Corrector Theory for Elliptic Equations with Long-range Correlated Random Potential

Guillaume Bal* Josselin Garnier† Yu Gu* Wenjia Jing*

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Abstract

We consider an elliptic pseudo-differential equation with a highly oscillating linear potential modeled as a stationary ergodic random field. The random field is a function composed with a centered long-range correlated Gaussian process. In the limiting of vanishing correlation length, the heterogeneous solution converges to a deterministic solution obtained by averaging the random potential. We characterize the deterministic and stochastic correctors. With proper rescaling, the mean-zero stochastic corrector converges to a Gaussian random process in probability and weakly in the spatial variables. In addition, for two prototype equations involving the Laplacian and the fractional Laplacian operators, we prove that the limit holds in distribution in some Hilbert spaces. We also determine the size of the deterministic corrector when it is larger than the stochastic corrector. Depending on the correlation structure of the random field and on the singularities of the Green's function, we show that either the deterministic or the random part of the corrector dominates.

Key words: Corrector theory, random homogenization, long-range correlations, Gaussian random field, weak convergence of probability measures

1 Introduction

We consider elliptic pseudo-differential equations with random potential of the form

$$P(x,D)u_\varepsilon + \tilde{q}_\varepsilon\left(x,\frac{x}{\varepsilon},\omega\right)u_\varepsilon = f(x), \tag{1.1}$$

for x in an open subset $X \subset \mathbb{R}^d$ with appropriate boundary conditions on ∂X if necessary. Here, $\tilde{q}_\varepsilon\left(x,\frac{x}{\varepsilon},\omega\right)$ is composed of a low frequency part $q_0(x)$ and a high frequency part $q\left(\frac{x}{\varepsilon},\omega\right)$, which is a re-scaled version of $q(x,\omega)$, a stationary mean zero random field defined on some abstract probability space $(\Omega,\mathcal{F},\mathbb{P})$ with (possibly multi-dimensional) parameter $x \in \mathbb{R}^d$. The equations are parametrized by the realization $\omega \in \Omega$ and by the small

*Department of Applied Physics & Applied Mathematics, Columbia University, New York, NY 10027 (gb2030@columbia.edu; yg2254@columbia.edu; wj2136@columbia.edu)

†Laboratoire de Probabilités et Modèles Aléatoires & Laboratoire Jacques-Louis Lions, Université Paris VII, 2 Place Jussieu, 75251 Paris Cedex 5, France (garnier@math.jussieu.fr)

parameter $0 < \varepsilon \ll 1$ modeling the correlation length of the random medium. We denote by $\mathbb{E}$ the mathematical expectation with respect to the probability measure $\mathbb{P}$. Equations with coefficients varying at a smaller scale than the scale at which the phenomenon is observed have many practical applications in the physical modeling of complex media. Prototypical examples include the Laplace operator $P(x, D) = -\Delta$, which is ubiquitous in physics, and the fractional Laplace operator $P(x, D) = (-\Delta)^{\beta/2}$ with $\beta \in (0, 2)$, which is used in, e.g., mathematical finance [7, 10]. Pseudo-differential equations like (1.1) may also be used to investigate partial differential equations with random boundary; see e.g. [4].

It is both mathematically and practically interesting to develop asymptotic theories for solutions to (1.1) because numerical solutions become prohibitively expensive computationally when $\varepsilon \to 0$. Homogenization theory and averaging theory aim at finding an effective or homogenized equation whose solution u_0 is the limit of u_ε as ε goes to zero. This theory is well developed. We refer the reader to, e.g., [18, 23, 24] for early work on linear second-order elliptic and parabolic equations with random conductivity tensors and e.g. to [13] and [9, 21] for work on random transport equation and fully nonlinear equations. The main assumption on the random coefficients is rather mild: all we need is stationarity and ergodicity.

The effective medium approximation is deterministic. In many settings, it is important to characterize the random fluctuations in the solutions, for instance to assess the influence of the random medium on given measurements. Corrector theory aims to capture the leading terms in the corrector $u_\varepsilon - u_0$. Compared to homogenization, correctors are characterized in very few settings, and their structure depends on finer properties of the random medium than the effective medium solution does. Two main properties of the heterogeneous equation influence the structure of the corrector. First, the *correlation function* of the random field matters. In the setting of a second-order elliptic boundary value problem in 1D, we have the following behavior. When the random coefficient $q(x, \omega)$ has an integrable correlation function $\mathbb{E}\{q(y, \omega)q(y+x, \omega)\}$, in which case we say that $q(x, \omega)$ has *short-range* correlation, then the corrector converges to a short-range correlated Gaussian process that may be written as a stochastic integral with respect to Brownian motion [8]. However, as shown in [2], the corrector is much larger and has a very different structure when $q(x, \omega)$ has *long-range* correlation in the sense that the correlation function does not decay fast enough to be integrable. Second, the *singularity* structure of Green's function is important. The results mentioned above hold when the Green's function is a little more than square integrable [15, 1]. When the Green's function is not square integrable, as we shall see, deterministic correctors may dominate and the structure of the corrector is thus affected.

The main objectives of this paper are twofold. First, we develop a corrector theory for (1.1) with long-range random potential in multi-dimensional space, generalizing the short-range cases and long-range case in one dimension developed [15, 1, 4, 2]. Second, we explore the influence of singularity of the Green's function, following results in the short-range case in [4], on the competition between the deterministic and mean-zero random parts of the corrector. We assume that $q(x, \omega)$ is the composition of a function with a stationary centered Gaussian field with a correlation function that decays like $|x|^{-\alpha}$ for $\alpha < d$. We find that the mean-zero corrector has an amplitude of order $\varepsilon^{\alpha/2}$, which is much larger than $\varepsilon^{d/2}$ obtained in the short-range case. Moreover, the corrector can be written as a stochastic integral with respect to an appropriate Gaussian random field that is no longer the standard multi-parameter Wiener process. We assume that the Green's function has a singularity of the form $G(x,y) \lesssim |x-y|^{-(d-\beta)}$ near the diagonal and we find the relationship between α and β for which the deterministic or the random parts of the corrector dominate.

The rest of the paper is organized as follows. We describe the problem setting and state our main results in section 2. In section 3, we prove the homogenization and convergence rate results, which depend on the de-correlation rate α and the singularity of the Green's function β. In section 4, assuming $\alpha < 4\beta$, we characterize the limiting distribution of the random corrector weakly in space. In section 5, in the one-dimensional case and under the assumptions that the Green's function is Lipschitz continuous and the solution to (1.1) is continuous, we prove that the convergence of the corrector holds in distribution in the space of continuous paths. In section 6, we show for specific examples that the convergence results of section 4 hold in a stronger sense, namely in distribution with paths in appropriate Hilbert spaces. A discussion on generalization to other cases of singularities and relations between α and β is presented in section 7. Several useful estimates on the convolution of potentials are recorded in the appendix.

2 Main results

In this section, we first describe our main assumptions and then state of main results. The rescaled random field $q\left(\frac{x}{\varepsilon}\right)$ is often denoted by $q_\varepsilon(x)$ and the dependence on the realization ω often ignored. We denote by $\|f\|_{p,X}$ the $L^p(X)$ norm of f. When then context is clear, we use the notation $\|f\|_p \equiv \|f\|_{p,X}$ and $\|f\|$ when $p=2$. We denote by $\langle \cdot\,,\,\cdot\rangle_{\mathcal{H}}$ the inner product on the Hilbert space $\mathcal{H}$ and omit the subscript if $\mathcal{H}=L^2$. We denote by $a\wedge b$ the minimum of a and b. We use C to denote constants that may vary from line to line. We say that C is *universal* when it only depends on dimension d and the domain of interest X.

Let us write (1.1) in detail as

$$\begin{cases} P(x,D)u_\varepsilon(x,\omega) + \left(q_0(x)+q_\varepsilon(x,\omega)\right)u_\varepsilon(x,\omega) = f(x), & x\in X,\\ u_\varepsilon(x,\omega)=0, & x\in\partial X. \end{cases} \tag{2.1}$$

We assume that the pseudo-differential operator $P(x,D)+q_0+q_\varepsilon$ is invertible from $L^2(X)$ to the appropriate functional sub-space of $L^2(X)$ as long as q_0+q_ε is non-negative. We assume further that the operator norm of $(P(x,D)+q_0+q_\varepsilon)^{-1}$ can be bounded independent of the potential. When $P(x,D)=-\Delta$, these assumptions follow by applying the Lax-Milgram theorem [14]. For $P(x,D)=(-\Delta)^{\beta/2}$, these assumptions also hold [6, 10].

We assume that $q_0(x)$ is a smooth function bounded from below by a positive constant γ. Then the inverse of $P(x,D)+q_0$, denoted by $\mathcal{G}$, is well defined. The operator norm $\|\mathcal{G}\|_{\mathcal{L}(L^2)}$ is bounded by some universal constant C. For simplicity, we assume that $\mathcal{G}$, as a transform on $L^2(X)$, is self-adjoint. Finally, we assume that the Green's function $G(x,y)$ associated to $\mathcal{G}$ satisfies:

$$|G(x,y)| \le \frac{C}{|x-y|^{d-\beta}}, \tag{2.2}$$

for some universal constant C and some real number $\beta \in (0, d)$, which measures how singular the Green's function is near the diagonal $x = y$. The examples mentioned above satisfy such properties [14, 11, 4].

The main assumptions on the random process $q(x, \omega)$ are as follows.

(A1) $q(x)$ is defined as $q(x) = \Phi(g(x))$, where $g(x)$ is a centered stationary Gaussian random field with unit variance. Furthermore, the correlation function of $g(x)$ has heavy tail of the form:

$$R_g(x) := \mathbb{E}\{g(y)g(y+x)\} \sim \kappa_g |x|^{-\alpha} \text{ as } |x| \to \infty, \tag{2.3}$$

for some positive constant κ_g and some real number $\alpha \in (0, d)$.

(A2) The function $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies $|\Phi| \le \gamma \le q_0$ and

$$\int_{\mathbb{R}} \Phi(s) e^{-\frac{s^2}{2}} ds = 0. \tag{2.4}$$

The upper bound of Φ above ensures that $|q(x)| \le \gamma$. Consequently, $q_0 + q_\varepsilon$ is non-negative, and (2.1) is well-posed almost surely with solution operator bounded uniformly with respect to q. Due to the construction above and (2.4), $q(x)$ is mean-zero and stationary, and has long-range correlation function that decays like $|x|^{-\alpha}$ as we show later.

