Correlations of heterogeneous wave fields propagating in homogeneous media

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Abstract

The paper concerns the analysis of the correlations of wave fields propagating in heterogeneous media. We show that selected field-field correlations asymptotically propagate in non-scattering media, i.e., propagate as if the underlying medium was homogeneous. Such correlations are therefore good candidates for imaging strategies in heterogeneous media based on back-propagations in a homogeneous medium.

keywords: inverse problem, imaging, correlations, radiative transfer, random media.

1 Introduction

Imaging in heterogeneous media using wave field measurements has a long tradition for instance in seismic applications or non-destructive testing [16, 23]. Popular imaging methods such as Kirchhoff migration (KM) or matched field imaging are based on the back-propagation of the measured wave fields. A model for the back-propagation (on the computer to form an image) is necessary, typically a back-propagation in a known homogeneous medium with known properties. The main difficulty with such inversions is that the reconstructed image strongly depends on the neglected fluctuations in the heterogeneous medium, which are modeled as random since they are not known.

A solution to the statistical instability of wave field back-propagation is to consider imaging procedures based on field-field correlations instead. Still using a homogeneous medium to perform the back-propagation of the correlations, coherent interferometry

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(CINT) was shown to have superior reconstruction capabilities to KM in several regimes of wave propagation [12, 14]. The back-propagation in a homogeneous medium is accurate for relatively weak disorder in the underlying medium. For stronger disorder with distances of propagation larger than the mean free path, more complex models for the back-propagation of the correlations, for instance based on transport equations, need to be used; see for instance [3, 9]. Such models require that the statistical properties of the underlying medium be known or estimated prior to the reconstructions.

The main advantage of correlation-based imaging functionals such as CINT is that they are statistically stable in some regimes of wave propagation. This means that the reconstructed images weakly depend on the specific realization of the random medium. This is in contrast to the Kirchhoff imaging functional [14]. The main reason for such a statistical stability is that random phases of fields cancel when appropriate correlations are considered; see [1, 3, 9, 13].

This paper considers the analysis of such correlations. We show that there exists a set of correlations that we will call coherent correlations, which can be constructed from field measurements and which propagate in a homogeneous medium. This is in contrast to other correlations, whose description involves an incoherent contribution that underwent scattering by the random heterogeneities. These coherent correlations are still attenuated during the propagation and hence are still affected by the heterogeneities. However, they do not scatter and as such form a good class of candidates for back-propagation in a homogeneous medium as is done in imaging functionals such as CINT.

To be more specific, we consider a heterogeneous medium modeled by a random sound speed $c_\varepsilon(x)$ centered at c_0 and let us define two propagating fields in that random medium, solutions of

$$\frac{\partial^2 u_{\varepsilon,m}}{\partial t^2} - c_\varepsilon^2(x)\Delta u_{\varepsilon,m} = 0, \qquad t>0,\ x\in\mathbb{R}^d,\ m=1,2, \tag{1}$$

augmented with two sets of initial conditions indexed by m. Here $d \geq 2$ is spatial dimension. We want to calculate some spatio-temporal correlations of these two random fields. Identifying $u_{\varepsilon,m}$ as a pressure potential, we introduce the pressure and velocity fields as

$$p_{\varepsilon,m} = \frac{\partial}{\partial t}u_{\varepsilon,m}, \qquad v_{\varepsilon,m} = \frac{1}{\rho}\nabla u_{\varepsilon,m}, \tag{2}$$

where ρ is a density we assume to be constant to simplify and the compressibility is defined as $\kappa_\varepsilon(x) = (\rho c_\varepsilon^2(x))^{-1}$ with average κ_0 so that the average sound speed $c_0 = (\kappa_0\rho)^{-\frac{1}{2}}$. We assume we are in the weak coupling regime [15] in which the fluctuations of the compressibility are modeled by

$$\kappa_\varepsilon(x) = \kappa_0 + \sqrt{\varepsilon}\kappa_1\left(x, \frac{x}{\varepsilon}\right), \tag{3}$$

where κ_1 is a mean zero stationary random field with given correlation function. $\varepsilon \ll 1$ is the rescaled wavelength, and the factor $\sqrt{\varepsilon}$ is the rescaled strength of the fluctuations. Let us define the vectors

$$\mathbf{w}_{\varepsilon,m}(t,x) = \left(\sqrt{\kappa_\varepsilon(x)}v_{\varepsilon,m}(t,x)\,,\ \sqrt{\rho}p_{\varepsilon,m}\right)^t. \tag{4}$$

The field-field correlation is then defined by

$$C_\varepsilon(t,s,x,y) = \frac{1}{2}\mathbf{w}_{\varepsilon,1}(t-\frac{\varepsilon s}{2}, x-\frac{\varepsilon y}{2})\cdot\mathbf{w}_{\varepsilon,2}(t+\frac{\varepsilon s}{2}, x+\frac{\varepsilon y}{2}). \tag{5}$$

This "correlation" is not a statistical correlation (see below) as its average does not vanish. We still refer to C_ε as a (field-field) correlation.

It turns out that the above correlation function asymptotically satisfies a closed-form equation only when it is written in the frequency domain for the offset variable s. This is done by defining

$$\hat{C}_\varepsilon(t,\omega,x,y) = \int_{\mathbb{R}} e^{-i\omega s} C_\varepsilon(t,s,x,y)\frac{ds}{\pi}. \tag{6}$$

The unusual normalization will simplify notation in the sequel. The main result of this work is that there exists a class of offset values y, which depends on ω and in general on (t,x), such that the corresponding correlation $\hat{C}_\varepsilon(t,\omega,x,y)$ asymptotically propagates in an absorbing but non-scattering medium. Such coherent correlations are immune to scattering by the heterogeneities and are therefore good candidates for imaging functionals based on back-propagation in a homogeneous medium.

We will provide numerical evidence that these coherent correlations exist and propagate as indicated above. The derivation of such correlations is based on the oscillatory properties of correlations in highly disordered media and is obtained by an analysis of the transport equation satisfied by the space-time Wigner transform of the wave field. Such oscillations were studied e.g. in [2] and were verified experimentally in e.g. [19, 20].

The use of such correlations in practical imaging scenarios requires further studies. The coherent correlations do not undergo scattering when they are *statistically stable*, which may require that they are averaged over sufficiently large detectors. In imaging settings where such spatial averaging is feasible when the imaging functional is applied, then we expect the coherent correlations to enhance the reconstructions. In the setting of imaging of small inclusions, then resolution is typically limited by statistical fluctuations [1] of the field-field correlations, which are not addressed in detail in this paper.

The paper is structured as follows: our main result is presented in section 2. Its derivation is given in sections 3 and 4. Section 3 is devoted to the derivation of transport equations for the correlations and section 4 to the filtering of the incoherent part of the correlations. An example of application is given in section 5, where we address the problem of the reconstruction of a source buried in a heterogeneous medium. In section 6, we show numerically the existence of the coherent correlations. Some conclusions are given in section 7.

