsilon\rho_\varepsilon - \mathcal{G}\bar\rho)f\|^2\} \lesssim \varepsilon. \tag{4.9}$$

So far, using $\|\mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\tilde q_\varepsilon\|_{\mathcal{L}} \leq r < 1$, we have thus obtained the following result:

LEMMA 4.2. *Let u_ε be the solution to the heterogeneous problem* (2.9) *and $u_0 = \bar\rho\mathcal{G}f$ the solution to the corresponding homogenized problem. Then we have that*

$$\big(\mathbb{E}\{\|u_\varepsilon - u_0\|^2\}\big)^{\frac12} \lesssim \sqrt{\varepsilon}\|f\|. \tag{4.10}$$

The estimate (3.14) with $d=1$ is thus verified in the context of the elliptic equation (2.9). As a consequence, we find that $\mathbb{E}\{\|u_\varepsilon-u_0\|^2\} \lesssim \varepsilon$ so that by Cauchy-Schwarz and (4.9), $\mathbb{E}\{\|\mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\tilde q_\varepsilon(u_\varepsilon-u_0)\|\} \lesssim \varepsilon$. It remains to exhibit the term of order $\sqrt{\varepsilon}$ in $u_\varepsilon - u_0$. Let us introduce the decomposition

$$\begin{aligned} u_\varepsilon - u_0 &= \Big[\mathcal{G}_{1\varepsilon}\bar\rho + \mathcal{G}\delta\rho_\varepsilon - \mathcal{G}\tilde q_\varepsilon\mathcal{G}\bar\rho\Big]f + s_\varepsilon, \qquad (4.11)\\ s_\varepsilon &= (\delta\mathcal{G}_\varepsilon\delta\rho_\varepsilon + \mathcal{R}_\varepsilon\bar\rho)f - (\mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\rho_\varepsilon - \mathcal{G}\tilde q_\varepsilon\mathcal{G}\bar\rho)f + \mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\tilde q_\varepsilon(u_\varepsilon-u_0) + \mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}_\varepsilon\tilde q_\varepsilon\mathcal{G}f. \end{aligned}$$

LEMMA 4.3. *Let $f \in L^2(D)$. We have*

$$\mathbb{E}\{\|s_\varepsilon\|\} \lesssim \varepsilon\|f\|. \tag{4.12}$$

Proof. Because $G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1))$ is uniformly bounded, the proof of Lemma 3.5 generalizes to show that $\mathbb{E}\{\|\mathcal{G}_\varepsilon\tilde{q}_\varepsilon\mathcal{G}_\varepsilon\tilde{q}_\varepsilon\mathcal{G}f\|^2\} \lesssim \varepsilon^2\|f\|^2$. We already know that $\mathbb{E}\{\|\mathcal{R}_\varepsilon\|\} \lesssim \varepsilon$. It remains to address the terms $I_1 = \mathcal{G}_{1\varepsilon}\delta\rho_\varepsilon f$, $I_2 = \mathcal{G}\tilde{q}_\varepsilon\mathcal{G}_{1\varepsilon}\rho_\varepsilon f$, $I_3 = \mathcal{G}_{1\varepsilon}\tilde{q}_\varepsilon\mathcal{G}\rho_\varepsilon f$, and $I_4 = \mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\delta\rho_\varepsilon f$. Because ρ_ε is uniformly bounded $\mathbb{P}-$a.s., the first three terms are handled in a similar way. Let us consider $\mathbb{E}\{I_1^2\}$, which is bounded by the sum of three terms of the form:

$$\mathbb{E}\{\int_{D^3} \delta z_\varepsilon(v_1(x,y))H(x,y)\delta z_\varepsilon(v_2(x,\zeta))H(x,\zeta)\delta\rho_\varepsilon(y)\delta\rho_\varepsilon(\zeta)f(y)f(\zeta)dxdyd\zeta\},$$

where $v_k(x,y)$ is either x, y, or 1 for $k = 1,2$, and $H(x,y)$ is a uniformly bounded function. Using the definition of δz_ε, we recast the above integral as

$$\int_{D^3}\int_0^{v_1}\int_0^{v_2} \mathbb{E}\{b_\varepsilon(t_1)b_\varepsilon(t_2)\delta\rho_\varepsilon(y)\delta\rho_\varepsilon(\zeta)\}dt_1dt_2H(x,y)H(x,\zeta)f(y)f(\zeta)dxdyd\zeta.$$

Using (3.2), we see that the above integral is bounded by terms of the form

$$\int_{D^3}\int_0^{v_1}\int_0^{v_2} \varphi^{\frac12}\Big(\frac{u_1-u_2}{\varepsilon}\Big)\varphi^{\frac12}\Big(\frac{u_3-u_4}{\varepsilon}\Big)dt_1dt_2|H(x,y)H(x,\zeta)||f(y)||f(\zeta)|dxdyd\zeta,$$

where $(u_1,u_2,u_3,u_4) = (u_1,u_2,u_3,u_4)(t_1,t_2,y,\zeta)$ is an arbitrary (fixed) permutation of (t_1,t_2,y,ζ). Because $\varphi(r)$ is integrable, the Cauchy-Schwarz inequality shows that the above term is $\lesssim \varepsilon^2\|f\|^2$. The term $\mathbb{E}\{I_4^2\}$ is given by

$$\mathbb{E}\Big\{\int_{D^4} G(x,y)G(x,\zeta)\tilde{q}_\varepsilon(y)\tilde{q}_\varepsilon(\zeta)G(y,z)G(\zeta,\xi)\delta\rho_\varepsilon(z)\delta\rho_\varepsilon(\xi)f(z)f(\xi)d[xyz\zeta\xi]\Big\}.$$