The first main theorem concerns the homogenization of (2.1). It shows, in particular, how the competition between the de-correlation rate α and the Green's function singularity β affects the convergence rate of homogenization.

Theorem 2.1. *Let u_ε be the solution to (2.1) and u_0 be the solution to the same equation with q_ε replaced by its zero average. Assume that $q(x)$ is constructed as in (A1) and (A2) and that $f \in L^2(X)$. Then, assuming $2\beta < d$, we have*

$$\mathbb{E}\, \|u_\varepsilon - u_0\|^2 \le \|f\|^2 \times \begin{cases} C\varepsilon^{\alpha}, & \alpha < 2\beta, \\ C\varepsilon^{2\beta} |\log \varepsilon|, & \alpha = 2\beta, \\ C\varepsilon^{2\beta}, & \alpha > 2\beta. \end{cases} \tag{2.5}$$

The constants α and β are defined in (2.3) and (2.2) respectively. When $2\beta \ge d$, the result on the first line above holds. The constant C depends on α, β, γ and the uniform bound on the solution operator of (2.1).

This theorem states u_ε and u_0 are close in the energy norm $L^2(\Omega, L^2(X))$. The corrector, defined as the difference between these two solutions, is decomposed as follows:

$$u_\varepsilon - u_0 = (\mathbb{E}\{u_\varepsilon\} - u_0) + (u_\varepsilon - \mathbb{E}\{u_\varepsilon\}). \tag{2.6}$$

We call the first first part the *deterministic corrector*, and the second mean-zero part the *stochastic corrector*. For the deterministic corrector, we have the following estimates on its size, which depend on α and β.

Theorem 2.2. *Let u_ε, u_0, $q(x)$ and f be as in the previous Theorem. Then for an arbitrary function $\varphi \in L^2(X)$, we have,*

$$|\langle \mathbb{E}\{u_\varepsilon\} - u_0, \varphi\rangle| \le \|f\|\|\varphi\| \times \begin{cases} C\varepsilon^\alpha, & \alpha < \beta, \\ C\varepsilon^\beta |\log \varepsilon|, & \alpha = \beta, \\ C\varepsilon^\beta, & \alpha > \beta. \end{cases} \tag{2.7}$$

The constant C depends on the same factors as in the previous theorem.

The magnitude of the stochastic corrector is always of order $\varepsilon^{\frac{\alpha}{2}}$, as we shall see later in the paper. We deduce from the above theorem that the deterministic corrector can therefore be larger than the stochastic corrector when $\alpha > 2\beta$. To describe the stochastic corrector more precisely, we characterize its limiting distribution. We need to impose the following additional assumptions:

(A3) The function Φ satisfies

$$\int_{\mathbb{R}} |\hat{\Phi}(\xi)|\big(1 + |\xi|^3\big) < \infty, \tag{2.8}$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ.

This condition allows one to derive a (non-asymptotic) estimate, (A.4) in the appendix, for the fourth-order moments of $q(x)$, which is a technicality one encounters often in corrector theory. With this assumption, we have:

Theorem 2.3. *Let u_ε and u_0 solve* (2.1) *and the homogenized equation, respectively. Assume $f \in L^2(X)$ and $q(x)$ is constructed by* (A1-A2) *with Φ satisfying* (A3). *Further, assume $\alpha < 4\beta$. Then:*

$$\frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\varepsilon^{\alpha/2}} \xrightarrow[\varepsilon \to 0]{\text{distribution}} - \int_X G(x,y) u_0(y) W^\alpha(dy), \tag{2.9}$$

where $W^\alpha(dy)$ is formally defined to be $\dot{W}^\alpha(y)dy$ and $\dot{W}^\alpha(y)$ is a Gaussian random field with covariance function given by $\mathbb{E}\{\dot{W}^\alpha(x)\dot{W}^\alpha(y)\} = \kappa|x-y|^{-\alpha}$. Here, $\kappa = \kappa_g \left(\mathbb{E}\{g_0\Phi(g_0)\}\right)^2$ where κ_g, Φ and g_0 are defined in (2.3), *and we assume that $\kappa > 0$. The convergence is understood in probability distribution and weakly in space; see the following remark.*

Remark 2.4. We refer the reader to [17] for the theory on multi-parameter random processes. What we mean by convergence in probability distribution weakly in space is as follows. We fix an arbitrary natural number N and a set of test functions $\{\varphi_i;\ 1 \le i \le N\}$ in $\mathcal{C}(\overline{X})$. Define $I_i^\varepsilon := \langle \varphi_i, \varepsilon^{-\alpha/2}(u_\varepsilon - \mathbb{E}\{u_\varepsilon\})\rangle$, for $i = 1, \cdots, N$. What (2.9) means is that the N-dimensional random vector $(I_1^\varepsilon, \cdots, I_N^\varepsilon)$ converges in distribution to a centered N-dimensional Gaussian vector $(I_1, \cdots, I_N)$, whose covariance matrix Σ_{ij} is given by

$$\Sigma_{ij} := \int_{X^2} \frac{\kappa}{|y-z|^\alpha} (u_0 \mathcal{G}\varphi_i)(y)(u_0\mathcal{G}\varphi_j)(z) dy dz. \tag{2.10}$$

Here $\mathcal{G}$ is the solution operator of the homogenized equation defined above (2.2). By the definition of the stochastic integral above, we see I_i is precisely the inner product of φ_i with the right hand side of (2.9).

We deduce from Theorem 2.2 that when $\alpha < 2\beta$ we can replace $\mathbb{E}\{u_\varepsilon\}$ in (2.9) by u_0, since the deterministic corrector is asymptotically smaller. This is no longer the case for $\alpha \geq 2\beta$. The condition $\alpha < 4\beta$ in Theorem 2.3 is due to technical reasons which we explain later. The conclusion of the theorem holds in general if we can prove an estimate on high-order (more than four-order) moments of q, which is not considered in this paper. □

In the one dimensional case, with the assumption that u_ε and u_0 have continuous paths and that the Green's function is Lipschitz continuous, we can show convergence of the corrector in distribution in the space of continuous paths, as in [8, 2].

Theorem 2.5. *Let X be the unit interval $[0,1]$ in $\mathbb{R}$. Assume that the Green's function $G(x,y)$ is Lipschitz continuous in x with Lipschitz constant $\mathrm{Lip}(G)$ uniform in y. Let u_ε be the solution to (2.1) and u_0 be the homogenized solution. Assume $q(x)$ is constructed as in* (A1)-(A3)*. Then*

$$\frac{u_\varepsilon - u_0}{\varepsilon^{\alpha/2}}(x) \xrightarrow[\varepsilon\to 0]{\text{distribution}} -\sqrt{\frac{\kappa}{H(2H-1)}} \int_0^1 G(x,y)u_0(y)dW_H(y), \tag{2.11}$$

where W_H is the standard fractional Brownian motion with Hurst index $H = 1 - \frac{\alpha}{2}$.

Remark 2.6. We refer the reader to [25] for a review on the definitions of fractional Brownian motions and of the stochastic integral with respect to them. In particular, the random process on the right hand side of (2.11) is a mean-zero Gaussian process which, if designated as $I_H(x)$, has the following covariance function:

$$\mathrm{Cov}[I_H](x,y) = \kappa \int_0^1 \int_0^1 \frac{G(x,t)u_0(t)G(y,s)u_0(s)}{|t-s|^{2(1-H)}} dtds. \tag{2.12}$$

In higher dimensional spaces, for the prototypes where $P(x,D)$ is the Laplacian or fractional Laplacian, we can show that the limit in Theorem 2.3 actually holds in distribution in appropriate Hilbert spaces. More precisely, we consider the pseudo-differential equation:

$$\left[(-\Delta)_{\mathrm{D}}^{\frac{\beta}{2}} + q_0 + q_\varepsilon(x)\right] u_\varepsilon(x) = f(x). \tag{2.13}$$

Here the exponent $\beta \in (0,2]$. The subscription D denotes "Dirichlet boundary" on X. When $\beta = 2$, the boundary condition is in the usual sense, but when β is less than two and hence the equation is pseudo-differential, the boundary condition is $u_\varepsilon = 0$ on X^c, the whole complement of X. This is necessary because the fractional Laplacian is non-local.

It turns out that the above equation admits a set of pairs $(\lambda_n^\beta, \phi_n^\beta)$, $1 \leq n \leq \infty$, where λ_n^β is an eigenvalue and ϕ_n^β is the corresponding eigenfunction. That is,

$$(-\Delta)_{\mathrm{D}}^{\frac{\beta}{2}} \phi_n^\beta = \lambda_n^\beta \phi_n^\beta. \tag{2.14}$$

Without loss of generality we can assume that $\{\phi_n^\beta\}$ is orthonormal in $L^2(X)$. We can then define a system of Hilbert spaces as follows, with $\mathscr{D}'$ denoting the space of Schwartz distributions,

$$\mathcal{H}_\beta^s := \left\{ f \in \mathscr{D}' \ : \ \sum_{n=1}^\infty \Big(\langle f, \phi_n^\beta \rangle (\lambda_n^\beta)^s \Big)^2 < \infty \right\}, \quad s \in \mathbb{R}. \tag{2.15}$$

The inner product and norm on $\mathcal{H}_\beta^s$ is implied in the definition. We observe from the definition that $\mathcal{H}_\beta^{-s}$ is the dual space of $\mathcal{H}_\beta^s$. Moreover, when s is an integer, $\mathcal{H}_\beta^s$ consists of distributions f such that $((-\Delta)_{\mathrm{D}}^{\beta/2})^s f$ is in $L^2(X)$.

We can view the corrector $u_\varepsilon - u_0$ as $\mathcal{H}_\beta^s$-valued random variables for certain s. With the natural metric on $\mathcal{H}_\beta^s$, we can consider the weak convergence of the probability measures on $\mathcal{H}_\beta^s$ (equipped with its Borel σ-algebra) induced by the random variables $\{u_\varepsilon - \mathbb{E}u_\varepsilon\}_{\varepsilon \in (0,1)}$, as ε goes to zero, and in the sense of [5]. That is, the laws of these random variables converges to the law of the limiting process.