2 Main result

The *statistical correlation* associated to $C_\varepsilon(t,s,x,y)$ is defined by

$$\begin{aligned} \delta C(t,s,x,y) &= C_\varepsilon(t,s,x,y) - C_{\varepsilon,0}(t,s,x,y) \\ C_{\varepsilon,0}(t,s,x,y) &= \frac{1}{2}\mathbb{E}\big\{\mathbf{w}_{\varepsilon,1}(t-\tfrac{\varepsilon s}{2}, x-\tfrac{\varepsilon y}{2})\big\}\ \mathbb{E}\big\{\mathbf{w}_{\varepsilon,2}(t+\tfrac{\varepsilon s}{2}, x+\tfrac{\varepsilon y}{2})\big\}, \end{aligned} \tag{7}$$

where $\mathbb{E}$ denotes ensemble average over all possible realizations of the random medium. We drop the dependency in ε in the definition to simplify the notation. In fact, the statistical correlation function of the two wave fields $\mathbf{w}_{\varepsilon,1}$ and $\mathbf{w}_{\varepsilon,2}$ evaluated at $(t-\frac{\varepsilon s}{2}, x-\frac{\varepsilon y}{2})$ and $(t+\frac{\varepsilon s}{2}, x+\frac{\varepsilon y}{2})$, respectively, should be the two point correlation function $\mathbb{E}\{\delta C(t,s,x,y)\}$. However, we shall recall in (9) below that C_ε, and hence δC are approximately, independent of the realization so that that $\delta C \sim \mathbb{E}\{\delta C\}$. We thus refer to δC as the statistical correlation function.

Let $\mathbf{u}_m = (v_m, p_m)^t$ (still dropping the dependency in ε), where v_m and p_m are defined in (2). When the two fields $\mathbf{u}_1 = \mathbf{u}_2$ are identical and the offsets $s = y = 0$, then the correlation measures the energy density

$$C_\varepsilon(t,0,x,0) \equiv \mathcal{E}_\varepsilon(t,x) = \frac{1}{2}\big(\rho p^2(t,x) + \kappa_\varepsilon(x)|\nabla v|^2(t,x)\big),$$

whose spatial integration $\int_{\mathbb{R}^d} \mathcal{E}_\varepsilon(t,x)dx$ is a conserved quantity of the dynamics as a function of time. When $u_{1,\varepsilon} = u_{2,\varepsilon}$, then $C_\varepsilon(t,s,x,y)$ measures the two point (in space-time) correlation of the random field. More generally, the correlation we have defined above models the two point (in space-time) correlation of two fields propagating in the *same* random medium. An example of two different fields $\mathbf{u}_1$ and $\mathbf{u}_2$ is obtained when the initial condition for $\mathbf{u}_1$ is a shifted copy of that for $\mathbf{u}_2$:

$$\mathbf{u}_1(t=0,x) = \mathbf{u}_2(t=0,x+\varepsilon\tau), \qquad \tau \in \mathbb{R}^d. \tag{8}$$

This corresponds to shifting the location of the initial source term by $\varepsilon\tau$. The shift occurs at the scale of the wavelength, here modeled by the small parameter ε. In such a setting, the correlations are denoted by $C_\varepsilon(t,s,x,y;\tau)$ and $\delta C(t,s,x,y;\tau)$.

As we shall further explain later, $C_{\varepsilon,0}(t,s,x,y;\tau)$ corresponds to the coherent component of the correlation $C_\varepsilon(t,s,x,y;\tau)$. As a consequence, $C_{\varepsilon,0}(t,s,x,y;\tau)$ propagates in a *homogeneous medium* and the effect of the random heterogeneities only appear by means of an absorption factor. We denote by $\widehat{\delta C}(t,\omega,x,y;\tau)$ and $\hat{C}_{\varepsilon,0}(t,\omega,x,y;\tau)$ the Fourier transform of $\delta C(t,s,x,y;\tau)$ and $C_\varepsilon(t,s,x,y;\tau)$ with respect to the variable s as in (6). The crucial property of statistical stability is adressed in the following paragraph.

Statistical stability. We place ourselves in regimes of wave propagation where the correlations are statistically stable (or self-averaging) in the limit $\varepsilon \to 0$, i.e., such that

$$C_\varepsilon(t,s,x,y;\tau) \sim \mathbb{E}\{C_\varepsilon(t,s,x,y;\tau)\} \qquad \text{as } \varepsilon \to 0. \tag{9}$$

The statistical stability is not affected by the presence of a shift τ [10]. Stability results have been established for several regimes of wave propagation that may be seen as simplifications of the regime considered here. We refer the reader to e.g. [6, 10, 21] and the review paper [5].

In all these results, it is shown that the correlations C_ε or δC are not stable pointwise, but rather after *sufficient* spatio-temporal averaging. The domain of averaging has to be large compared to the wavelength ε or the corresponding period of the propagating wave-fields. The limit in (9) thus has to be understood in some weak sense. How large the domain of integration has to be or how much averaging occurs when an imaging functional is applied to the measured correlations is a crucial aspect of the resolution capabilities of imaging functionals [1]. This important aspect is not considered here. What we are interested in this paper is the set of parameters (y, τ) such that $C_\varepsilon \sim E\{C_\varepsilon\}$ is modeled by a homogeneous medium.

Our main result is the following and concerns the statistical correlation $\widehat{\delta C}$:

Result 1 *Given a measurement point (t, x) and a frequency ω, there exists a subset $\mathfrak{Y} = \mathfrak{Y}(t, x, \omega)$ of $\mathbb{R}^d \times \mathbb{R}^d$, such that, for all $(y, \tau) \in \mathfrak{Y}$:*

$$\lim_{\varepsilon \to 0} \widehat{\delta C}(t, \omega, x, y; \tau) = 0. \tag{10}$$

The set $\mathfrak{Y}$ will be characterized in several geometries and regimes of wave propagation in subsequent sections.

In the limit $\varepsilon \to 0$ and for $(y, \tau) \in \mathfrak{Y}$, the above result implies that $\hat{C}_\varepsilon \equiv \hat{C}_{\varepsilon,0}$, the coherent correlation, which we will see propagates in a homogeneous medium. These specific correlations can then be back-propagated in a homogeneous medium without any approximation, for instance to construct correlation-based imaging functionals when such correlations $\hat{C}(t, \omega, x, y; \tau)$ can be estimated sufficiently accurately from available measurements. All other correlations satisfy models of propagation that involve scattering coefficients. When such correlations are used in imaging, more complicated model for inversion should be used such as those for instance in [3, 9, 8].

In some particular configurations, such as the echo mode or the diffusive regime, we will see in section 4 that the set $\mathfrak{Y}$ may depend only on the frequency ω and not on (t, x).

The above result is formal and based on derivations of radiative transfer equations that are themselves formal in the regime of propagation considered in this paper. The mathematical properties of the set $\mathfrak{Y}$, and in particular the characterization of its dependence on (t, x, ω), is understood in simple settings.

The first step to obtain the above result is to derive a model for the propagation of correlations. Such a task is carried out in the next section.

3 Radiative transfer models for correlations

The correlations introduced earlier in (5) satisfy closed-form equations in the limit of vanishing wavelength, i.e., $\varepsilon \to 0$. These equations are radiative transfer equations or phase-space transport equations, whose derivation is briefly recalled below. Their solution may then be decomposed into a coherent part modeling the coherent correlation and an incoherent part modeling the scattering component of the correlation. Our objective in section 4 will be to find the set of (y, τ) such that the latter component vanishes.