Since $G(x,y)$ is uniformly bounded on D, we again use (3.2) as above to obtain a bound of the form $\varepsilon^2\|f\|^2$. □

It remains to analyze the convergence of the contribution $[\mathcal{G}_{1\varepsilon}\bar{\rho}+\mathcal{G}\delta\rho_\varepsilon-\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\bar{\rho}]f$. As in (3.18), we define $u_{1\varepsilon}(x,\omega) = \frac{1}{\sqrt{\varepsilon}}\Big[\mathcal{G}_{1\varepsilon}\bar{\rho}+\mathcal{G}\delta\rho_\varepsilon-\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\bar{\rho}\Big]f(x)$, which we recast as

$$u_{1\varepsilon}(x,\omega) = \frac{1}{\sqrt{\varepsilon}}\int_0^1 \Big[b\Big(\frac{t}{\varepsilon}\Big)H_b(x,t) + \delta\rho\Big(\frac{t}{\varepsilon}\Big)H_\rho(x,t) - \tilde{q}\Big(\frac{t}{\varepsilon}\Big)H_q(x,t)\Big]dt, \qquad (4.13)$$

with the kernels defined in (2.13). We have the following result:

THEOREM 4.4. *Let $f \in L^\infty(0,1)$. The process $u_{1\varepsilon}(x,\omega)$ converges weakly and in distribution in the space of continuous paths $\mathcal{C}(D)$ to the limit $u_1(x,\omega)$ given by*

$$u_1(x,\omega) = \int_0^1 \sigma(x,t)dW_t, \qquad (4.14)$$

where W_t is standard Brownian motion and $\sigma(x,t)$ is defined in (2.12).

The corrector to homogenization thus satisfies that:

$$\frac{u_\varepsilon - u_0}{\sqrt{\varepsilon}}(x) \xrightarrow{\text{dist.}} u_1(x,\omega), \qquad \text{as } \varepsilon \to 0, \qquad (4.15)$$

in the space of integrable paths $L^1(D)$.

Note that we may recast $u_{1\varepsilon}(x,\omega)$ as $u_{1\varepsilon}(x,\omega) = \sum_{k=1}^3 \frac{1}{\sqrt{\varepsilon}}\int_D p_k(\frac{t}{\varepsilon})H_k(x,t)dt$, where the p_k are mean zero processes and the kernels $H_k(x,t)$ are given in (2.13). The corrector in (4.14) may then be rewritten as

$$u_1(x) = \sum_{k=1}^3 \int_D \sigma_k(x,t)dW_t^j, \qquad (4.16)$$

with three correlated standard Brownian motions such that $dW_t^j dW_t^k = \rho_{jk} dt$, where:

$$\begin{aligned} \sigma_k(x,t) &= H_k(x,t)\sqrt{2}\Big(\int_0^\infty \mathbb{E}\{p_k(0)p_k(\tau)\}d\tau\Big)^{\frac{1}{2}} \\ \rho_{jk} &= \frac{\displaystyle\int_0^\infty \mathbb{E}\{p_j(0)p_k(\tau)+p_k(0)p_j(\tau)\}d\tau}{2\Big(\displaystyle\int_0^\infty \mathbb{E}\{p_j(0)p_j(\tau)\}d\tau \int_0^\infty \mathbb{E}\{p_k(0)p_k(\tau)\}d\tau\}\Big)^{\frac{1}{2}}}. \end{aligned} \tag{4.17}$$

That (4.14) and (4.16) are equivalent comes from the straightforward calculation that both processes are mean zero Gaussian processes with the same correlation function. The new equation (4.16) shows the linearity of $u_1(x)$ with respect to $f(x)$.

Proof. We recast $u_{1\varepsilon}(x,\omega)$ as

$$u_{1\varepsilon}(x,\omega) = \sum_k \frac{1}{\sqrt{\varepsilon}} \int_D q_k(\frac{t}{\varepsilon}) H_k(x,t) dt,$$

with a decomposition similar to but different from (4.13) above and where the q_k are mean-zero processes. We verify that we can choose the terms $H_k(x,t)$ in the above decomposition so that all of them are uniformly (in t) Lipschitz in x, except for one term, say $H_1(x,t)$, which is of the form

$$H_1(x,t) = \chi_x(t) L_1(x,t), \qquad L_1(x,t) = \int_0^1 \frac{\partial}{\partial x} G(x,y;1)\bar{\rho} f(y) dy,$$

where $L_1(x,t)$ is uniformly (in t) Lipschitz in x. This results from the fact that $G(x,y;1)$ is Lipschitz continuous and that its partial derivatives are bounded and piecewise Lipschitz continuous; we leave the tedious details to the reader.