Theorem 2.7. *Let u_ε be the solution of the pseudo-differential equation (2.13) with Laplacian exponent $\beta \in (0, 2]$, and let u_0 be the homogenized solution. Suppose that q_0 and f are smooth enough so that u_0 is continuous on $\overline{X}$. Suppose also the random coefficient $q(x, \omega)$ satisfies the conditions in Theorem 2.3; in particular, assume the decorrelation rate α is less than 4β. Set $\mu = [d/2\beta]$, the integer part of $d/2\beta$. Then we have that (2.9) holds in distribution in the space $\mathcal{H}_\beta^{-\mu}$.*

This theorem is stronger than Theorem 2.3. In particular, when $P(x, D) = -\Delta$ and $d \le 3$, the Hilbert space above can be chosen as $L^2(X)$.

3 Convergence to the homogenized solution

In this section, we prove Theorem 2.1, which says that the homogenized equation for (2.1) is obtained by averaging q_ε.

We first verify that the random field $q(x)$ constructed in (A1) and (A2) has the same heavy tail as the underlying Gaussian random field. By stationarity, the correlation function of $q(x)$ is given by

$$R(x) := \mathbb{E}\{q(y)q(y + x)\} = \mathbb{E}\{\Phi(g_0)\Phi(g_x)\}. \tag{3.1}$$

We show that $R(x)$ has the same asymptotic behavior as R_g in (2.3).

Lemma 3.1. *Let $q(x)$ be the random field above. Define $V_1 = \mathbb{E}\{g_0\Phi(g_0)\}$ where g_x is the underlying Gaussian random field. There exist some $T, C > 0$ such that the autocorrelation function $R(x)$ of q satisfies*

$$|R(x) - V_1^2 R_g(x)| \le C R_g^2(x), \quad \textit{for all } |x| \ge T, \tag{3.2}$$

where R_g is the correlation function of g. Further,

$$|\mathbb{E}\{g(y)q(y + x)\} - V_1 R_g(x)| \le C R_g^2(x), \quad \textit{for all } |x| \ge T. \tag{3.3}$$

Proof. A proof of this lemma can be found in [2]; we record it here for the reader's convenience.

$$R(x) = \frac{1}{2\pi\sqrt{1-R_g^2(x)}} \int_{\mathbb{R}^2} \Phi(g_1)\Phi(g_2) \exp\left(-\frac{g_1^2+g_2^2-2R_g(x)g_1g_2}{2(1-R_g^2(x))}\right) dg_1 dg_2.$$

For large $|x|$, the coefficient $R_g(x)$ is small and we can expand the value of the double integral in powers of $R_g(x)$. The zeroth order term is the integration of $\Phi(g_1)\Phi(g_2)$ with respect to $\exp(-|g|^2/2)dg$ where dg is short for dg_1dg_2; this term vanishes due to (2.4). The first order term is integration of $\Phi(g_1)\Phi(g_2)g_1g_2$ with respect to the $\exp(-|g|^2/2)dg$, which gives $V_1^2 R_g(x)$.

Similarly, for the second item in the lemma, we first write

$$\mathbb{E}\{g(y)\Phi(g(y+x))\} = \frac{1}{2\pi\sqrt{1-R_g^2(x)}} \int_{\mathbb{R}^2} g_1\Phi(g_2) \exp\left(-\frac{g_1^2+g_2^2-2R_g(x)g_1g_2}{2(1-R_g^2(x))}\right) dg_1 dg_2.$$

Then we expand the value of the double integral in powers of R_g and characterize the first two orders as before. □

It follows that $R(x)$ behaves like $\kappa|x|^{-\alpha}$, where $\kappa = V_1^2\kappa_g$, for large $|x|$. In particular, there exists some constant C so that $|R(x)| \le C|x|^{-\alpha}$. Meanwhile, $|R(x)|$ is uniformly bounded, say by $|\Phi|^2 \le \gamma^2$ according to assumption (A2).

Lemma 3.2. *Let $\mathcal{G}$ be the Green's operator and $q(x)$ be the random field above. Let f be an arbitrary function in $L^2(X)$. Assume $2\beta < d$. Then, we have:*

$$\mathbb{E}\, \|\mathcal{G}q_\varepsilon f\|^2 \le \|f\|^2 \times \begin{cases} C\varepsilon^\alpha, & \alpha < 2\beta, \\ C\varepsilon^{2\beta}|\log\varepsilon|, & \alpha = 2\beta, \\ C\varepsilon^{2\beta}, & \alpha > 2\beta. \end{cases} \tag{3.4}$$

The constant C depends only on α, β, X, $\|q\|_\infty$ and the bound for $\|\mathcal{G}_\varepsilon\|_{\mathcal{L}}$. If $2\beta \ge d$, then only the first case is necessary.

Proof. The L^2 norm of $\mathcal{G}q_\varepsilon f$ has the following expression:

$$\|\mathcal{G}q_\varepsilon f\|^2 = \int_X \left(\int_X G(x,y)q_\varepsilon(y)f(y)dy\right)^2 dx.$$

After writing the integrand as a double integral and taking expectation, we have

$$\mathbb{E}\|\mathcal{G}q_\varepsilon f\|^2 = \int_{X^3} G(x,y)G(x,z)R_\varepsilon(y-z)f(y)f(z)dydzdx. \tag{3.5}$$

Use (2.2) to bound the Green's functions. Integrate over x and apply Lemma A.1. We get

$$\mathbb{E}\|\mathcal{G}q_\varepsilon f\|^2 \le C\int_{X^2} \frac{1}{|y-z|^{d-2\beta}}|R_\varepsilon(y-z)f(y)f(z)|dydz. \tag{3.6}$$

Change variable $(y, y-z) \mapsto (y, z)$. The above integral becomes

$$\int_X \int_{y-X} \frac{1}{|z|^{d-2\beta}} |R_\varepsilon(z) f(y) f(y-z)| dy dz.$$

We can further bound the integral from above by enlarging the domain $y - X$ to some finite ball $B(2\rho)$ where $\rho = \sup_{x \in X} |x|$, because the translated region $y - X$ is included in this ball for every y. After this replacement, integrate over y first, and we have:

$$\mathbb{E}\|\mathcal{G} q_\varepsilon f\|^2 \leq C\|f\|^2 \int_{B(2\rho)} \frac{|R_\varepsilon(z)|}{|z|^{d-2\beta}} dz. \tag{3.7}$$

Decompose the integration region into two parts:

$$\begin{cases} D_1 := \{|x\varepsilon^{-1}| \leq T\} \cap B(2\rho), & \text{on which we have } |R_\varepsilon| \leq \gamma^2; \\ D_2 := \{|x\varepsilon^{-1}| > T\} \cap B(2\rho), & \text{on which we have} |R_\varepsilon| \leq C\varepsilon^\alpha |x|^{-\alpha}. \end{cases}$$

The integration on D_1 can be carried out explicitly. The restriction $|x| \leq T\varepsilon$ yields that this term is of order $\varepsilon^{2\beta}$. The integration over D_2 is

$$C \int_{\varepsilon T}^{2\rho} \frac{\varepsilon^\alpha |z|^{d-1}}{|z|^{d-2\beta+\alpha}} d|z|.$$

When $2\beta = \alpha$, the integral equals $C\varepsilon^\alpha(\log(2\rho) - \log(T\varepsilon))$, and is of order $\varepsilon^\alpha |\log \varepsilon|$. When $2\beta \neq \alpha$, the integral equals $C\varepsilon^\alpha((2\rho)^{2\beta-\alpha} - (T\varepsilon)^{2\beta-\alpha})$. This estimate proves the other two cases of the lemma.

The same analysis can be done for $2\beta \geq d$. In this case, the singular term $|y-z|^{-(d-2\beta)}$ in (3.6) should be replaced by either $|\log|y-z||$ or C, which is much smoother. Consequently, $\mathbb{E}\|\mathcal{G} q_\varepsilon f\|^2$ is of order ε^α. $\square$

Proof of Theorem 2.1. The homogenized solution satisfies $(P(x,D) + q_0)u_0 = f$. Define $\chi_\varepsilon = -\mathcal{G} q_\varepsilon u_0$, that is the solution of $(P(x,D) + q_0)\chi_\varepsilon = -q_\varepsilon u_0$. Compare these two equations with the one for u_ε, i.e. (2.1). We get

$$(P(x,D) + q_0 + q_\varepsilon)(\xi_\varepsilon - \chi_\varepsilon) = -q_\varepsilon \chi_\varepsilon,$$

where ξ_ε denotes $u_\varepsilon - u_0$. Since this equation is well-posed a.e. in Ω, we have $\xi_\varepsilon = \chi_\varepsilon - \mathcal{G}_\varepsilon q_\varepsilon \chi_\varepsilon$, which implies

$$\|\xi_\varepsilon\| \leq \|\chi_\varepsilon\| + \|\mathcal{G}_\varepsilon\|_{\mathcal{L}(L^2)} \|q\|_\infty \|\chi_\varepsilon\|. \tag{3.8}$$

Recall that the operator norm $\|\mathcal{G}_\varepsilon\|_{\mathcal{L}(L^2)}$ can be bounded uniformly in Ω; so the right hand side above is further bounded by $C\|\chi_\varepsilon\|$. Since χ_ε is of the form of $\mathcal{G} q_\varepsilon f$, we take expectation and apply the previous lemma to complete the proof. $\square$

We decompose the corrector into the deterministic corrector $\mathbb{E}\{u_\varepsilon\} - u_0$ and the stochastic corrector $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$. We consider their sizes and limits only in the weak sense, that is after pairing with test functions. We have the following formula for u_ε,

$$u_\varepsilon - u_0 = -\mathcal{G} q_\varepsilon u_0 + \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u_0 + \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon (u_\varepsilon - u_0). \tag{3.9}$$

Pairing this with an arbitrary test function $\varphi \in \mathcal{C}(\overline{X})$, we have

$$\langle u_\varepsilon - u_0, \varphi\rangle = -\langle \mathcal{G} q_\varepsilon u_0, \varphi\rangle + \langle \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u_0, \varphi\rangle + \langle \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon (u_\varepsilon - u_0), \varphi\rangle. \tag{3.10}$$

Now the deterministic corrector $\langle \mathbb{E}\{u_\varepsilon\} - u_0, \varphi\rangle$ is precisely the expectation of the expression above. In the following, we estimate the size of this corrector using the analysis developed in the proof of Lemma 3.2.