3.1 Transport models for the correlations

The matrix-valued spatio-temporal Wigner transform of the fields $\mathbf{u}_1$ and $\mathbf{u}_2$ is defined as

$$W_\varepsilon[\mathbf{u}_1, \mathbf{u}_2](t, \omega, x, k) = \int_{\mathbb{R}^{d+1}} e^{i(y \cdot k - \omega s)} \mathbf{u}_1(t - \frac{\varepsilon s}{2}, x - \frac{\varepsilon y}{2}) \mathbf{u}_2^*(t + \frac{\varepsilon s}{2}, x + \frac{\varepsilon y}{2}) \frac{ds dy}{(2\pi)^{d+1}}. \tag{11}$$

By applying the inverse Fourier transform, we can decompose the correlations in the phase space using the Wigner transform:

$$\mathbf{u}_1(t-\frac{\varepsilon s}{2}, x-\frac{\varepsilon y}{2})\mathbf{u}_2^*(t+\frac{\varepsilon s}{2}, x+\frac{\varepsilon y}{2}) = \int_{\mathbb{R}^{d+1}} e^{i(-y\cdot k+\omega s)} W_\varepsilon[\mathbf{u}_1, \mathbf{u}_2](t,\omega,x,k)d\omega dk. \quad (12)$$

In the limit $\varepsilon \to 0$, the Wigner transform satisfies a closed form radiative transfer equation, which may be used to analyze the correlation introduced in (5). More precisely, let us define the $d+1$ diagonal matrix $A_\varepsilon = \mathrm{diag}(\kappa_\varepsilon, \dots, \kappa_\varepsilon, \rho)$. Then we find that

$$C_\varepsilon(t,s,x,y) = \frac{1}{2}\mathrm{Tr}\big(A_\varepsilon \mathbf{u}_1(t-\frac{\varepsilon s}{2}, x-\frac{\varepsilon y}{2})\mathbf{u}_2^*(t+\frac{\varepsilon s}{2}, x+\frac{\varepsilon y}{2})\big). \quad (13)$$

Following [2, 22], the spatio-temporal Wigner transform $W_\varepsilon[\mathbf{u}_1, \mathbf{u}_2]$ admits in the limit $\varepsilon \to 0$ the decomposition

$$W(t,\omega,x,k) = \sum_{p=\pm} a_p(t,x,k)\delta(\omega - pc_0|k|)\mathbf{b}_p(k)\mathbf{b}_p^*(k), \quad (14)$$

where $\mathbf{b}_\pm(k) = (\frac{\pm 1}{\sqrt{2\rho}}\hat{k}, \frac{1}{\sqrt{2\kappa}})^t$ with $\hat{k} = \frac{k}{|k|}$ and where $a_p(t,x,k)$ for $p = \pm$ solves the following radiative transfer equation:

$$\frac{\partial a_p}{\partial t} + pc_0\hat{k}\cdot\nabla_x a_p + \Sigma(x,|k|)a_p = \int_{\mathbb{R}^d} \sigma(x,k,k')a_p(t,x,k')\delta(c_0|k| - c_0|k'|)dk'. \quad (15)$$

The derivation of the above radiative transfer equation for correlations may be found in e.g. [2, 8, 22]. The solutions of the radiative transfer equation are deterministic, which is a consequence of the self-averaging properties of the random Wigner transform in the limit $\varepsilon \to 0$. The optical parameters in the above kinetic equation are given in the regime of weak-coupling [22] by

$$\sigma(x,k,k') = \frac{\pi c_0^2|k|^2}{2(2\pi)^d}\hat{R}(x,k-k'), \qquad \Sigma(x,|k|) = \int_{\mathbb{R}^d} \sigma(x,k,k')\delta(c_0|k| - c_0|k'|)dk'. \quad (16)$$

Here, $\hat{R}(x,k)$ is the power spectrum associated to the fluctuations of the sound speed:

$$\hat{R}(x,k) = \int_{\mathbb{R}^d} e^{-ik\cdot y}\mathbb{E}\{\kappa_1(x,y)\kappa_1(x,0)\}dy.$$

Note that $W^*(-\omega,-k) = W(\omega,k)$ by construction, which implies [2] that $a_-^*(-k) = a_+(k)$. Let $A_0 = \mathrm{diag}(\kappa_0, \dots, \kappa_0, \rho)$ be the average of A_ε. Using (12), (13) and (14), and the fact that $\mathrm{Tr}(A_0\mathbf{b}_p\mathbf{b}_p^*) = 1$ for $p = \pm$, we find that

$$C_0(t,s,x,y) := \lim_{\varepsilon\to 0} C_\varepsilon(t,s,x,y) \quad = \quad \Re\int_{\mathbb{R}^d} e^{i(-y\cdot k + c_0|k|s)}a_+(t,x,k)dk. \quad (17)$$

Here, $\Re$ stands for real part. Note that a_+ may be complex-valued because of its initial conditions, which we have not specified yet. The correlation C_ε however is always real-valued by construction. Owing (6), we verify that $\hat{C}_\varepsilon^*(t,-\omega,x,y) = \hat{C}_\varepsilon(t,\omega,x,y)$ since $W^*(-\omega,-k) = W(\omega,k)$ so that

$$\hat{C}_0(t,\omega,x,y) := \lim_{\varepsilon\to 0}\hat{C}_\varepsilon(t,\omega,x,y) = \int_{\mathbb{R}^d} e^{-iy\cdot k}a_+(t,x,k)\delta(\omega - c_0|k|)dk, \quad \omega > 0. \quad (18)$$

The above relationship may be recast as

$$\hat{C}_0(t,\omega,x,y) = \frac{\omega^{d-1}}{c_0^d} \int_{S^{d-1}} e^{-i\frac{\omega}{c_0}y\cdot\hat{k}} a_+\big(t,x,\frac{\omega}{c_0}\hat{k}\big) d\hat{k}. \tag{19}$$

Here, S^{d-1} denotes the $d-1$ dimensional unit sphere. As we mentioned earlier, the convergences in (17) and (19) hold in a weak sense, i.e. $C_\varepsilon(t,s,x,y)$ and $\hat{C}_\varepsilon(t,\omega,x,y)$ need to be integrated on domains in (s,x,y) and (ω,x,y) that are sufficiently large with respect to the wavelength ε.

3.2 Assessing the correlations from field measurements

The construction of the correlation function $\hat{C}_\varepsilon(t,\omega,x,y)$ for all ω may be performed by Fourier transforming $C_\varepsilon(t,s,x,y)$ for all values of s. This requires a significant number of measurements.

Alternatively, we may construct $a_+(t,x,k)$ directly from the wave fields as follows. We verify from (11) that

$$a_+(t,x,k) = \lim_{\varepsilon\to 0} \mathbf{b}_+^*(k) A_\varepsilon \int_{\mathbb{R}^d} e^{iy\cdot k} \mathbf{u}_1(t,x-\frac{\varepsilon y}{2}) \mathbf{u}_2^*(t,x+\frac{\varepsilon y}{2}) \frac{dy}{(2\pi)^d} \mathbf{b}_p(k).$$

In other words, $a_+(t,x,k)$ is the projection onto the propagating mode corresponding to $\omega = c_0|k|$ of the spatial Wigner transform (the integral of the spatio-temporal Wigner transform over the variable ω). Such a construction requires measuring the correlation of $\mathbf{u}_1$ and $\mathbf{u}_2$ for all values of y and therefore in practice requires wave field measurements over very large detector arrays.