Because of the presence of the term $H_1(x,t)$ in the above expression, it is not sufficient to consider second-order moments of $u_{1\varepsilon}$ to show tightness as in the proof of Theorem 3.7. Rather, we consider fourth-order moments as follows:

$$\begin{aligned} \mathbb{E}\{|u_{1\varepsilon}(x,\omega) - u_{1\varepsilon}(\xi,\omega)|^4\} = \; & \frac{1}{\varepsilon^2} \sum_{k_1,k_2,k_3,k_4} \int_{D^4} \mathbb{E}\{q_{k_1}(\frac{t_1}{\varepsilon}) q_{k_2}(\frac{t_2}{\varepsilon}) q_{k_3}(\frac{t_3}{\varepsilon}) q_{k_4}(\frac{t_4}{\varepsilon})\} \times \\ & \prod_{m=1}^4 (H_{k_m}(x,t_m) - H_{k_m}(\xi,t_m)) dt_1 dt_2 dt_3 dt_4. \end{aligned}$$

Using the mixing condition of the processes q_k and Lemma 3.1 (where each q in (3.2) may be replaced by q_k without any change in the result), we obtain that $\mathbb{E}\{|u_{1\varepsilon}(x,\omega) - u_{1\varepsilon}(\xi,\omega)|^4\}$ is bounded by a sum of terms of the form

$$\frac{1}{\varepsilon^2} \int_{D^4} \varphi^{\frac{1}{2}}(\frac{t_2-t_1}{\varepsilon}) \varphi^{\frac{1}{2}}(\frac{t_4-t_3}{\varepsilon}) \prod_{m=1}^4 (H_{k_m}(x,t_m) - H_{k_m}(\xi,t_m)) dt_1 dt_2 dt_3 dt_4,$$

whence is bounded by terms of the form

$$\Big(\frac{1}{\varepsilon} \int_{D^2} \varphi^{\frac{1}{2}}(\frac{t_2-t_1}{\varepsilon}) \prod_{m=1}^2 (H_{k_m}(x,t_m) - H_{k_m}(\xi,t_m)) dt_1 dt_2\Big)^2.$$

When all the kernels H_{k_m} are Lipschitz continuous, then the above term is of order $|x-\xi|^4$. The largest contribution is obtained when $k_1 = k_2 = 1$ because $H_1(x,t)$ is not uniformly Lipschitz continuous. We concentrate on that contribution and recast

$$H_1(x,t) - H_1(\xi,t) = (\chi_x(t) - \chi_\xi(t)) L_1(x,t) + \chi_\xi(t)(L_1(x,t) - L_1(\xi,t)).$$

Again, the largest contribution to the fourth moment of $u_{1\varepsilon}$ comes from the term $(\chi_x(t) - \chi_\xi(t))L_1(x,t)$ since $L_1(x,t)$ is Lipschitz continuous. Assuming that $x \geq \xi$ without loss of generality, we calculate that

$$\int_{D^2} (\chi_x(t) - \chi_\xi(t))L_1(x,t)(\chi_\xi(s) - \chi_\xi(s))L_1(\xi,s)\frac{1}{\varepsilon}\varphi^{\frac{1}{2}}(\frac{t-s}{\varepsilon})dtds$$
$$= \int_\xi^x \int_\xi^x L_1(x,t)L_1(\xi,s)\frac{1}{\varepsilon}\varphi^{\frac{1}{2}}(\frac{t-s}{\varepsilon})dtds \lesssim (x-\xi),$$

since $\varphi^{\frac{1}{2}}$ is integrable. Note that this term is not of order $|\xi - x|^2$. Nonetheless, we have shown that $\mathbb{E}\{|u_{1\varepsilon}(x,\omega) - u_{1\varepsilon}(\xi,\omega)|^4\} \lesssim |\xi - x|^2$, so that we can apply the Kolmogorov criterion in [9] and obtain tightness of $u_{1\varepsilon}(x,\omega)$ as a process with values in the space of continuous functions $\mathcal{C}(D)$.

The finite-dimensional distributions are treated as in the proof of Theorem 3.7 and are replaced by the analysis of random integrals of the form:

$$\frac{1}{\sqrt{\varepsilon}} \int_0^1 \Big[b\Big(\frac{t}{\varepsilon}\Big)m_b(t) + \delta\rho\Big(\frac{t}{\varepsilon}\Big)m_\rho(t) + \tilde{q}\Big(\frac{t}{\varepsilon}\Big)m_q(t)\Big]dt.$$

The functions m are continuous and can be approximated by m_h constant on intervals of size h so that we end up with M independent (in the limit $\varepsilon \to 0$) variables of the form: $\frac{\sqrt{h}}{\sqrt{N}} \sum_{j=1}^N m_{bh}b_j + m_{\rho h}\delta\rho_j + m_{qh}\tilde{q}_j$. It remains to apply the central limit theorem as in the proof of Theorem 3.7. The above random variable converges in distribution to

$$\mathcal{N}(0, h\sigma^2), \quad \sigma^2 = 2\int_0^\infty \mathbb{E}\{(m_{bh}b + m_{\rho h}\delta\rho + m_{qh}\tilde{q})(0)(m_{bh}b + m_{\rho h}\delta\rho + m_{qh}\tilde{q})(t)\}dt.$$