Proof of Theorem 2.2. Take expectation in (3.10). Since the first term on the right is mean zero, we have

$$\langle \mathbb{E}\{u_\varepsilon\} - u_0, \varphi\rangle = \mathbb{E}\langle \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u_0, \varphi\rangle + \mathbb{E}\langle \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon (u_\varepsilon - u_0), \varphi\rangle. \tag{3.11}$$

Let m denote the L^2 function $\mathcal{G}\varphi$. Rewrite the first term on the right as $\mathbb{E}\langle q_\varepsilon u_0, \mathcal{G} q_\varepsilon m\rangle$, which can be written as

$$\int_X G(x,y) R_\varepsilon(x-y) u_0(x) m(y) dx dy.$$

After controlling the green's function by $C|x-y|^{-d+\beta}$, we have an object similar to (3.6). Following the same procedure, we can show that $|\mathbb{E}\langle q_\varepsilon u_0, \mathcal{G} q_\varepsilon m\rangle|$ can be bounded as in (2.7). To complete the proof, we only need to control the remainder term in (3.11), which can be written as $\mathbb{E}\langle q_\varepsilon (u_\varepsilon - u_0), \mathcal{G} q_\varepsilon m\rangle$. We have:

$$\mathbb{E}|\langle q_\varepsilon (u_\varepsilon - u_0), \mathcal{G} q_\varepsilon m\rangle| \le \|q_\varepsilon\|_\infty \left(\mathbb{E}\|u_\varepsilon - u_0\|^2\right)^{1/2} \left(\mathbb{E}\|\mathcal{G} q_\varepsilon m\|^2\right)^{1/2}. \tag{3.12}$$

According to Theorem 2.1 and Lemma 3.2, this term can be bounded by the right hand side of (3.4). Therefore, the remainder is smaller than the quadratic term which gives the desired estimate. □

For any fixed test function φ, the random corrector $\langle u_\varepsilon - \mathbb{E}\{u_\varepsilon\}, \varphi\rangle$ is precisely the mean-zero part of the right hand side of (3.10). We are interested in its limiting distribution. The size of its variance is given by that of $-\langle \mathcal{G} q_\varepsilon u_0, \varphi\rangle$. We calculate

$$\mathrm{Var}\left(-\langle \mathcal{G} q_\varepsilon u_0, \varphi\rangle\right) = \mathrm{Var}\left(-\langle q_\varepsilon u_0, m\rangle\right) = \int_{X^2} R_\varepsilon(x-y) u_0 m(x) u_0 m(y) dx dy.$$

Estimating this integral by decomposing the domain as in the proof of Lemma 3.2, we verify that this object is of size ε^α independent of β. Therefore, a more accurate characterization of the stochastic corrector is to find the limiting distribution of $\varepsilon^{-\alpha/2}\langle u_\varepsilon - \mathbb{E}\{u_\varepsilon\}, \varphi\rangle$. This is the main task of Section 4.

4 Corrector theory in higher dimensional space

In this section, we consider the limiting distribution of the stochastic corrector. In the analyses we are going to develop, the following estimate proves very useful. Recall that R is uniformly bounded, and there exists some T so that $|R| \le C|x|^{-\alpha}$ when $|x| > T$.

Lemma 4.1. *Recall that $R(x)$ denotes the correlation function of the random field $q(x)$ constructed in* (A1) *and* (A2), *and that $R_\varepsilon(x)$ denotes $R(\varepsilon^{-1}x)$. Let $p \geq 1$; we have*

$$\|R_\varepsilon\|_{p,B(\rho)} \leq \begin{cases} C\varepsilon^{\alpha}, & \alpha p < d, \\ C\varepsilon^{\alpha}|\log\varepsilon|^{\frac{1}{p}}, & \alpha p = d, \\ C\varepsilon^{\frac{d}{p}}, & \alpha p > d. \end{cases} \tag{4.1}$$

Here, $B(\rho)$ is the open ball centered at zero with radius ρ. The constant C depends on ρ, dimension d, and the constant in the asymptotic behavior of $R(x)$.

Proof. We break the expression for $\|R_\varepsilon\|_p^p$ into two parts as follows:

$$\int_{B(\varepsilon T)} |R_\varepsilon(x)|^p dx + \int_{B(\rho)\setminus B(\varepsilon T)} |R_\varepsilon(x)|^p dx.$$

For the first term, we bound R_ε by its uniform norm and verify this term is of order ε^d. For the second term, which we call I_2, we use the asymptotic behavior of R and have

$$I_2 \leq C\int_{B(\rho)\setminus B(\varepsilon T)} \varepsilon^{\alpha p}|x|^{-\alpha p} dx \leq C\varepsilon^{\alpha p}\int_{T\varepsilon}^{\rho} r^{d-1-\alpha p} dr.$$

We carry out this integral and find that it is of order $\varepsilon^{\alpha p}|\log\varepsilon|$ if $\alpha p = d$ and of order $\varepsilon^{\alpha p\wedge d}$ otherwise.

Now combine the two parts; compare the orders case by case to get the bound for $\|R_\varepsilon\|_p^p$. Then take pth roots to complete the proof. □

Lemma 4.2. *Assume $q(x)$ constructed in* (A1-A2) *satisfies* (A3). *Let φ be an arbitrary test function in $\mathcal{C}(\overline{X})$. Then we have the following estimate of the variance of the second term in* (3.10)*:*

$$\operatorname{Var}\,\langle \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0, \varphi\rangle \ll C\|u_0\|^2\|\varphi\|_\infty^2\varepsilon^{\alpha}. \tag{4.2}$$

Again, the constant C only depend on the factors as stated in Theorem 2.1.

Proof. We observe first that $m := \mathcal{G}\varphi$ is a bounded function since φ is uniformly bounded; a useful fact in the sequel. To simplify notation, we denote by I the variance of $\langle \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0, \varphi\rangle$. It has the expression:

$$I = \int_{X^4} u_0(x)m(y)u_0(\xi)m(\eta)G(x,y)G(\xi,\eta)\times \\ \times\Big[\mathbb{E}\{q_\varepsilon(x)q_\varepsilon(y)q_\varepsilon(\xi)q_\varepsilon(\eta)\} - \mathbb{E}\{q_\varepsilon(x)q_\varepsilon(y)\}\mathbb{E}\{q_\varepsilon(\xi)q_\varepsilon(\eta)\}\Big] dxdyd\xi d\eta.$$

Apply Lemma A.2 to estimate the variance of the product of q_ε above and use the bound for the Green's functions. We have

$$I \leq C\int_{X^4} |u_0(x)m(y)u_0(\xi)m(\eta)| \frac{1}{|x-y|^{d-\beta}}\frac{1}{|\xi-\eta|^{d-\beta}}\times \\ \times \sum_{p\neq\{(1,2),(3,4)\}} |R_\varepsilon(x_{p(1)} - x_{p(2)})R_\varepsilon(x_{p(3)} - x_{p(4)})| dxdyd\xi d\eta.$$

Here, $p = \{(p_1, p_2), (p_3, p_4)\}$ denotes the possibilities of choosing two different pairs of indices from $\{1, 2, 3, 4\}$ in such a way that each pair contains different indices though the two pairs may share the same index. There are $C_6^2 = 15$ different choices for p; however, $p = \{(1, 2), (3, 4)\}$ is excluded from the sum above. Identifying (x_1, x_2, x_3, x_4) with (x, y, ξ, η), we see that there are 14 terms in the sum, and each of them is a product of two R_ε functions whose arguments are the difference vectors of points in $\{x, y, \xi, \eta\}$; more importantly, at most one of the R_ε functions shares the same argument as one of the Green's functions.

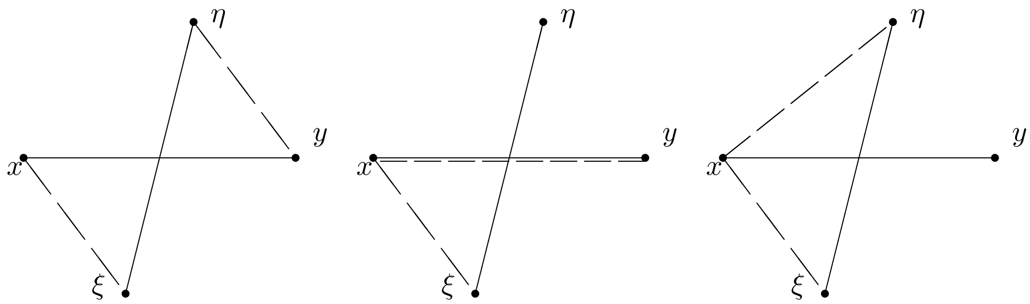

Figure 1: Difference vectors of four points. The *solid* lines represent arguments of the Green's functions, while the *dashed* lines represent those of the correlation functions.

We can divide the fourteen choices of p into three categories as shown in Figure 1. In the first category as illustrated by the first picture, the two vectors in the correlation functions are linear independent with both of the vectors in the Green's functions; in the second category, one of the Green's function shares the same argument with one of the correlation function; finally in the third category, the vector in one of the Green's function is a linear combination of the two vectors of the correlation functions.