As a third possibility to construct correlations from field measurements, we may use (18) and realize that $\hat{C}_0(t,\omega,x,y)$ corresponds to correlations of signals propagating with frequency ω. We can then filter the frequency ω in $\mathbf{u}_m$ for $m=1,2$ so that the Wigner transform for these fields is now of the form $a_+(t,x,k)\delta(|k|-\frac{\omega}{c_0})$. We then observe that

$$C_0(t,0,x,y) = \Re \int_{|k|=\frac{\omega}{c_0}} e^{-ik\cdot y} a_+(t,x,k) dk = \frac{1}{c_0} \Re \hat{C}_0(t,\omega,x,y).$$

In other words, we may construct the real part of $\hat{C}_0(t,\omega,x,y)$ from $C_0(t,0,x,y)$. The measurements performed on each shell separately does not allow us to separate the forward propagating modes a_+ from the backward propagating modes a_-. We thus cannot obtain the full measurement $\hat{C}_0(t,\omega,x,y)$ but rather only its real part. Yet such a construction comes at a much lower cost than the other measurements mentioned above as it avoids knowing all the temporal or all spatial correlations of the wave fields.

3.3 Initial conditions and spatial shifts

In order to separate the coherent and incoherent contributions in the correlations, we need to find initial conditions for $a_+(t,x,k)$. Let us assume that the initial conditions for the wave fields are of the form

$$p_{\varepsilon,m}(0,x) = p_{0,m}(x,\frac{x}{\varepsilon}), \qquad v_{\varepsilon,m}(0,x) = \phi_m(x)\nabla\psi_m(\frac{x}{\varepsilon}), \qquad m=1,2, \tag{20}$$

for smooth functions $p_{0,m}$, ϕ_m, and ψ_m. Let us introduce the scalar quantity

$$f_{\varepsilon,m}(x,z,k) = \sqrt{\frac{\rho}{2}}\phi_m(x)\nabla\psi_m(\frac{z}{\varepsilon})\cdot\hat{k} + \sqrt{\frac{\kappa_0}{2}}p_{0,m}(x,\frac{z}{\varepsilon}). \tag{21}$$

Following [22, (3.41)], obtain that the initial condition for $a(t,x,k)$ is given by

$$\begin{aligned} a_+(0,x,k) &= W[f_{\varepsilon,1},f_{\varepsilon,2}](x,k) \\ W[f_{\varepsilon,1},f_{\varepsilon,2}](x,k) &= \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{iy\cdot k} f_{\varepsilon,1}(x,x-\frac{\varepsilon y}{2},k)f_{\varepsilon,2}(x,x+\frac{\varepsilon y}{2},k)dy. \end{aligned} \tag{22}$$

To obtain a mode concrete transport model for the correlations, we consider a specific relationship between $\mathbf{u}_1(t=0,x)$ and $\mathbf{u}_2(t=0,x)$ characterized by the spatial shifts introduced in (8). This translates into the same shifts for the terms f_ε:

$$f_{\varepsilon,1}(x,z,k) = f_{\varepsilon,2}(x+\varepsilon\tau, z+\varepsilon\tau, k).$$

Let us define

$$f_\varepsilon(x,z,k) := f_{\varepsilon,1}(x-\frac{\varepsilon\tau}{2}, z-\frac{\varepsilon\tau}{2},k) = f_{\varepsilon,2}(x+\frac{\varepsilon\tau}{2}, z+\frac{\varepsilon\tau}{2},k)$$

$$W[f_{\varepsilon,1},f_{\varepsilon,2}](x,k) = \lim_{\varepsilon\to 0}\frac{e^{i\tau\cdot k}}{(2\pi)^d}\int_{\mathbb{R}^d} e^{iy\cdot k} f_\varepsilon(x+\frac{\varepsilon\tau}{2},x-\frac{\varepsilon y}{2},k)\cdot f_\varepsilon(x-\frac{\varepsilon\tau}{2},x+\frac{\varepsilon y}{2},k)dy.$$

Since f_ε is smooth in its first variable, the limit is not affected by replacing $x\pm\frac{\varepsilon\tau}{2}$ by x so that

$$W[f_{\varepsilon,1},f_{\varepsilon,2}](x,k) = e^{i\tau\cdot k}a_0(x,k), \qquad a_0(x,k) := W[f_\varepsilon,f_\varepsilon](x,k). \tag{23}$$

This is the main result of this paragraph. The initial conditions for the radiative transfer equation are $e^{i\tau\cdot k}$ times the initial conditions in the case where $\mathbf{u}_1(t=0,x) = \mathbf{u}_2(t=0,x)$ so that:

$$a_+(0,x,k)\delta(\omega - c_0|k|) = e^{i\tau\cdot k}W_0(\omega,x,k), \tag{24}$$

where we have defined

$$W_0(\omega,x,k) = a_0(x,k)\delta(\omega - c_0|k|). \tag{25}$$

In other words, $W_0(\omega,x,k)$ is the part of the initial conditions for the correlation function that picks the component of frequency $\omega = c_0|k|$.

Let us now define the Green's function $\mathfrak{G}(t,x,k;z,l)$ such that

$$a_+(t,x,k) = \int_{\mathbb{R}^{2d}}\mathfrak{G}(t,x,k;z,l)a_+(0,z,l)dzdl. \tag{26}$$

We verify that $\mathfrak{G}(t,x,k;z,l)$ solves the equation

$$\begin{aligned} &\frac{\partial\mathfrak{G}}{\partial t} + c_0\hat{k}\cdot\nabla_x\mathfrak{G} + \Sigma(x,|k|)\mathfrak{G} = \int_{\mathbb{R}^d}\sigma(x,k,k')\mathfrak{G}(t,x,k';z,l)\delta(c_0|k| - c_0|k'|)dk' \\ &\mathfrak{G}(0,x,k;z,l) = \delta(x-z)\delta(k-l). \end{aligned} \tag{27}$$

Going back to (18), we find

$$\hat{C}_0(t,\omega,x,y;\tau) = \int_{\mathbb{R}^{3d}} e^{i(\tau\cdot l - y\cdot k)}\mathfrak{G}(t,x,k;z,l)a_0(z,l)\delta(\omega - c_0|k|)dzdldk. \tag{28}$$

For an active source problem, where $a_0(z,l)$ is the energy radiated away by the active source, the objective of imaging functionals would be to reconstruct $a_0(z,l)$ from knowledge of $\hat{C}_0(t,\omega,x,y;\tau)$ for given values of (t,x), of (y,τ), and for all ω over $\mathbb{R}_+$.