This concludes our analysis of the convergence in distribution of $u_{1\varepsilon}$ to its limit in the space of continuous paths $\mathcal{C}(D)$. The convergence of $u_\varepsilon - u_0$ follows from the bound (4.12). ∎

5. Correctors for spectral problems. For $\omega \in \Omega$, let $A_\eta(\omega)$ be a sequence of bounded (uniformly in ω $\mathbb{P}-$a.s. and in $\eta > 0$), compact, self-adjoint operators, converging to a deterministic, compact, self-adjoint operator A as $\eta \to 0$ in the sense that the following error estimate holds:

$$\mathbb{E}\|A_\eta(\omega) - A\|^p \lesssim \eta^p, \qquad \text{for some } 1 \leq p < \infty, \tag{5.1}$$

where $\|A_\eta(\omega) - A\|$ is the $L^2(D)$ norm and D is an open subset of $\mathbb{R}^d$.

The operators A and $\mathbb{P}-$a.s. $A_\eta(\omega)$ admit the spectral decompositions (λ_n, u_n) and $(\lambda_n^\eta, u_n^\eta)$, where the real-valued eigenvalues are ordered in decreasing order of their absolute values and counted m_n times, where m_n is their multiplicity.

For λ_n, let μ_n be (one of) the closest eigenvalue of A that is different from λ_n. Let us then define the distance:

$$d_n = \frac{|\lambda_n - \mu_n|}{2}. \tag{5.2}$$

Following [31], we analyze the spectrum of A_η in the vicinity of λ_n. Let Γ be the circle of center λ_n and radius d_n in the complex plane and let $R(\zeta, A) = (A - \zeta)^{-1}$ be the resolvent of A defined for $\zeta \notin \sigma(A)$, the spectrum of A. The projection operator onto the spectral components of B inside the curve Γ is defined by

$$P_n[B] = -\frac{1}{2\pi i}\int_\Gamma R(\zeta, B)d\zeta. \tag{5.3}$$

Note that for all $\zeta \in \Gamma$, we have that $R(\zeta, A)P_n[A] = (\lambda_n - \zeta)^{-1}$. We have:

PROPOSITION 5.1. *Let A_η and A be the operators described above and let λ_n be fixed. Then, for η sufficiently small with respect to d_n, we can choose m_n eigenvalues λ_n^η of A_η in the vicinity of λ_n so that the following estimate holds:*

$$\mathbb{E}\{|\lambda_n - \lambda_n^\eta|^p\} + \mathbb{E}\{|\|u_n^\eta - u_n\|^p\} \lesssim \frac{\eta^p}{d_n^p} \wedge 1, \tag{5.4}$$

for a suitable labeling of the eigenvectors u_n^η of A^η associated to the eigenvalues λ_n^η.

Proof. It follows from [31, Theorem IV.3.18] that for those realizations ω such that $\|A_\eta(\omega) - A\| < d_n$, then there are exactly m_n eigenvalues of A_η in the d_n-vicinity of Γ. Since this also holds for every λ_m such that $d_m > d_n$, we can index the eigenvalues of A_η as the eigenvalues of A. Moreover, $|\lambda_n^\eta(\omega) - \lambda_n| \le \|A_\eta(\omega) - A\|$. For those realizations ω such that $\|A_\eta(\omega) - A\| \ge d_n$, we choose m_n eigenvalues of $A_\eta(\omega)$ arbitrarily among the eigenvalues that have not been chosen in the d_m-vicinity of λ_m for $|\lambda_m| > |\lambda_n|$. For all realizations, we thus obtain that

$$|\lambda_n^\eta(\omega) - \lambda_n| \lesssim \frac{\|A_\eta(\omega) - A\|}{d_n}.$$

It remains to take the pth power and average the above expression to obtain the first inequality of the proposition.

In order for the eigenvectors u_n^η and u_n to be close, we need to restrict the size of η further. To make sure the eigenvectors are sufficiently close, we need to ensure that

$$P_n[A_\eta] - P_n[A] = \frac{-1}{2\pi i}\int_\Gamma [R(\zeta, A_\eta) - R(\zeta, A)]d\zeta = \frac{1}{2\pi i}\int_\Gamma R(\zeta, A_\eta)(A_\eta - A)R(\zeta, A)d\zeta,$$

is sufficiently small. On the circle Γ and for $\|A - A_\eta\| < d_n$, we verify that

$$\sup_{\zeta\in\Gamma} \|R(\zeta, A)\| = \frac{1}{d_n}, \qquad \sup_{\zeta\in\Gamma} \|R(\zeta, A_\eta)\| \le \frac{1}{d_n - \|A - A_\eta\|},$$

by construction of d_n and by using $R^{-1}(\zeta, A_\eta) = R^{-1}(\zeta, A) + (A_\eta - A)$ and the triangle inequality $\|R^{-1}(\zeta, A_\eta)\| \ge \|R^{-1}(\zeta, A)\| - \|A_\eta - A\| \ge d_n - \|A_\eta - A\|$. Upon integrating the expression for $P_n[A_\eta] - P_n[A]$ on Γ, we find for $2\|A_\eta - A\| < d_n$ that

$$\rho := \|P_n[A_\eta] - P_n[A]\| \le \frac{\|A_\eta - A\|}{d_n - \|A_\eta - A\|} \le \frac{2}{d_n}\|A_\eta - A\| < 1.$$