For the first category, we consider a typical term of the form:

$$J_1 = \int_{X^4} |u_0(x)m(y)u_0(\xi)m(\eta)| \frac{1}{|x-y|^{d-\beta}} \frac{1}{|\xi-\eta|^{d-\beta}} |R_\varepsilon(x-\xi)R_\varepsilon(y-\eta)|. \tag{4.3}$$

Change variable as follows:

$$(x, x-y, x-\xi, y-\eta) \mapsto (x, y, \xi, \eta).$$

Bound m by its uniform norm. In terms of the new variables, we have

$$J_1 \le \|m\|_\infty^2 \int_X dx \int_{x-X} dy \int_{x-X} d\xi \int_{x-y-X} d\eta \, \frac{|u_0(x)u_0(x-\xi)R_\varepsilon(\xi)R_\varepsilon(\eta)|}{|y|^{d-\beta}|y-(\xi-\eta)|^{d-\beta}}.$$

We can replace the integration region of y and ξ by $B(2\rho)$, and replace that of η by $B(3\rho)$, where ρ as before denotes the maximum distance of a point in X and the origin. After doing this, we integrate over x first to get rid of the u_0 function; then integrate over y and apply Lemma A.1 to get

$$J_1 \le \|m\|_\infty^2 \|u_0\|^2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|R_\varepsilon \mathbf{1}_{B(2\rho)}(\xi)| \; |R_\varepsilon \mathbf{1}_{B(3\rho)}(\eta)|}{|\xi-\eta|^{d-2\beta}} d\xi d\eta. \tag{4.4}$$

Here, $\mathbf{1}_A$ is the indicator function of a subset $A \subset \mathbb{R}^d$. We considered the case $2\beta < d$; the other cases are easier. To estimate the integral above, we apply the Hardy-Littlewood-Sobolev inequality [20, Theorem 4.3]. With $p = 2d/(d+2\beta) > 1$, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|R_\varepsilon \mathbf{1}_{B(2\rho)}(\xi)| \; |R_\varepsilon \mathbf{1}_{B(3\rho)}(\eta)|}{|\xi - \eta|^{d-2\beta}} d\xi d\eta \le C(d, \beta, p) \|R_\varepsilon\|_{p,B(2\rho)} \|R_\varepsilon\|_{p,B(3\rho)}. \tag{4.5}$$

Now apply Lemma 4.1: If $\alpha p \le d$, we see J_1 is of order $\varepsilon^{2\alpha}$ or $\varepsilon^{2\alpha}|\log \varepsilon|^{2/p}$ which is much smaller than ε^α; if otherwise, J_1 is of order $\varepsilon^{2d/p} \ll \varepsilon^\alpha$ because by our choice of p we have $2d/p - \alpha = d + 2\beta - \alpha > 2\beta > 0$.

In the second category, we consider a typical term of the form:

$$J_2 = \int_{X^4} |u_0(x)m(y)u_0(\xi)m(\eta)| \frac{1}{|x-y|^{d-\beta}} \frac{1}{|\xi - \eta|^{d-\beta}} |R_\varepsilon(x-y)R_\varepsilon(x-\xi)|. \tag{4.6}$$

This time we use the following change of variables,

$$(x, x-y, x-\xi, \xi-\eta) \mapsto (x, y, \xi, \eta).$$

With this change and bounding m, we have

$$J_2 \le \|m\|_\infty^2 \int_X dx \int_{x-X} dy \int_{x-X} d\xi \int_{x-\xi-X} d\eta \; \frac{|u_0(x)u_0(x-\xi)R_\varepsilon(\xi)R_\varepsilon(y)|}{|y|^{d-\beta}|\eta|^{d-\beta}}.$$

Enlarge the integration region of y, ξ, η as before, and then integrate over x and η. We have

$$J_2 \le \|m\|_\infty^2 \|u_0\|^2 \int_{B^2(2\rho)} \frac{1}{|y|^{d-\beta}} |R_\varepsilon(y)||R_\varepsilon(\xi)| d\xi dy. \tag{4.7}$$

The integration over ξ yields a term of size ε^α; meanwhile, the integration over y can be estimated as in the integral in (3.7), and is of size given in (2.7). Therefore, $J_2 \ll \varepsilon^\alpha$.

For the third category, we consider a typical term of the form:

$$J_3 = \int_{X^4} |u_0(x)m(y)u_0(\xi)m(\eta)| \frac{1}{|x-y|^{d-\beta}} \frac{1}{|\xi - \eta|^{d-\beta}} |R_\varepsilon(x-\xi)R_\varepsilon(x-\eta)|. \tag{4.8}$$

Change variables according to

$$(x, x-y, x-\xi, x-\eta) \mapsto (x, y, \xi, \eta).$$

After the routine of enlarging integration domains, bounding m, and integrating the non-singular terms, we have

$$J_3 \le \|m\|_\infty^2 \|u_0\|^2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|R_\varepsilon \mathbf{1}_{B(2\rho)}(\xi)||R_\varepsilon \mathbf{1}_{B(2\rho)}(\eta)|}{|\xi - \eta|^{d-\beta}} d\xi d\eta. \tag{4.9}$$

This term can be estimated exactly as what we have done for (4.4). In particular, it is much smaller than ε^α. This completes the proof. □

To prove Theorem 2.3, we essentially consider the law of random vectors of the form $(J_1^\varepsilon(\omega), \cdots, J_N^\varepsilon(\omega))$, where

$$J_j^\varepsilon(\omega) := -\frac{1}{\varepsilon^{\alpha/2}} \int_X q_\varepsilon(y)\psi_j(y)dy, \tag{4.10}$$

for some collection of $L^2(X)$ functions $\{\psi_k(x);\ 1 \le k \le N\}$. We have the following result characterizing their limiting joint law:

Lemma 4.3. *The random vector $(J_1^\varepsilon, J_2^\varepsilon, \cdots, J_N^\varepsilon)$ converges in distribution to the centered Gaussian random vector $(J_1, J_2, \cdots, J_N)$ whose covariance matrix is given by*

$$C_{ik} = \mathbb{E}\{J_i J_k\} = \int_{X^2} \frac{\kappa\ \psi_i(y)\psi_k(z)}{|y-z|^\alpha} dydz. \tag{4.11}$$

Moreover, the random variable J_k admits the following stochastic integral representation:

$$J_k = -\int_X \psi_k(y) W^\alpha(dy). \tag{4.12}$$

Here $W^\alpha(dy)$ is formally defined in Theorem 2.3.

Proof. We want to show that $\forall t_1, t_2, \cdots, t_N \in \mathbb{R}$, $\sum_{i=1}^N t_i J_i^\varepsilon$ converges in distribution to $\sum_{i=1}^N t_i J_i$. Since

$$\sum_{i=1}^N t_i J_i^\varepsilon = -\frac{1}{\varepsilon^{\alpha/2}} \int_X q_\varepsilon(y) \sum_{i=1}^N t_i \psi_i(y) dy,$$

$$\sum_{i=1}^N t_i J_i = -\int_X (\sum_{i=1}^N t_i \psi_i(y)) W^\alpha(dy),$$

and $\sum_{i=1}^N t_i\psi_i(y) \in L^2(X)$, we only need to show

$$-\frac{1}{\varepsilon^{\alpha/2}} \int_X q_\varepsilon(y) f(y) dy \xrightarrow[\varepsilon\to 0]{\text{distribution}} -\int_X f(y) W^\alpha(dy) \tag{4.13}$$

for any $f \in L^2(X)$.

We prove this convergence in two steps: First, we show it holds when $q(x) = g(x)$, i.e., q is a centered stationary Gaussian field. Second, we generalize the result to the case when $q(x) = \Phi(g(x))$.

The Gaussian case. When $q(x) = g(x)$, the random variable $-\varepsilon^{-\alpha/2}\int_X q_\varepsilon(y)f(y)dy$ is centered, Gaussian, with variance $S_\varepsilon := \varepsilon^{-\alpha}\int_{X^2} R_g(\frac{y-z}{\varepsilon}) f(y)f(z)dydz$, so it suffices to show

$$S_\varepsilon \longrightarrow \int_{X^2} \frac{\kappa_g f(y)f(z)}{|y-z|^\alpha} dydz =: \text{Var}\left(-\int_X f(y)W^\alpha(dy)\right) \tag{4.14}$$

as $\varepsilon \to 0$. The equality above holds by the definition of our stochastic integral. Note that in this case, $q(x) = \Phi(g(x))$ with $\Phi(s) = s$; consequently, the κ in the covariance function of W^α in Theorem 2.3 is precisely κ_g, because $\mathbb{E}\{g(0)\Phi(g(0))\} = \mathbb{E}\{g(0)^2\} = 1$.

Since $R_g(x) \sim \kappa_g |x|^{-\alpha}$, for any $\delta > 0$, there exists an $M > 0$ so that $|x| > M$ implies $|R_g(x) - \kappa_g |x|^{-\alpha}| < \delta \kappa_g |x|^{-\alpha}$. According to this, we have

$$\Big| S_\varepsilon - \int_{X^2} \frac{\kappa_g f(y) f(z)}{|y-z|^\alpha} dydz \Big| \leq \int_{|y-z|>M\varepsilon} \frac{\delta \kappa_g |f(y)f(z)|}{|y-z|^\alpha}\, dydz + \\ + \int_{|y-z|\leq M\varepsilon} |f(y)f(z)| \left(\varepsilon^{-\alpha} + \frac{\kappa_g}{|y-z|^\alpha} \right) dydz := (I) + (II) + (III).$$

We have used the fact $\|R\|_\infty = 1$. It is easy to see that $(I) \leq C\delta$, $(II) + (III) \leq C\varepsilon^{d-\alpha}$. First let $\varepsilon \to 0$, then let $\delta \to 0$, we prove (4.14).

The case of a function of the Gaussian field. In this case, $q(x) = \Phi(g(x))$ for more general Φ. Recall that $V_1^2 = \mathbb{E}\{g(0)\Phi(g(0))\}$ and V_1 is assumed to be positive. we claim that the difference between the random variables $\varepsilon^{-\alpha/2} \int_X q_\varepsilon(y) f(y) dy$ and $\varepsilon^{-\alpha/2} \int_X V_1 g_\varepsilon(y) f(y) dy$ converges to zero in probability. Then (4.13) follows from this, the Gaussian case, and the fact $\kappa = \kappa_g V_1^2$.