Change to angular variables. We introduce some simplified notation for the correlation that emphasizes the angular variables. Let ω be fixed. With

$$|l|^{d-1}\delta(l-k) = \delta(|l|-|k|)\delta(\hat{l}-\hat{k}),$$

we introduce

$$\mathfrak{G}(t,x,k;z,l) = |l|^{1-d}\delta(|l|-|k|)\mathfrak{G}(t,x,\hat{k};z,\hat{l};c_0|k|),$$

with $\mathfrak{G}(t,x,\hat{k};z,\hat{l};\omega)$ solving the transport equation

$$\begin{aligned} &\frac{\partial\mathfrak{G}}{\partial t} + c_0\hat{k}\cdot\nabla_x\mathfrak{G} + \Sigma(x,\tfrac{\omega}{c_0})\mathfrak{G} = \int_{S^{d-1}} \sigma(x,\hat{k},\hat{k}')\mathfrak{G}(t,x,\hat{k}';y,\hat{l};\omega)d\hat{k}', \\ &\mathfrak{G}(0,x,\hat{k};z,\hat{l};\omega) = \delta(x-y)\delta(\hat{k}-\hat{l}) \end{aligned} \tag{29}$$

where we have defined

$$\sigma(x,\theta,\theta';\omega) = \frac{\pi c_0(\frac{\omega}{c_0})^{d+1}}{2(2\pi)^d}\hat{R}(x,\frac{\omega}{c_0}(\theta-\theta')), \qquad \Sigma(x) = \int_{S^{d-1}}\sigma(x,\theta,\theta')d\theta'. \tag{30}$$

We use the same notation $\mathfrak{G}$ and σ for the coefficients written in terms of the angular variables. The new Green's function depends on ω since scattering does. We then write

$$\begin{aligned} &\int \mathfrak{G}(t,x,k;z,l)W_0(\omega,z,l)dzdl = \int \mathfrak{G}(t,x,k;z,l)\delta(\omega-c_0|l|)a_0(x,\frac{\omega}{c_0}\hat{l})dz|l|^{d-1}d|l|d\hat{l} \\ &= \delta(\omega-c_0|k|)\int \mathfrak{G}(t,x,\hat{k};z,\hat{l};\omega)a_0(x,\frac{\omega}{c_0}\hat{k})dzd\hat{l}. \end{aligned}$$

With this notation, we find that the limiting correlation is given by

$$\begin{aligned} \hat{C}_0(t,\omega,x,y;\tau) &= \int_{\mathbb{R}^{2d}\times S^{d-1}} e^{i\frac{\omega}{c_0}(\tau\cdot\hat{l}-y\cdot\hat{k})}\delta(\omega-c_0|k|)\mathfrak{G}(t,x,\hat{k};z,\hat{l};\omega)a_0(x,\frac{\omega}{c_0}\hat{k})dzd\hat{l}dk \\ &= \frac{\omega^{d-1}}{c_0^d}\int_{\mathbb{R}^d\times S^{d-1}\times S^{d-1}} e^{i\frac{\omega}{c_0}(\tau\cdot\hat{l}-y\cdot\hat{k})}\mathfrak{G}(t,x,\hat{k};z,\hat{l};\omega)a_0(x,\frac{\omega}{c_0}\hat{k})dzd\hat{l}d\hat{k}. \end{aligned}$$

This is the final result of this section. The correlation corresponding to various values of τ and y may be expressed as the Fourier transform of the initial conditions $a_0(x,\frac{\omega}{c_0}\hat{k})$ with frequencies ω propagated by the radiative transport kernel $\mathfrak{G}(t,x,\hat{k};z,\hat{l};\omega)$.

4 Correlation filtering and derivation of Result 1

We now derive Result 1 and show that the statistical correlations $\widehat{\delta C}$ vanish in the limit $\varepsilon\to 0$ for an appropriate choice of offsets y and τ. Furthermore, we investigate in section 4.2 different configurations in which such a choice can be made explicit.

4.1 Derivation of Result 1

The Green function $\mathfrak{G}$ is decomposed into $\mathfrak{G} := \mathfrak{G}_0 + \mathfrak{G}_s$, where $\mathfrak{G}_0(t,x,\theta;y,\hat{l};\omega)$ solves the equation

$$\begin{aligned} &\frac{\partial\mathfrak{G}_0}{\partial t} + c_0\theta\cdot\nabla_x\mathfrak{G}_0 + \Sigma(x,\tfrac{\omega}{c_0})\mathfrak{G}_0 = 0, \\ &\mathfrak{G}_0(0,x,\theta;y,\hat{l};\omega) = \delta(x-y)\delta(\theta-\hat{l}). \end{aligned} \tag{31}$$

In other words, $\mathfrak{G}_0$ is the ballistic component of $\mathfrak{G} := \mathfrak{G}_0 + \mathfrak{G}_s$, which is still attenuated because of scattering of coherent waves into incoherent signals. The coherent correlation verifies

$$\begin{aligned}\hat{C}_{00}(t,\omega,x,y) &:= \lim_{\varepsilon\to 0} \hat{C}_{\varepsilon,0}(t,\omega,x,y), \\ &= \frac{|\omega|^{d-1}}{c_0^d} \int_{S^{d-1}\times S^{d-1}\times\mathbb{R}^d} e^{i\frac{\omega}{c_0}(\tau\cdot\hat{l}-y\cdot\theta)} \mathfrak{G}_0(t,x,\theta;z,\hat{l};\omega) a_0(z,\frac{\omega}{c_0}\hat{l}) d\hat{l}d\theta dz. \quad (32)\end{aligned}$$

Indeed, the coherent part of the wavefield $\mathbb{E}\{\mathbf{w}\}$ asymptotically propagates in an effective medium with attenuation related to the power spectrum, see e.g. [18]. This implies that the limiting correlation C_{00} satisfies the decomposition (14) with a_+ solution to the transport equation

$$\frac{\partial a_+}{\partial t} + c_0\theta\cdot\nabla_x a_+ + \Sigma(x,\frac{\omega}{c_0})a_+ = 0,$$

augmented with the initial condition $a_+(t=0,x,\frac{\omega}{c_0}\theta) = e^{i\tau\frac{\omega}{c_0}\theta} a_0(x,\frac{\omega}{c_0}\theta)$, where a_0 is defined in (23). When Σ is independent of the spatial variable, we verify that

$$\mathfrak{G}_0(t,x,\theta;y,\hat{l};\omega) = e^{-\Sigma(\frac{\omega}{c_0})t}\delta(x-c_0t\theta-y)\delta(\theta-\hat{l}), \quad (33)$$

so that

$$\hat{C}_{00}(t,\omega,x,y;\tau) = \frac{|\omega|^{d-1}}{c_0^d} e^{-\Sigma(\frac{\omega}{c_0})t} \int_{S^{d-1}} e^{i|k|(\tau-y)\cdot\theta} a_0(x-c_0t\theta,\frac{\omega}{c_0}\theta)d\theta. \quad (34)$$

Thus, the statistical correlation may be decomposed as

$$\widehat{\delta C}(t,\omega,x,y;\tau) = \frac{|\omega|^{d-1}}{c_0^d} \int_{S^{d-1}\times S^{d-1}\times\mathbb{R}^d} e^{i\frac{\omega}{c_0}(\tau\cdot\hat{l}-y\cdot\theta)} \mathfrak{G}_s(t,x,\theta;z,\hat{l};\omega) a_0(z,\frac{\omega}{c_0}\hat{l}) d\hat{l}d\theta dz. \quad (35)$$

Under fairly generic hypotheses, the above correlation δC *vanishes* for a co-dimension one submanifold of values y,τ. The submanifold *strongly depends* on the frequency ω. This is the reason for the frequency selection obtained by Fourier transforming $C_\varepsilon(t,s,x,y)$ into $\hat{C}_\varepsilon(t,\omega,x,y)$. For such values of y,τ, we obtain that $\hat{C}_0 \equiv \hat{C}_{00}$, which propagates in a homogeneous medium as may be seen in (32). It is therefore those specific correlations that should be back-propagated in a homogeneous medium.