For self-adjoint operators A and A_η, the above bound on the distance ρ between the eigenspaces is sufficient to characterize the distance between the corresponding eigenvectors. We follow [31, I.4.6 & II.4.2] and construct the unitary operator

$$U_n^\eta = \Big(I - (P_n[A_\eta] - P_n[A])^2\Big)^{-\frac{1}{2}} \Big(P_n[A_\eta]P_n[A] + (I - P_n[A_\eta])(I - P_n[A])\Big). \tag{5.5}$$

Let $u_{n,k}$, $1 \le k \le m_n$ be the eigenvectors associated to λ_n, $n \ge 1$. The eigenspace associated to λ_n^η admits for an orthonormal basis the eigenvectors defined by [31]:

$$u_{n,k}^\eta = U_n^\eta u_{n,k}, \qquad 1 \le k \le m_n. \tag{5.6}$$

The relation (5.5) may be recast as

$$U_n^\eta = (I - R_n^\eta)\big(I + P_n[A_\eta](P_n[A_\eta] - P_n[A]) + (P_n[A_\eta] - P_n[A])P_n[A_\eta]\big),$$

where $\|R_n^\eta\| \lesssim \rho^2$. This shows that $\|U_n^\eta - I\| \lesssim \rho$ and $\|u_{n,k}^\eta - u_{n,k}\| \lesssim \rho \lesssim d_n^{-1}\|A_\eta - A\|$, $1 \leq k \leq m_n$, whenever $d_n^{-1}\|A_\eta - A\| < \mu$ for μ sufficiently small. When $d_n^{-1}\|A_\eta - A\| \geq \mu$, we find that $\|u_{n,k} - u_{n,k}^\eta\| \lesssim 2\mu\|A_\eta(\omega) - A\|/d_n$, where the vectors $u_{n,k}^\eta$ are constructed as an arbitrary orthonormal basis of the eigenspace associated to λ_n^η. Upon taking pth power and ensemble averaging, we obtain (5.4). □

5.1. Correctors for eigenvalues and eigenvectors. Let (λ_n, u_n) be a solution of $Au_n = \lambda_n u_n$ and let λ_n^η and u_n^η be the solution of $A_\eta u_n^\eta = \lambda_n^\eta u_n^\eta$ defined in Proposition 5.1. We assume that (5.1) holds with $p = 2$. We calculate that

$$\frac{\lambda_n^\eta - \lambda_n}{\eta} = \Big(u_n, \frac{A_\eta - A}{\eta} u_n\Big) + \frac{1}{\eta}\Big(u_n^\eta - u_n, \big((A_\eta - \lambda_n^\eta) - (A - \lambda_n)\big)u_n\Big).$$

The last term, which we denote by $r_n^\eta(\omega)$ is bounded by $O(\eta)$ in $L^1(\Omega)$ using the results of Proposition 5.1 with $p = 2$ and the Cauchy-Schwarz inequality. Thus, $r_n^\eta(\omega)$ converges to 0 in probability. Let us assume that the eigenvectors are defined on a domain $D \subset \mathbb{R}^d$ and that for a smooth function $M(\mathbf{x})$, we have:

$$\Big(M(\mathbf{x}), \frac{A_\eta - A}{\eta} u_n(\mathbf{x})\Big) \xrightarrow{\text{dist.}} \int_{D^2} M(\mathbf{x})\sigma_n(\mathbf{x},\mathbf{y}) dW_{\mathbf{y}} d\mathbf{x} \qquad \text{as } \eta \to 0. \tag{5.7}$$

Using this result, and provided that the eigenvectors $u_n(\mathbf{x})$ are sufficiently smooth, we obtain that

$$\frac{\lambda_n^\eta - \lambda_n}{\eta} \xrightarrow{\text{dist.}} \int_{D^2} u_n(\mathbf{x})\sigma_n(\mathbf{x},\mathbf{y}) dW_{\mathbf{y}} d\mathbf{x} := \int_D \Lambda_n(\mathbf{y}) dW_{\mathbf{y}} \qquad \text{as } \eta \to 0. \tag{5.8}$$

The eigenvalue correctors are therefore Gaussian variables, which may conveniently be written as a stochastic integral that is quadratic in the eigenvectors since $\sigma_n(\mathbf{x},\mathbf{y})$ is a linear functional of u_n. The correlations between different correctors may also obviously be obtained as

$$\mathbb{E}\Big\{\frac{\lambda_n^\eta - \lambda_n}{\eta} \frac{\lambda_m^\eta - \lambda_m}{\eta}\Big\} \xrightarrow{\eta \to 0} \int_D \Lambda_n(\mathbf{x})\Lambda_m(\mathbf{x}) d\mathbf{x}. \tag{5.9}$$