To show the convergence in probability, we estimate the second moment as follows:

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{\varepsilon^{\alpha/2}} \int_X (q_\varepsilon(y) - V_1 g_\varepsilon(y)) f(y) dy \right)^2 \\ = \; & \frac{1}{\varepsilon^\alpha} \int_{X^2} \mathbb{E}\{(q_\varepsilon(y) - V_1 g_\varepsilon(y))(q_\varepsilon(z) - V_1 g_\varepsilon(z))\} f(y) f(z) dydz. \end{aligned}$$

The expectation term inside the integral can be written as

$$R_\varepsilon(y-z) - V_1^2 (R_g)_\varepsilon(y-z) + V_1[V_1 (R_g)_\varepsilon(y-z) - \mathbb{E}\{g_\varepsilon(y) q_\varepsilon(z)\}] \\ + V_1[(R_g)_\varepsilon(y-z) - \mathbb{E}\{g_\varepsilon(z) q_\varepsilon(y)\}].$$

Recall (3.3) of Lemma 3.1 to estimate these terms. We can bound the second moment above by

$$C\varepsilon^{-\alpha} \int_{|y-z|\leq T\varepsilon} |f(y)f(z)| dydz + C\varepsilon^{-\alpha} \int_{|y-z|>T\varepsilon} \frac{\varepsilon^{2\alpha} |f(y)f(z)|}{|y-z|^{2\alpha}} dydz := (I) + (II).$$

Carrying out the routine analysis we have developed for this type of integrals, we verify that $(I) \leq C\varepsilon^{d-\alpha}$ and (II) is of order ε^α if $2\alpha < d$, of order $\varepsilon^\alpha |\log \varepsilon|$ if $2\alpha = d$, and of order $\varepsilon^{d-\alpha}$ if $2\alpha > d$. In all cases, we have $(I) + (II)$ converges to zero, which completes the proof. □

According to the interpretation in Remark 2.4, the lemma above implies that $\mathcal{G} q_\varepsilon u_0$ converges to the limit in (2.9). The other terms in the stochastic corrector $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$ are controlled by Lemmas 3.2 and 4.2. These are sufficient to prove Theorem 2.3 as follows.

Proof of Theorem 2.3. Recall the expression (3.9) for the corrector. We see its random part, i.e. $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$, can be decomposed as

$$-\mathcal{G} q_\varepsilon u_0 + (\mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u_0 - \mathbb{E}\{\mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u_0\}) + (\mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon (u_\varepsilon - u_0) - \mathbb{E}\{\mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon (u_\varepsilon - u_0)\}). \quad (4.15)$$

By (4.2), for any test function $\varphi \in \mathcal{C}(\overline{X})$, we have

$$\left\langle \frac{\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0 - \mathbb{E}\{\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0\}}{\varepsilon^{\alpha/2}}, \varphi \right\rangle \xrightarrow[\varepsilon \to 0]{\text{probability}} 0. \tag{4.16}$$

Recall estimate (3.12) and apply (2.5) and (3.4). We find that when $\alpha < 4\beta$, the size of $\mathbb{E}|\langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0), \varphi\rangle|$ is much smaller than $\varepsilon^{\alpha/2}$, which implies

$$\left\langle \frac{\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0)}{\varepsilon^{\alpha/2}}, \varphi \right\rangle \xrightarrow[\varepsilon \to 0]{\text{probability}} 0. \tag{4.17}$$

The leading term in the random corrector is therefore $\langle -\mathcal{G}q_\varepsilon u_0, \varphi\rangle$.

Consider an arbitrary set of test functions $\{\varphi_i, 1 \le i \le N\}$. By the same argument above we can verify that the vectors $(Q_1^\varepsilon, \cdots, Q_N^\varepsilon)$, where

$$Q_i^\varepsilon := \varepsilon^{-\alpha/2} \langle \varphi_i, \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0 + \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0)\rangle,$$

converge in probability to zero vectors. On the other hand, by Lemma 4.3 and the fact that $u_0(y)\mathcal{G}\varphi(y) \in L^2(X)$, we verify that $(I_\varepsilon^i, \cdots, I_\varepsilon^N)$ converges in distribution to $(I_1, \cdots, I_N)$, where

$$I_\varepsilon^i := \varepsilon^{-\alpha/2} \langle \varphi_i, -\mathcal{G}q_\varepsilon u_0\rangle,$$

and $(I_1, \cdots, I_N)$ is the centered Gaussian with covariance matrix given by (2.10). Combining this convergence result with (4.16) and (4.17), we see that $(I_1^\varepsilon, \cdots, I_N^\varepsilon)$, where $I_i^\varepsilon := \varepsilon^{-\alpha/2}\langle u_\varepsilon - \mathbb{E}\{u_\varepsilon\}, \varphi_i\rangle$ as defined in Remark 2.4, converges in distribution to $(I_1, \cdots, I_N)$. This completes the proof. □

5 Corrector in one dimensional space

In this section, we restrict the dimension to be one. With further assumptions that the Green's function is Lipschitz continuous and the solution to (2.1) has continuous path, we derive a stronger convergence result of $u_\varepsilon - u_0$, in probability distribution in the space of continuous paths. The proof resembles and depends on [2] largely.

For concreteness, let $X = (0, 1)$. For simplicity we assume that the solution to (2.1) has continuous path. This is the case for the steady diffusion problem, where solutions belong to $H_0^1(X) \subset \mathcal{C}(X)$. We also assume that the Green's function $G(x, y)$ is Lipschitz in x with Lipschitz constant uniform in y. Again, this is the case for the steady diffusion problem. However, it is not the case for the Robin boundary equation, where even in $1D$, the Green's function has a logarithmic singularity. With these assumptions, we characterize the limiting distribution of $(u_\varepsilon - u_0)/\varepsilon^{\alpha/2}$ in the space of continuous paths $\mathcal{C}(X)$.

Recall the decomposition in (3.9) and write

$$\frac{u_\varepsilon - u_0}{\varepsilon^{\alpha/2}}(x) = -\varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon u_0(x) + \varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0(x) + \varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0)(x). \tag{5.1}$$

We call the first time on the right hand side $I_\varepsilon(x)$, the second term $Q_\varepsilon(x)$, and the third one $r_\varepsilon(x)$. We verify also that the sum of the last two terms is $\varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_\varepsilon(x)$, which we call $Q^\varepsilon(x)$.

Our plan is as follows: First, we show that $I_\varepsilon(x)$ has the limiting distribution in $\mathcal{C}(X)$ as desired in (2.11). Second, we show that $Q^\varepsilon(x)$ converges in distribution $\mathcal{C}(X)$ to the zero function. Since the zero process is deterministic, the convergence in fact holds in probability [5, p.27]; the conclusion of Theorem 2.5 follows immediately.

To show convergence of $I_\varepsilon(x)$ and $Q^\varepsilon(x)$, we apply the following standard result on weak convergence of probability measures, whose proof can be found for instance in [16, p.64].

Proposition 5.1. *Suppose $\{M_\varepsilon\}_{\varepsilon\in(0,1)}$ is a family of random processes parametrized by $\varepsilon \in (0,1)$ with values in the space $\mathcal{C}(X)$ of continuous functions on X, and $M_\varepsilon(0) = 0$. Then M_ε converges in distribution to M_0 as $\varepsilon \to 0$ if the following holds:*

(i) (Finite-dimensional distributions) *for any $0 \le x_1 \le \cdots \le x_k \le 1$, the joint distribution of $(M_\varepsilon(x_1), \cdots, M_\varepsilon(x_k))$ converges to that of $(M_0(x_1), \cdots, M_0(x_k))$ as $\varepsilon \to 0$.*

(ii) (Tightness) *The family $\{M_\varepsilon\}_{\varepsilon\in(0,1)}$ is a tight sequence of random processes in $\mathcal{C}(X)$. A sufficient condition is the Kolmogorov criterion: $\exists \delta, \beta, C > 0$ such that*

$$\mathbb{E}\left\{|M_\varepsilon(s) - M_\varepsilon(t)|^\beta\right\} \le C|t-s|^{1+\delta}, \tag{5.2}$$

uniformly in ε and $t, s \in (0,1)$.

Proof of Theorem 2.5. We carry out the aforementioned two-step plan. Let us denote by $I(x)$ the Gaussian process on the right hand side of (2.11).

Convergence of $I_\varepsilon(x)$ to $I(x)$. We first show convergence of finite dimensional distributions. Fix an arbitrary natural number N, an N-tuple $(x_1, \cdots, x_N)$, we need to show that the joint law of $(I_\varepsilon(x_1), \cdots, I_\varepsilon(x_N))$ converges to that of $(I(x_1), \cdots, I(x_N))$. It suffices to show that for arbitrary N-tuple $(\xi_1, \cdots \xi_N) \in \mathbb{R}^N$, we have

$$\sum_{i=1}^N \xi_i I_\varepsilon(x_i) \xrightarrow[\varepsilon\to 0]{\text{distribution}} \sum_{i=1}^N \xi_i I(x_i),$$

as convergence in distribution of random variables. Recalling the exact form of I_ε and I, our goal is to show, with $\sigma_H := \sqrt{\kappa/(H(2H-1))}$, that

$$\frac{1}{\varepsilon^{\alpha/2}} \int_X \sum_{i=1}^N \xi_i G(x_i, y) q_\varepsilon(y) dy \xrightarrow[\varepsilon\to 0]{\text{distribution}} \sigma_H \int_X \sum_{i=1}^N \xi_i G(x_i, y) u_0(y) dW_H(y). \tag{5.3}$$

Set $F_x(y) = \sum_{i=1}^N \xi_i G(x_i, y) u_0(y)$. We verify that $F_x \in L^1 \cap L^\infty(\mathbb{R})$ and apply the following convergence result:

$$\frac{1}{\varepsilon^{\alpha/2}} \int_X F(y) q_\varepsilon(y) dy \xrightarrow[\varepsilon\to 0]{\text{distribution}} \sigma_H \int_X F(y) dW_H(y), \quad \text{for } F \in L^1 \cap L^\infty, \tag{5.4}$$

which is Theorem 3.1 of [2]. This proves the convergence of finite dimensional distributions.

To show tightness of $I_\varepsilon(x)$, we calculate $\mathbb{E}|I_\varepsilon(x) - I_\varepsilon(y)|^2$ which we denote by J_1. Calculation shows:

$$\begin{aligned} J_1 &= \frac{1}{\varepsilon^\alpha} \mathbb{E}\left(\int_X [G(x,z) - G(y,z)] q_\varepsilon(z) u_0(z) dz\right)^2 \\ &= \frac{1}{\varepsilon^\alpha} \int_{X^2} [G(x,z) - G(y,z)][G(x,\xi) - G(y,\xi)] R_\varepsilon(z-\xi) u_0(z) u_0(\xi) dz d\xi. \end{aligned}$$

Use the assumption on the Lipschitz continuity of G to obtain

$$J_1 \le (\mathrm{Lip} G)^2 |x-y|^2 \frac{1}{\varepsilon^\alpha} \int_X |R_\varepsilon(z-\xi) u_0(z) u_0(\xi)| dz d\xi \le C|x-y|^2. \tag{5.5}$$

We used the fact that the integral above has size ε^α, which can be easily proved as before. This shows tightness and complete the first step.