Let us assume that $a_0(y,\frac{\omega}{c_0}\hat{l}) = \delta(y-y_0)$, which corresponds to a localized source term radiating at frequency $\frac{1}{\varepsilon}\omega$ uniformly in all directions $\hat{l}$. Let us also assume to simplify the expressions that Σ is constant and that $\sigma = \sigma(x)$ is independent of directions θ and θ'. This is the case when $\hat{R}(q)$ is constant over a ball $|q| \le 2|k|$ for instance. The correlation is then given by

$$\widehat{\delta C}(t,\omega,x,y;\tau) = \frac{|k|^{d-1}}{c_0} \int_{S^{d-1}\times S^{d-1}} e^{i|k|(\tau\cdot l-y\cdot\theta)} \mathfrak{G}_s(t,x,\theta;y_0,l;\omega) dl d\theta. \quad (36)$$

Under such hypotheses, we verify that

$$e^{i|k|(\tau\cdot l-y\cdot\theta)} \mathfrak{G}_s(t,x,\theta;y_0,\hat{l};\omega),$$

is a continuous function in θ after integration in $\hat{l}$ and a continuous function in $\hat{l}$ after integration in θ. To fix notation, consider

$$I(\theta) = \int_{S^{d-1}} e^{i|k|\tau\cdot\hat{l}}\, \mathfrak{G}_s(t, x, \theta; y_0, \hat{l}; \omega) d\hat{l}.$$

This is a continuous function in θ on the unit sphere. We want to find values of y such that

$$\int_{S^{d-1}} I(\theta) e^{-i|k|y\cdot\theta} d\theta = 0. \tag{37}$$

There are an infinite number of $d-1$-dimensional shells of values of y such that the above occurs. When $I(\theta) \equiv 1$, the values of y are given by the zeros of

$$\mathfrak{J}_d(|k||y|) := \int_{S^{d-1}} e^{-i|k|y\cdot\theta} d\theta = 0. \tag{38}$$

We have $\mathfrak{J}_2(\mu) = J_0(\mu)$, the zeroth-order Bessel function of the first kind in dimension $d = 2$ and the sinc function $\mathfrak{J}_3(\mu) = \frac{\sin(\mu)}{\mu}$ in dimension $d = 3$. The latter zeros are given by $|k||y| = m\pi$, i.e., $|y| = m\frac{\lambda}{2}$ for $m \in \mathbb{N}^*$, where $\lambda = \frac{2\pi}{|k|} = \frac{2\pi c_0}{\omega}$. This shows that the choice of $y = y(\omega)$ very much depends on ω.

For more general functions $I(\theta)$, the shells may be obtained asymptotically for large values of $|y|$ by stationary phase. Note that the values of y and τ such that (37) holds depends on $\mathfrak{G}_s(t, x, k; y_0, l; \omega)$, i.e., on the measurement point (t, x) and on the geometry.

The argument carries over to more general initial conditions a_0 and power spectra $\hat{R}$, the main point being that, for (t, x, ω) fixed, the function

$$I(\theta) = \int_{S^{d-1}\times\mathbb{R}^d} e^{i\frac{\omega}{c_0}\tau\cdot\hat{l}}\, \mathfrak{G}_s(t, x, \theta; z, \hat{l}; \omega) a_0(z, \frac{\omega}{c_0}\hat{l}) d\hat{l} dz$$

is continuous in θ thanks to the double integration in $\hat{l}$ and z and the regularizing properties of the collision operator. We have thus obtained the existence of a subset $\mathfrak{Y}(t, x, \omega)$ of $\mathbb{R}^d \times \mathbb{R}^d$ such that (10) holds true.

4.2 Filtering of correlations in different regimes.

We provide now some examples in which a more explicit characterization of $\mathfrak{Y}$ can be obtained. Let us assume again that $a_0(y, \frac{\omega}{c_0}\hat{l}) = \delta(y - y_0)$.

Echo mode. Let us consider first the case of echo measurements, where $x = y_0$, and assume that the scattering medium is totally uniform, i.e., σ and Σ are constant independent of the spatial position. Then we observe that because of the invariance of the domain by rotation, $\mathfrak{G}_s(t, y_0, k; y_0, l)$ depends only on the angle between k and l. In other words,

$$\widehat{\delta C}(t, \omega, x, y; \tau) = \frac{|k|^{d-1}}{c_0} \int_{S^{d-1}\times S^{d-1}} e^{i|k|(\tau\cdot\hat{l} - y\cdot\theta)}\, \tilde{\mathfrak{G}}_s(t, \theta\cdot\hat{l}) d\hat{l} d\theta,$$

with $\tilde{\mathfrak{G}}_s(t,\theta\cdot\hat{l}) = \mathfrak{G}_s(t,y_0,\theta;y_0,\hat{l};\omega)$. We want to cancel the above relation independent of $\tilde{\mathfrak{G}}_s$. Let us decompose $\tilde{\mathfrak{G}}_s(\theta\cdot l)$ as a superposition of delta functions $\delta(\theta-\mathfrak{R}(l))$, where $\mathfrak{R}$ is a rotation in the group $SO(d)$. We want to impose that

$$\int_{S^{d-1}} e^{i(\tau\cdot\hat{l}-y\cdot\mathfrak{R}(\hat{l}))}d\hat{l} = \int_{S^{d-1}} e^{i(\mathfrak{R}(\tau)-y)\cdot\hat{l}}d\hat{l} = 0, \qquad \text{for all } \mathfrak{R}\in SO(d). \tag{39}$$

This implies that $|\mathfrak{R}(\tau)-y|$ is independent of $\mathfrak{R}$, i.e., that $\tau=0$ or $y=0$. The above relation is observed when $\tau=0$ and $\mathfrak{J}_d(|k||y|)=0$ and when $y=0$ and $\mathfrak{J}_d(|k||\tau|)=0$. When such a relation is verified, then $\widehat{\delta C}(t,\omega,x,y;\tau)$ vanishes simultaneously at *all points* (t,x). Note that the choice of τ and y strongly depends on ω.

Diffusive regime. In the diffusive regime [17, 22], the Green's function $\mathfrak{G}$ becomes independent of the angular variables θ and l so that $\widehat{\delta C}$ is proportional to

$$\mathfrak{J}_d(|k||\tau|)\,\mathfrak{J}_d(|k||y|) \;=\; \int_{S^{d-1}} e^{i|k|\tau\cdot l}dl \int_{S^{d-1}} e^{-i|k|\tau\cdot\theta}d\theta.$$

Any choice of τ or y that cancels one of the two above integrals will cancel $\widehat{\delta C}(t,\omega,x,y;\tau)$ for all points (t,x). In the diffusive regime, it is therefore relatively easy to cancel the correlations. The diffusive regime is valid when $\Sigma t \gg 1$. Since the ballistic signal decays like $e^{-\Sigma t}\ll 1$ as may be seen in (33), it is therefore extremely faint in the diffusive regime.

Single scattering regime. When σt is relatively small though not so small that propagation in a homogeneous medium may be used, we may approximate $\mathfrak{G}_s$ by considering only single scattering. We then obtain that

$$\mathfrak{G}_1(t,x,\theta;z,\hat{l};\omega) = \sigma e^{-\Sigma(\frac{\omega}{c_0})t}\int_0^t \delta(x-z-c_0(t-u)\theta-c_0u\hat{l})du. \tag{40}$$

This shows that the single scattering contribution to the correlation (with σ constant) is given, with $y_0=0$ to simplify, by

$$\widehat{\delta C_1}(t,\omega,x,y;\tau) \;=\; \frac{|k|^{d-1}}{c_0}\sigma e^{-\Sigma t}\int_{S^{d-1}\times S^{d-1}\times(0,t)} e^{i|k|(\tau\cdot\hat{l}-y\cdot\theta)}\delta(x-c_0(t-u)\theta-c_0u\hat{l})d\hat{l}d\theta du. \tag{41}$$

When $\tau=0$ or $y=0$, we retrieve that the above integral vanishes when $x=0$, which yields the result in the echo mode.