Let us now turn to the corrector for the eigenvectors. Note that $\|u_n - u_n^\eta\|^2 = 2(1 - (u_n, u_n^\eta))$, so that (u_n, u_n^η) is equal to 1 plus an error term of order $O(\eta^2)$ on average. The construction of the eigenvectors in (5.6) shows that $u_n - u_n^\eta$ is of order $O(\eta^2)$ in the whole eigenspace associated to the eigenvalue λ_n. It thus remains to analyze the convergence properties of $(u_n - u_n^\eta, u_m)$ for all $m \neq n$. A straightforward calculation similar to the one obtained for the eigenvalue corrector shows that

$$\Big(\frac{u_n^\eta - u_n}{\eta}, (A - \lambda_n)u_m\Big) = -\Big(\frac{(A_\eta - \lambda_n^\eta) - (A - \lambda_n)}{\eta} u_n, u_m\Big) - \frac{1}{\eta}((A_\eta - A)(u_n^\eta - u_n), u_m).$$

The last term converges to 0 in probability (and is in fact of order $O(\eta)$ in $L^1(\Omega)$ as above). We thus find that

$$\Big(\frac{u_n^\eta - u_n}{\eta}, u_m\Big) \xrightarrow{\text{dist.}} \frac{1}{\lambda_n - \lambda_m} \int_{D^2} u_m(\mathbf{x})\sigma_n(\mathbf{x},\mathbf{y}) dW_{\mathbf{y}} d\mathbf{x}. \tag{5.10}$$

The Fourier coefficients of the eigenvector correctors converge to Gaussian random variables. As in the case of eigenvalues, this allows us to estimate the cross-correlations of the Fourier coefficients corresponding to (possibly) different eigenvectors.

5.2. Applications. The first application pertains to the following problem:

$$A_\varepsilon = (P(\mathbf{x}, D) + q_\varepsilon)^{-1}, \qquad A = P(\mathbf{x}, D)^{-1}. \tag{5.11}$$

Lemma 3.4 and its corollary (3.14) show that (5.1) holds with $p = 2$ and $\eta = \varepsilon^{\frac{d}{2}}$. The operators A_ε and A are also compact and self-adjoint for a large class of operators $P(\mathbf{x}, D)$ which includes the Helmholtz operator $P(\mathbf{x}, D) = -\Delta + q_0(\mathbf{x})$. Let $(\lambda_n^\varepsilon, u_n^\varepsilon)$ be the solutions of $\lambda^\varepsilon P_\varepsilon u_\varepsilon = u_\varepsilon$ and (λ_n, u_n) the solutions of $\lambda P u = u$. Then,

$$\frac{\lambda_n^\varepsilon - \lambda_n}{\varepsilon^{\frac{d}{2}}} \xrightarrow{\text{dist.}} -\lambda_n \sigma \int_{D^2} u_n(\mathbf{x}) G(\mathbf{x}, \mathbf{y}) u_n(\mathbf{y}) dW_{\mathbf{y}} d\mathbf{x} = -\lambda_n^2 \sigma \int_D u_n^2(\mathbf{y}) dW_{\mathbf{y}}, \tag{5.12}$$

or equivalently, that for the eigenvalues of P_ε and P, we have:

$$\frac{(\lambda_n^\varepsilon)^{-1} - \lambda_n^{-1}}{\varepsilon^{\frac{d}{2}}} \xrightarrow{\text{dist.}} \sigma \int_D u_n^2(\mathbf{y}) dW_{\mathbf{y}}. \tag{5.13}$$

The Fourier coefficients of the eigenvectors satisfy similar expressions.

The second example is the one-dimensional elliptic equation (2.9). Still setting $\eta = \varepsilon^{\frac{1}{2}}$, we find that $\varepsilon^{-\frac{1}{2}}(A_\varepsilon - A)u_n \xrightarrow{\text{dist.}} \int_D \sigma_n(x, t) dW_t$, where $\sigma_n(x, t)$ is defined in (2.12) with the source term f in (2.13) being replaced by $u_n(x)$. The operators A_ε and A satisfy (5.1) with $p = 2$ thanks to Lemma 4.2 and its corollary (3.14). The expressions for the eigenvalue and eigenvector correctors are thus directly given by (2.16) and (2.18), respectively.

5.3. Correctors for time dependent problems. As an application of the preceding theory, let us now consider an evolution problem of the form $u_t + \theta P u = 0$, for $t >$ with $u(0) = u_0$, where $\theta = 1$ or $\theta = i$, and P is a symmetric pseudodifferential operator with domain $\mathcal{D}(P) \subset L^2(D)$ for some subset $D \subset \mathbb{R}^d$ and with a compact inverse $A = P^{-1}$, which we assume, without loss of generality, has positive eigenvalues.