Convergence of $Q^\varepsilon(x)$ to zero function. For convergence of the finite distributions, we show that $\sum_{i=1}^N \xi_i Q^\varepsilon(x_i)$ converges to zero in $L^2(\Omega, \mathbb{P})$, which is stronger. Since we can group $\sum_{i=1}^N \xi_i G(x_i, y)$ together as in (5.3), it suffices to show $\sup_{x \in X} \mathbb{E}|Q^\varepsilon(x)| \to 0$.

We prove this by showing $\sup_{x\in X} \mathbb{E}|Q_\varepsilon(x)|^2 \to 0$ and $\sup_{x\in X} \mathbb{E}|r_\varepsilon(x)| \to 0$. The first term, i.e., $\mathbb{E}|Q_\varepsilon(x)|^2$, has the following expression,

$$\varepsilon^{-\alpha} \int_{X^4} G(x,y)G(y,z)G(x,\xi)G(\xi,\eta)u_0(z)u_0(\eta)\mathbb{E}\{q_\varepsilon(y)q_\varepsilon(z)q_\varepsilon(\xi)q_\varepsilon(\eta)\} d\xi d\eta dz dy. \tag{5.6}$$

Bound the Green's functions and u_0 by their uniform norms. Then apply Lemma A.2 to get

$$\mathbb{E}|Q_\varepsilon(x)|^2 \le C\varepsilon^{-\alpha} \|G\|_\infty^4 \|u_0\|_\infty^2 \int_{X^4} \sum_p |R_\varepsilon(x_{p(1)} - x_{p(2)}) R_\varepsilon(x_{p(3)} - x_{p(4)})|. \tag{5.7}$$

This time p runs over all 15 possible ways to choose two pairs from $\{1,2,3,4\}$. Since R_ε is bounded by $C\varepsilon^\alpha |x|^{-\alpha}$, we verify each item in the sum has a contribution of size $\varepsilon^{2\alpha}$ and so does the sum. Consequently, $\mathbb{E}|Q_\varepsilon(x)|^2 \le C\varepsilon^\alpha$ and converges to zero uniformly in x.

For $r_\varepsilon(x)$, we use Cauchy-Schwarz to get

$$|r_\varepsilon(x)| \le \varepsilon^{-\frac{\alpha}{2}} \left(\int_X |q_\varepsilon(z)(u_\varepsilon - u_0)(z)|^2 dz \right)^{\frac12} \left(\int_X \left(\int_X G(x,y) q_\varepsilon(y) G(y,z) dy \right)^2 dz \right)^{\frac12}.$$

Bound q_ε in the first integral by its uniform norm. Take expectation afterwards. We verify that $\mathbb{E}|r_\varepsilon(x)|$ is bounded by

$$C\varepsilon^{-\frac{\alpha}{2}} \left(\mathbb{E}\|u_\varepsilon - u_0\|^2 \right)^{\frac12} \left(\mathbb{E} \int_{X^3} G(x,y)G(y,z)G(x,\xi)G(\xi,z) q_\varepsilon(y) q_\varepsilon(\xi) dy d\xi dz \right)^{\frac12}.$$

The integral above can be estimated as before and is of size ε^α. Expectation of $\|u_\varepsilon - u_0\|^2$ is also of size α as shown before. As a result, $\mathbb{E}|r_\varepsilon(x)| \le C\varepsilon^\alpha$ and converges to zero uniformly with respect to x.

It suffices now to prove tightness of $Q^\varepsilon(x)$. To this end, we calculate $\mathbb{E}|Q^\varepsilon(x) - Q^\varepsilon(y)|^2$ which we denote by J_2.

$$J_2 = \mathbb{E} \left(\varepsilon^{-\frac{\alpha}{2}} \int_{X^2} [G(x,z) - G(y,z)] q_\varepsilon(z) G(z,\xi) q_\varepsilon(\xi) u_\varepsilon(\xi) d\xi dz \right)^2.$$

Use Cauchy-Schwarz and the uniform bound on q_ε; we get

$$J_2 \le \varepsilon^{-\alpha} \mathbb{E} \left\{ (\|q\|_\infty \|u_\varepsilon\|)^2 \int_X \left(\int_X [G(x,z) - G(y,z)] q_\varepsilon(z) G(z,\xi) dz \right)^2 d\xi \right\}.$$

The term $\|u_\varepsilon\|$ can be bounded uniformly with respect to ω because the operator norm of $\mathcal{G}_\varepsilon$ is. Therefore, we have

$$J_2 \le C\mathbb{E}\int_{X^3}[G(x,z)-G(y,z)][G(x,\eta)-G(y,\eta)]q_\varepsilon(z)q_\varepsilon(\eta)G(z,\xi)G(\eta,\xi)dzd\eta d\xi.$$

Use the Lipschitz continuity and the uniform bound of G to get

$$J_2 \le C\varepsilon^{-\alpha}\int_{X^3}(\mathrm{Lip}G)^2|x-y|^2R_\varepsilon(z-\eta)\|G\|_\infty^2 dzd\eta d\xi \le C|x-y|^2. \tag{5.8}$$

The second inequality holds because the integral is of size ε^α as we have seen many times. This completes the proof of Q^ε converging to zero functions. Recall the argument above Proposition 5.1 to complete the proof of the theorem. □

Remark 5.2. We assume that the random field $q(x)$ satisfies (A3) to take advantage of Lemma A.2. However, this assumption is not necessary for Theorem 2.5 to hold. Indeed, with (A1) and (A2), we can derive the asymptotic behavior of the fourth order moment $\mathbb{E}\{q(x_1)q(x_2)q(x_3)q(x_4)\}$ when the four points are mutually far away from each other. We can use this fact to estimate (5.6) instead. The argument involves routine decomposition of integration domains, which is tedious so we omit it here.

6 Weak convergence in the Hilbert space $\mathcal{H}_\beta^{-\mu}$

In this section, we prove Theorem 2.7, which states that the limit in Theorem 2.3 holds in a stronger sense. Namely, viewed as $\mathcal{H}_\beta^{-\mu}$-valued processes, $\{u_\varepsilon - \mathbb{E}u_\varepsilon\}_{\varepsilon\in(0,1)}$ converges in distribution to the right hand side of (2.9). In some cases, $\mathcal{H}_\beta^{-\mu}$ can be chosen as $L^2(X)$.

Let $\mathcal{H}$ denotes a separable Hilbert space with an orthonormal basis $\{\phi_n\}_{n=1}^\infty$. To prove convergence in law of $\mathcal{H}$-valued process $\{Y_\varepsilon\}_{\varepsilon\in(0,1)}$ to a $\mathcal{H}$-valued random variable Y_0, we need to show that any finite dimensional distribution of Y_ε converges to that of Y_0 and that the family of laws of $\{Y_\varepsilon\}_{\varepsilon\in(0,1)}$ is tight. The first condition boils down to

$$\left(\langle Y_\varepsilon,\phi_{i_1}\rangle,\cdots,\langle Y_\varepsilon,\phi_{i_k}\rangle\right) \xrightarrow[\varepsilon\to 0]{\text{distribution}} \left(\langle Y_0,\phi_{i_1}\rangle,\cdots,\langle Y_0,\phi_{i_k}\rangle\right), \tag{6.1}$$

as $\mathbb{R}$-valued random variables, for any $k\in\mathbb{N}$, and any k-tuple $(i_1,\cdots,i_k)$. The technicality lies in the tightness of the family $\{Y_\varepsilon\}_{\varepsilon\in(0,1)}$. A sufficient condition is the following.

Proposition 6.1. *Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{\phi_n\}_{n=1}^\infty$. For an integer $n\ge 1$, let P_n denote the projection into the space spanned by the first n basis functions.*

A family of $\mathcal{H}$-valued random variables $\{Y_\varepsilon\}_{\varepsilon\in(0,1)}$ is tight *if*

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}\|Y_\varepsilon\|_{\mathcal{H}} < \infty, \tag{6.2}$$

and

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}\|Y_\varepsilon - P_NY_\varepsilon\|_{\mathcal{H}} \xrightarrow{N\to\infty} 0. \tag{6.3}$$

This proposition follows from the definition of tightness of general probability measure on metric spaces, and the structure of separable Hilbert spaces; see [5, 22].

Proof of Theorem 2.7. The Laplacian case. We first consider the case $P(x, D) = -\Delta$, and hence $\beta = 2$. For simplicity, let us denote the eigenvalues and corresponding eigenfunctions of $(-\Delta)_{\mathrm{D}}$ by $(\nu_n, \phi_n)_{n=1}^{\infty}$; let us also simplify the notation $\mathcal{H}^s_2$ by $\mathcal{H}^s$.