When $|x|>c_0t$ no information can propagate to the detectors and $\widehat{\delta C}\equiv 0$ independent of y and τ. When $|x|<c_0t$, we have [4]

$$u=u(\theta)=\frac{|x-c_0t\theta|^2}{2(c_0t\theta-x)\cdot c_0\theta},\qquad \hat{l}=\hat{l}(\theta)=\frac{x-c_0(t-s)\theta}{sc_0}.$$

Moreover, the change of variables dx to $d\theta ds$ yields a weight so that the above integral is equal to

$$\int_{S^{d-1}} \frac{2^{d-1}((x-c_0t\theta)\cdot\theta)^{d-3}}{|S^{d-1}||x-c_0t\theta|^{2d-4}}e^{i|k|(\tau\cdot l(\theta)-y\cdot\theta)}d\theta, \tag{42}$$

which is the Fourier transform of a (complicated) function restricted to the unit sphere. For each value of τ, we may thus find surfaces of values of y such that the above integral vanishes.

Such integrals are however rather complicated and the surfaces of y depend on the measurement point (t, x) in a non-trivial manner. Even though the cancellation of the single scattering effects are thus theoretically feasible, they involve that the zeros of non-explicit functions be evaluated.

5 Application to the imaging of sources

As an application of the result derived in the preceding section, we consider the reconstruction of sources buried in heterogeneous media. We assume we have access to the measurements $\hat{C}_\varepsilon(t, \omega, x, y; \tau)$, which are approximated by $\hat{C}_0(t, \omega, x, y; \tau)$ given in (28), for some values of (t, x, ω, τ) and $y \in D$. The objective is to reconstruct the initial condition $a_0(z, l)$.

Homogeneous medium. At first, we consider the simplified setting in which all correlations are assumed propagate in a homogeneous medium but that may still be absorbed. In the absence of scattering in the medium, the cross-section $\sigma \equiv 0$ and we assume to simplify that $\Sigma(x, |k|) = \Sigma(|k|)$. Since propagation occurs in a homogeneous medium, we observe that $\mathfrak{G}(t, x, k; z, l)$ is equal to

$$\mathfrak{G}_0(t, x, k; z, l) = e^{-\Sigma(|k|)t}\delta(k - l)\delta(x - tc_0\hat{k} - z). \tag{43}$$

In such a context, we find that

$$e^{\Sigma(\frac{\omega}{c_0})t}\hat{C}_0(t, \omega, x, y; \tau) = \int_{\mathbb{R}^d} e^{i(\tau - y)\cdot k} W_0(\omega, x - tc_0\hat{k}, k)dk := R_t W_0(\omega, x, y),$$

where W_0 is defined in (25). Let us assume that $\tau = 0$ and that we measure $\hat{C}_0(t, \omega, x, y; 0)$ at a fixed t, for all $x \in \mathbb{R}^d$ to simplify the presentation, at $y \in D$, and for all $\omega \in \mathbb{R}^+$. The imaging of the source term from knowledge of $\hat{C}_0(t, \omega, x, y; 0)$ is then performed by applying the adjoint operator R_t^* to the data, or equivalently, by back-propagating the available measurements in the homogeneous medium, to get

$$\begin{aligned} J(x, k) &= R_t^*(e^{\Sigma(\frac{\omega}{c_0})t}\hat{C}_0)(x, k) = \int_{\mathbb{R}^+}\int_D e^{iy\cdot k} e^{\Sigma(\frac{\omega}{c_0})t}\hat{C}_0(t, \omega, x + c_0 t\hat{k}, y)d\omega dy \\ &= \int_{\mathbb{R}^+}\int_D\int_{\mathbb{R}^d} e^{iy\cdot(k-q)} W_0(\omega, x + c_0 t(\hat{k} - \hat{q}), q)d\omega dy dq \\ &= \int_{\mathbb{R}^d}\Big(\int_D e^{iy\cdot(k-q)}dy\Big) a_0(x + c_0 t(\hat{k} - \hat{q}), q)dq. \end{aligned} \tag{44}$$

When $D = \mathbb{R}^d$ so that all correlations are measured and all propagate in a non-scattering medium, then we find that

$$J(x, k) = (2\pi)^d a_0(x, k), \tag{45}$$

so that the reconstruction is exact after proper rescaling. When $D = \{0\}$ so that correlations are measured only for $y = 0$, which corresponds to measurements of the energy density, we have instead:

$$J(x,k) = \int_{\mathbb{R}^d} a_0(x + c_0 t(\hat{k} - \hat{q}), q) dq.$$

At $t = 0$, we retrieve the correct energy $\mathcal{E}(x)$ of the initial condition for all values of k.

Let us revisit the above derivation and write

$$\begin{aligned} e^{\Sigma(\frac{\omega}{c_0})t} \hat{C}_0(t,\omega,x,y) &= \int_{\mathbb{R}^d} e^{-iy\cdot k} W_0(\omega, x - tc_0\hat{k}, k) dk \\ &= \frac{\omega^{d-1}}{c_0^d} \int_{S^{d-1}} e^{-i\frac{\omega}{c_0} y\cdot\hat{k}} a_0(x - tc_0\hat{k}, \frac{\omega}{c_0}\hat{k}) d\hat{k} := \frac{\omega^{d-1}}{c_0^d} R_{t,\omega} a_0(t,\omega,x,y), \end{aligned}$$

where we now assume that ω is *fixed*. We may then obtain an image by back-propagating the above measurements at frequency ω, which corresponds to applying the operator $R^*_{t,\omega}$ to the above data, and obtain

$$\begin{aligned} I(x, \tfrac{\omega}{c_0}\hat{k}) &= e^{\Sigma(\frac{\omega}{c_0})t} \int_D \hat{C}_0(t,\omega,x + c_0 t\hat{k}, y) dy \\ &= \frac{\omega^{d-1}}{c_0^d} \int_D \int_{S^{d-1}} e^{i\frac{\omega}{c_0} y\cdot(\hat{k}-\hat{q})} a_0(x + tc_0(\hat{k} - \hat{q}), \frac{\omega}{c_0}\hat{q}) d\hat{q}. \end{aligned} \tag{46}$$

When $D = \mathbb{R}^d$, we recover that $I(x,k) = (2\pi)^d a_0(x,k)$ provided that ω is known on the frequency support of $a_0(x,k)$. When D is a smaller domain than $\mathbb{R}^d$, we observe that $I(x, \frac{\omega}{c_0}\hat{k})$ is no longer proportional to $a(x, \frac{\omega}{c_0}\hat{k})$. Rather, the source term is blurred by the kernel in (46), which becomes a Dirac distribution when $D = \mathbb{R}^d$. Hence, we recover the classical property that in a homogeneous medium, the larger the domain D, the more accurate the reconstruction will be.