We then consider the randomly perturbed problem $u_t^\eta + \theta P_\eta u_\eta = 0$, for $t > 0$ with $u_\eta(0) = u_0$, where $P_\eta(\omega)$ verifies the same hypotheses as P with compact inverse $A_\eta = P_\eta^{-1}$. We assume that (5.1) holds. We denote by λ_n and λ_n^η the eigenvalues of A and A_η and by u_n and u_n^η the corresponding eigenvectors. We have that

$$u(t) = e^{-\theta t P} u_0 = \sum_n e^{-\theta \lambda_n t} (u_n, u_0) u_n := \sum_n \alpha_n(t) u_n, \qquad \alpha_n(t) = e^{-\theta \lambda_n t}(u_n, u_0).$$

and $u_\eta(t) = \sum_n \alpha_n^\eta(t) u_n^\eta$, and $\alpha_n^\eta(t) = e^{-\theta \lambda_n^\eta t}(u_\eta^n, u_0)$. We verify that, for $|s_\eta| \xrightarrow{\eta \to 0} 0$:

$$\frac{\alpha_n^\eta - \alpha_n}{\eta} = e^{-\theta \lambda_n t} \theta t \frac{\lambda_n - \lambda_n^\eta}{\eta} (u_n, u_0) + e^{-\theta \lambda_n t} (\frac{u_n^\eta - u_n}{\eta}, u_0) + s_\eta. \tag{5.14}$$

The above difference thus converges to a mean zero Gaussian random variable whose variance may easily be estimated from the results obtained in the preceding section. We do not control the convergence of the eigenvectors for arbitrary values of n and thus cannot obtain the law of the full corrector $u_\eta(t) - u(t)$. We can, however, obtain a corrector for the low frequency parts $u_N^\eta(t)$ and $u^N(t)$ of $u_\eta(t)$ and $u(t)$, respectively, where only the first N terms are kept in the summation. We may also estimate the corrector for $(u_N^\eta(t) - u_N(t), u_m)$ using the above expansion for the Fourier coefficients and the results obtained in the preceding section. We again obtain that the corrector is a mean zero Gaussian variable whose variance may be calculated explicitly.

Other time-dependent equations may be treated in a similar way. For instance, the wave equation $u_{tt} + Pu = 0$, with $u(0) = u_0$ and $u_t(0) = g_0$, and P a symmetric operator with compact and positive definite inverse, may be recast as

$$w_t - Aw = 0, \quad w(0) = w_0, \qquad w = \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ P & 0 \end{pmatrix}. \tag{5.15}$$

We verify that the eigenvalues λ_n of A are purely imaginary and equal to $\pm i\sqrt{\lambda_P}$, where λ_P are the positive eigenvalues of P. We thus obtain that

$$\begin{pmatrix} u \\ u_t \end{pmatrix}(t) = \sum_\lambda e^{-\lambda t}\Pi_{A,\lambda}\begin{pmatrix} u_0 \\ g_0 \end{pmatrix}, \quad \text{with} \quad \Pi_{A,\lambda} = \begin{pmatrix} \Pi_{P,-\lambda^2} & 0 \\ 0 & \lambda\Pi_{P,-\lambda^2} \end{pmatrix}, \tag{5.16}$$

the orthogonal projector onto the nth eigenspace of A. A similar expression may be used for the perturbed problem $u_\eta(t)$, where P is replaced by P_η. The results presented earlier in this section easily generalize to provide an estimate for the low frequency component of $u(t) - u_\eta(t)$.

REFERENCES

[1] G. Allaire and R. Brizzi, *A multiscale finite element method for numerical homogenization*, Multiscale Model. Simul., 3 (2005), pp. 790–812.

[2] T. Arbogast, *Analysis of a two-scale, locally conservative subgrid upscaling for elliptic problems*, SIAM J. Numer. Anal., 42 (2004), pp. 576–598.

[3] I. Babuška, R. Tempone, and G. E. Zouraris, *Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 1251–1294.

[4] G. Bal, *Central limits and homogenization in random media*, arXiv:0710.0363, (2007).

[5] ———, *Homogenization in random media and effective medium theory for high frequency waves*, Disc. Cont. Dyn. Syst. B, 8 (2007), pp. 473–492.

[6] G. Bal, J. Garnier, S. Motsch, and V. Perrier, *Random integrals and correctors in homogenization*, submitted, (2008).

[7] G. Bal and K. Ren, *in preparation*.

[8] A. Bensoussan, J.-L. Lions, and G. C. Papanicolaou, *Boundary layers and homogenization of transport processes*, Res. Inst. Math. Sci., Kyoto Univ., 15 (1979), pp. 53–157.

[9] P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons, New York, 1999.

[10] P. Blankenship and G. C. Papanicolaou, *Stability and control of stochastic systems with wide-band noise disturbances*, SIAM J. Appl. Math., 34 (1978), pp. 437–476.

[11] E. Bolthausen, *On the central limit theorem for stationary mixing random fields*, Ann. Probab., 10 (1982), pp. 1047–1050.

[12] A. Bourgeat, A. Mikelić, and S. Wright, *Stochastic two-scale convergence in the mean and applications*, J. reine angew. Math, 456 (1994), pp. 19–51.

[13] A. Bourgeat and A. Piatnitski, *Estimates in probability of the residual between the random and the homogenized solutions of one-dimensional second-order operator*, Asympt. Anal., 21 (1999), pp. 303–315.

[14] ———, *Approximations of effective coefficients in stochastic homogenization*, Ann. I. H. Poincaré, 40 (2004), pp. 153–165.

[15] R. C. Bradley, *On the spectral density and asymptotic normality of weakly dependent random fields*, J. Theoret. Probab., 5 (1992), pp. 355–373.

[16] L. Breiman, *Probability*, Classics in Applied Mathematics, SIAM, Philadelphia, 1992.