We denote by $\{Y_\varepsilon(x)\}$ the $\mathcal{H}^{-\mu}$-valued sequence $\varepsilon^{-\alpha/2}(u_\varepsilon - \mathbb{E}u_\varepsilon)$ and by $I(x)$ the process in (2.9). According to the remark preceding this proof, Theorem 2.3 proves convergence of finite-dimensional distributions of Y_ε to those of I. It remains to show that $\{Y_\varepsilon\}$ is a tight sequence in $\mathcal{H}^{-\mu}$. To this end, we apply the proposition above. We first decompose Y_ε into three parts: $Y_{1\varepsilon} := -\varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon u_0$ and

$$Y_{2\varepsilon} := \frac{\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0 - \mathbb{E}\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0}{\varepsilon^{\frac{\alpha}{2}}}, \quad Y_{3\varepsilon} := \frac{\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0) - \mathbb{E}\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)}{\varepsilon^{\frac{\alpha}{2}}}.$$

Both criteria in the proposition concerns $\mathcal{H}^{-\mu}$ norms, so we express those of $Y_{i\varepsilon}$ explicitly, using the orthonormal basis given by $\{\nu_n^\mu \phi_n\}_{n=1}^\infty$. We have

$$\|Y_{1\varepsilon}\|^2_{\mathcal{H}^{-\mu}} = \sum_{n=1}^\infty \langle Y_{1\varepsilon}, \nu^\mu \phi_n\rangle^2_{\mathcal{H}^{-\mu}} = \sum_{n=1}^\infty \frac{1}{\nu^{2\mu}} \langle Y_{1\varepsilon}, \phi_n\rangle^2. \tag{6.4}$$

Recall the definition of χ_ε; we have that $Y_{1\varepsilon} = \varepsilon^{-\alpha/2}\chi_\varepsilon$. Since χ_ε satisfies

$$-\Delta\chi_\varepsilon + q_0\chi_\varepsilon = -q_\varepsilon u_0,$$

we have

$$\langle Y_{1\varepsilon}, \phi_n\rangle = \left\langle \frac{(-\Delta)_{\mathrm{D}}^{-1}(-q_\varepsilon u_0 - q_0\chi_\varepsilon)}{\varepsilon^{\frac{\alpha}{2}}}, \phi_n \right\rangle = \frac{1}{\nu_n} \left\langle \frac{-q_\varepsilon u_0 - q_0\chi_\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, \phi_n \right\rangle.$$

Now write

$$\left\langle \frac{-q_\varepsilon u_0 - q_0\chi_\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, \phi_n \right\rangle = -\frac{1}{\varepsilon^{\frac{\alpha}{2}}} \int_X q\left(\frac{x}{\varepsilon}\right) [u_0(x)\phi_n(x) - u_0(x)m(x)]dx,$$

with $m(x) = \mathcal{G}(q_0\phi_n)(x)$. It follows then that the mean square of this item can be bounded by $\|u_0\|_{L^\infty}, \|q_0\|_{L^\infty}$, with uniform bound in ε and n. That is,

$$\mathbb{E}\langle Y_{1\varepsilon}, \phi_n\rangle^2 \le C/\nu_n^2,$$

with some constant C uniform in ε and n. This shows that

$$\sup_{\varepsilon\in(0,1)} \|Y_{1\varepsilon}\|^2_{\mathcal{H}^{-\mu}} \le \sum_{n=1}^\infty \frac{C}{\nu_n^{2(\mu+1)}} \le C.$$

Here we used the fact that $\nu_n \ge Cn^{2/d}$ for some C universal in n. This fact follows from the Weyl's law on counting the eigenvalues of the Dirichlet Laplacian (cf. the remark after [**?**, Theorem 17.5.3]; the sharp constant is obtained in [19]). The series above converges because asymptotically the elements in the series are $1/n^{4(\mu+1)/d}$ and μ is chosen so that

$4(\mu+1)/d > 1$. This proves (6.2) for $Y_{1\varepsilon}$. Since $Y_{1\varepsilon} - P_N Y_{1\varepsilon}$ precisely consists of the coordinates with indices larger than N, the second criterion follows from the same lines above.

Now for $Y_{2\varepsilon}$ and $Y_{3\varepsilon}$, we repeat the above proof for $Y_{1\varepsilon}$. The only modification is:

$$\mathbb{E}\langle Y_{2\varepsilon}, \phi_n\rangle^2 = \varepsilon^{-\alpha}\mathrm{Var}\langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0, \varphi\rangle = \nu_n^{-2}\varepsilon^{-\alpha}\mathrm{Var}\langle q_\varepsilon \mathcal{G} q_\varepsilon u_0, \phi_n - m\rangle,$$

again with $m = \mathcal{G}q_0\phi_n$. The last equality can be shown by introducing $\chi_{2\varepsilon} = \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0$ and following the trick we did with χ_ε above. Now in Lemma 4.2, let u_0 play the role of φ of the lemma, and bound the L^2 norm of $\phi_n - m$ by some uniform constant. This implies $\sup_{\varepsilon\in(0,1)} \mathbb{E}\langle Y_{2\varepsilon}, \phi_n\rangle^2 \le C/\nu_n^2$. Then the criteria (6.2)-(6.3) follows for $Y_{2\varepsilon}$.

For $Y_{3\varepsilon}$, we can introduce $\chi_{3\varepsilon} = \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)$ and argue as above, and use estimate (3.12), again with the roles of u_0 and $\phi_n - m$ exchanged. Since $\alpha < 4\beta$, this estimate is enough to prove the criteria for $Y_{3\varepsilon}$.

Combining the above arguments, we finally proved that $\{Y_\varepsilon\}_{\varepsilon\in(0,1)}$ is tight in $\mathcal{H}^{-\mu}$. Therefore, we proved the theorem for the case of $P(x,D)$ being the Laplacian.

The fractional Laplacian case. We use the fact that λ_n^β, the eigenvalue of $(-\Delta)_{\mathrm{D}}^{\beta/2}$, is comparable to a fractional power of ν_n, the eigenvalue of $(-\Delta)_{\mathrm{D}}$:

$$C^{-1}\nu_n^{\frac{\beta}{2}} \le \lambda_n^\beta \le C\nu_n^{\frac{\beta}{2}}, \tag{6.5}$$

for some constant C [12]. Combining this with the aforementioned estimate on the eigenvalues of the Dirichlet Laplacian, we see that $\lambda_n^\beta \sim n^{\beta/d}$. Then the same procedure above works. This completes the proof. □

7 Conclusions and further discussions

We considered the deterministic stochastic correctors for equation (2.1), where the coefficient in the potential term is constructed as a function of a long-range correlated Gaussian random field. We found that the stochastic corrector had magnitude $\varepsilon^{\alpha/2}$ and its limiting distribution can be characterized by a Gaussian random process in some weak sense. The deterministic corrector, however, may be larger than the stochastic corrector. We find that the threshold for this to happen is $\alpha = \beta$.

In our analysis, we assumed that the Green's function $G(x,y)$ had a singularity of the type $|x-y|^{-(d-\beta)}$ near the diagonal $x = y$. Other types of singularities, such as $G(x,y) \sim \log|x-y|$, can be analyzed using similar techniques. For the logarithmic singularity, which occurs for the steady diffusion problem when $d = 2$ and the Robin boundary equation when $d = 1$, our results still hold. The deterministic corrector is then of order ε^α while the stochastic corrector has an amplitude of order $\varepsilon^{\alpha/2}$.

To prove the convergence in distribution of the stochastic corrector, we have assumed $\alpha < 4\beta$. This is a technical reason related to the fact that only in this case is the estimate (3.12) enough to control the remainder term in (3.10). Generalizations to $\alpha > 4\beta$ require that we estimate sufficiently high-order moments of $q(x)$. Once we have a good estimate on the sixth-order moments for instance, we can perform one more iteration in (3.10) to get

$$\begin{aligned}\langle u_\varepsilon - u_0, \varphi\rangle = -\langle \mathcal{G}q_\varepsilon u_0, \varphi\rangle + \langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0, \varphi\rangle - \langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0, \varphi\rangle \\ - \langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0), \varphi\rangle.\end{aligned}$$

Supposing that the sixth-order moment estimate is sufficiently accurate to control the variance of the third item on the right, and that the estimate on four-order moments is sufficient to control the remaining terms, then the same results as stated in Theorem 2.3 hold for a larger range of values of α. We do not carry out the details of such derivations here.

A Two useful lemmas

A.1 Estimates of convolution of potentials

Here, we record a lemma which estimates the convolution of two potential functions, or the convolution of a potential function with a logarithmic function. Its proof can be found in the appendix of [3].

Lemma A.1. *Let X be an open and bounded subset in $\mathbb{R}^d$, and $x \neq y$ two points in X. Let α, β be positive numbers in $(0, d)$. We have the following convolution results.*

$$\int_X \frac{1}{|z-x|^\alpha} \cdot \frac{1}{|z-y|^\beta} dz \le \begin{cases} C|x-y|^{d-(\alpha+\beta)}, & \alpha+\beta > d, \\ C(\log|x-y|+1), & \alpha+\beta = d, \\ C, & \alpha+\beta < d. \end{cases} \tag{A.1}$$

The convolution of logarithms with a weak singular potential turns out to be finite as follows:

$$\int_X |\log|z-x|| \frac{1}{|z-y|^\alpha} dz \le C. \tag{A.2}$$

A.2 Fourth-order moments of $q(x, \omega)$

The following lemma provides a non-asymptotic estimate of the four-moments of $q(x)$ constructed in (A1-A2), with the additional assumption (A3). In the following, we set $F = \{1, 2, 3, 4\}$ and denote by $\mathcal{U}$ the collections of two pairs of unordered numbers in F, i.e.,

$$\mathcal{U} = \big\{p = \{\big(p(1), p(2)\big), \big(p(3), p(4)\big)\} \mid p(i) \in F, p(1) \neq p(2), p(3) \neq p(4)\big\}. \tag{A.3}$$

As members in a set, the pairs $(p(1), p(2))$ and $(p(3), p(4))$ are required to be distinct; however, they can have one common index. There are three elements in $\mathcal{U}$ whose indices $p(i)$ are all different. They are precisely $\{(1,2),(3,4)\}$, $\{(1,3),(2,4)\}$ and $\{(1,4),(2,3)\}$. Let us denote by $\mathcal{U}_*$ the subset formed by these three elements, and its complement by $\mathcal{U}^*$.

Lemma A.2. *Let $q(x,\omega)$ be the random field constructed in* (A1)-(A3)*. Fix four arbitrary points $\{x_i \in \mathbb{R}^d;\ 1 \le i \le 4\}$. Then we have the following.*

$$\begin{aligned}&\left|\mathbb{E}\prod_{i=1}^{4} q(x_i) - \sum_{p\in\mathcal{U}_*} R(x_{p(1)} - x_{p(2)})R(x_{p(3)} - x_{p(4)})\right|\\ \le& C\sum_{p\in\mathcal{U}^*} R(x_{p(1)} - x_{p(2)})R(x_{p(3)} - x_{p(4)}).\end{aligned} \tag{A.4}$$

The constant C is the one in (2.8) *raised to the fourth power.*

For a proof of this lemma, we refer the reader to [4, Proposition 4.1].

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