[17] L. A. Caffarelli, P. E. Souganidis, and L. Wang, *Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media*, Comm. Pure Appl. Math., 58 (2005), pp. 319–361.

[18] R. Cameron and W. Martin, *The orthogonal development of nonlinear functionals in series of fourier-hermite functionals*, Ann. of Math., 48 (1947), pp. 385–392.

[19] J. G. Conlon and A. Naddaf, *On homogenization of elliptic equations with random coefficients*, Electron. J. Probab., 5 (2000), pp. 9–58.

[20] M. Cudna and T. Komorowski, *A finite dimensional approximation of the effective diffusivity for a symmetric random walk in a random environment*, J. Comp. Appl. Math., (2007).
[21] W. E, P. Ming, and P. Zhang, *Analysis of the heterogeneous multiscale method for elliptic homogenization problems*, J. Amer. Math. Soc., 1 (2005), pp. 121–156.
[22] Y. R. Efendiev, T. Y. Hou, and X.-H. Wu, *Convergence of a nonconforming multiscale finite element method*, SIAM J. Numer. Anal., 37 (2000), pp. 888–910.
[23] S. N. Ethier and T. G. Kurtz, *Markov processes. Characterization and convergence*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986.
[24] R. Figari, E. Orlandi, and G. Papanicolaou, *Mean field and Gaussian approximation for partial differential equations with random coefficients*, SIAM J. Appl. Math., 42 (1982), pp. 1069–1077.
[25] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave Propagation and Time Reversal in Randomly Layered Media*, Springer, New York, 2007.
[26] J.-P. Fouque and E. Merzbach, *A limit theorem for linear boundary value problems in random media*, Ann. Appl. Probab., 4 (1994), pp. 549–569.
[27] P. Frauenfelder, C. Schwab, and R. A. Todor, *Finite elements for elliptic problems with stochastic coefficients*, Comput. Methods Appl. Mech. Eng., 194 (2005), pp. 205–228.
[28] R. G. Ghanem and P. D. Spanos, *Stochastic finite elements: a spectral approach*, Springer-Verlag, New York, 1991.
[29] T. Y. Hou, W. Luo, B. Rozovskii, and H.-M. Zhou, *Wiener chaos expansions and numerical solutions of randomly forced equations of fluid mechanics*, J. Comput. Phys., 216 (2006), pp. 687–706.
[30] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, New York, 1994.
[31] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.
[32] R. Z. Khasminskii, *Stochastic stability of differential equations*, Sijthoff & Noordhoff, 1980.
[33] D. Khoshnevisan, *Multiparameter processes. An introduction to random fields*, Springer Verlag, New York, 2002.
[34] S. M. Kozlov, *The averaging of random operators*, Math. USSR Sb., 109 (1979), pp. 188–202.
[35] H. J. Kushner, *Approximation and weak convergence method for random processes, with applications to stochastic systems theory*, MIT Press, Cambridge, MA, 1984.
[36] A. Lejay, *Homogenization of divergence-form operators with lower-order terms in random media.*, Probab. Theory Related Fields, 120 (2001), pp. 255–276.
[37] P.-L. Lions and P. E. Souganidis, *Homogenization of "viscous" Hamilton-Jacobi equations in stationary ergodic media*, Comm. Partial Differential Equations, 30 (2005), pp. 335–375.
[38] H. G. Matthies and A. Keese, *Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 1295–1331.
[39] G. W. Milton, *The theory of composites*, Cambridge Monographs on Applied and Computational Mathematics, 6, Cambridge University Press, Cambridge, 2002.
[40] H. Owhadi and L. Zhang, *Homogenization of parabolic equations with a continuum of space and time scales*, Arxiv, math.AP/0512504, (2005).
[41] G. Papanicolaou, D. Stroock, and S. R. Varadhan, *Martingale approach to some limit theorems*, Statistical Mechanics and Dynamical Systems 1-120, Duke Univ. Press, 1977.
[42] G. C. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, in Random fields, Vol. I, II (Esztergom, 1979), Colloq. Math. Soc. János Bolyai, 27, North Holland, New York, 1981, pp. 835–873.
[43] E. Pardoux and A. Piatnitski, *Homogenization of a singular random one dimensional PDE*, GAKUTO Internat. Ser. Math. Sci. Appl., 24 (2006), pp. 291–303.
[44] C. R. Vogel, *Computational Methods for Inverse Problems*, SIAM Frontiers in Applied Mathematics, Philadelphia, 2002.
[45] B. White and J. Franklin, *A limit theorem for stochastic two-point boundary value problems of ordinary differential equations*, Comm. Pure Appl. Math., 32 (1979), pp. 253–276.
[46] N. Wiener, *The homogeneous chaos*, Amer. J. Math., 60 (1938), pp. 897–936.
[47] V. V. Yurinskii, *Averaging of symmetric diffusion in a random medium*, Siberian Math. J., 4 (1986), pp. 603–613. English translation of: Sibirsk. Mat. Zh. 27 (1986), no. 4, 167–180 (Russian).
[48] D. Zhang and Z. Lu, *An efficient, high-order perturbation approach for flow in random porous media via Karhunen-Loève and polynomial expansion*, J. Comput. Phys., 194 (2004), pp. 773–794.