Heterogeneous medium. When is the medium is heterogeneous, the situation is more complicated. Correlations for most values of the shift y do not propagate in a homogeneous medium, and back-propagating them with the operator $R_{t,\omega}$ generates errors (in the form of a bias) in the image. In such a setting, one would like only to back-propagate the coherent correlations obtained for the set $D = \mathfrak{Y}(t,x,\omega)$ such that $\hat{C}_0(t,\omega,x,y;\tau) \equiv \hat{C}_{00}(t,\omega,x,y;\tau)$, or at least those correlations for which scattering is minimal. Since $\mathfrak{Y}(t,x,\omega) \neq \mathbb{R}^d$ in general, the resulting image will be blurred by a kernel similar to the one obtained in (46). In order to understand the properties of this blurring, we assume that $a_0(x,k)$ is independent of $\hat{k}$ and that $\mathfrak{Y}(t,x,\omega)$ is independent of (t,x) as in the echo mode or the diffusion regime. We then average $I(x, \frac{\omega}{c_0}\hat{k})$ over S^{d-1} as well. We may then pass to the Fourier domain $x \to \xi$ and observe that

$$\hat{I}(\xi, \frac{\omega}{c_0}) := \int_{S^{d-1}} \hat{I}(\xi, \frac{\omega}{c_0}\hat{k}) d\hat{k} = \frac{\omega^{d-1}}{c_0^d} \hat{a}_0(\xi, \frac{\omega}{c_0}) M_\omega(\xi). \tag{47}$$

where we have defined

$$M_\omega(\xi) = \int_{S^{d-1}\times S^{d-1}\times\mathfrak{Y}(\omega)} e^{i(\frac{\omega}{c_0}y + t\xi)(\hat{k}-\hat{q})} d\hat{k} d\hat{q} dy = \int_{\mathfrak{Y}(\omega)} \Big|\mathfrak{J}_d\Big(|t\xi + \frac{\omega}{c_0}y|\Big)\Big|^2 dy. \tag{48}$$

Here $\mathfrak{J}_d$ is defined in (38). The above multiplier is always non-negative after integration in $d\hat{k}$ which makes $M(\xi)$ the symbol of a symmetric operator of the form R^*R. However, we verify that $M(\xi)$ may vanish. Some frequencies of the source term are therefore lost in the data.

When $D = \{0\}$, which corresponds to selecting only $y = 0$, we obtain that $M(\xi)$ is proportional to $\mathfrak{J}_d(|t\xi|)^2$, which vanishes for ξ on a countable number of shells. Integrating over several values of t when measurements are available for several values of t then makes the Fourier multiplier positive at all frequencies. Note, however, that the Fourier multiplied $M(\xi)$ decays with ξ like $|\xi|^{1-d}$. In other words, the image that is being reconstructed is roughly $d-1$ times smoother than the real image $a(x, \frac{\omega}{c_0}\hat{k})$. Note that when $\mathfrak{Y}(\omega)$ is such that $\mathfrak{J}_d(\frac{\omega}{c_0}y) = 0$, then $M(0) = 0$ independent of t. Therefore, an inversion based on choosing $y(\omega)$ as indicated above annihilates the average of the source term.

The outcome of this simple example is that filtering the incoherent correlations brings a blurring to the image when back-propagating only the coherent correlations. Such a blurred image is to be compared with the one obtained by propagating all correlations, which is a superposition of a perfect reconstruction and errors due to the back-propagation of the incoherent correlations. The amount of noise is proportional to the strength of scattering. We therefore expect that there exists a threshold for the scattering intensity under which all correlations should be back-propagated, and above which only the coherent ones should be used.

6 Numerical simulations

This section presents numerical simulations that validate the theoretical results on coherent correlations presented earlier in the paper. We show that it is indeed possible to cancel the incoherent component of the correlations by appropriately choosing the shifts. To demonstrate this, we simulate the propagation of mono-frequency waves in a random medium over times that are long enough for the diffusion approximation to be accurate. We then compare the numerical simulations with the theoretical predictions.

In the simulations, we discretize the first-order hyperbolic system for pressure $p(t, x)$ and velocity $v(t, x)$ augmented with suitable initial conditions and surrounded by a perfectly matched layer [11] using a standard finite difference scheme of second order. We assume that $\rho_0 = 1$, $\kappa_0 = 1$ and denote by R the two-point correlation function $R(y) = \mathbb{E}\{\kappa_1(x+y)\kappa_1(x)\}$, where κ_1 is defined in (3). The average sound speed in thus $c_0 = 1$. The fluctuations of the compressibility $\kappa_1(y)$ have been carefully modeled to verify prescribed power spectra as in [7]. When we say that a medium has fluctuations of order $x\%$, we refer to the standard deviation of κ_1 (with respect to $\kappa_0 \equiv 1$). The initial condition is of the form

$$\mathbf{u}_0(x) = \Big(\mathbf{0}, \exp\Big(-\frac{|x-x_0|^2}{2\sigma^2}\Big)\, J_0(k_0|x-x_0|)\Big) \tag{49}$$

where J_0 is the zero-th order Bessel function of the first kind. The exponential term is chosen here to localize the source term. However, it has sufficiently slow variations in order not to interfere with the highly oscillatory Bessel function. Here, σ is chosen to be on the order of ten wavelengths so that the frequency content of $\mathbf{u}_0$ is primarily that of a single wavenumber k_0. We set the frequency ν_0 equal to one, so that the wavelength $\lambda_0 = \frac{c_0}{\nu_0}$ is also equal to one and $k_0 = 2\pi$. The corresponding initial condition at the transport level is

$$a_0(x,k) = \delta(x - x_0)\delta(|k| - k_0)k_0^{-1}. \tag{50}$$

The random medium fluctuations are set to 5%. For the chosen frequency ν_0, the corresponding mean free path is approximately $100\lambda_0$. The initial condition is localized at the center of a domain of size $200\lambda_0 \times 200\lambda_0$, so that $x_0 = (100, 100)$, see figure 1. We use 30 points per wavelength for the simulations.

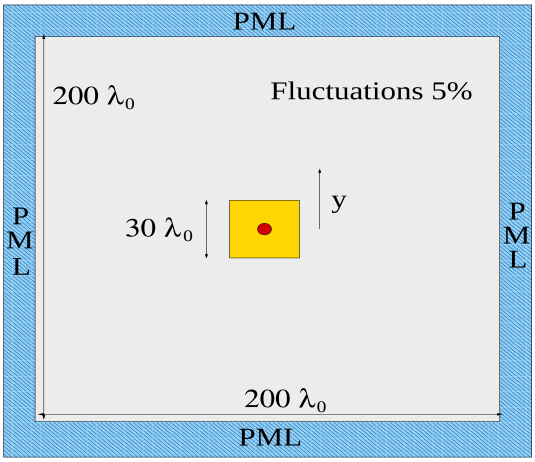

Figure 1: Setting for computing the spatial correlations

In the diffusive limit, the amplitude $a_+(t,x,k)$ is independent of the direction, so that, together with the fact that the initial condition is concentrated around wavenumbers with modulus k_0, we have approximately

$$a_+(t,x,k) \simeq a_+(t,x,|k|) \simeq U(t,x,k_0)\delta(|k| - k_0),$$

for some function U solution to a diffusion equation. The correlation $C_0(t,s,x,y)$ defined in (17) can therefore be written as

$$C_0(t,s,x,y) = \frac{k_0}{c_0} U(t,x,k_0)\, \Re\, e^{ic_0 k_0 s} \int_{S^1} e^{-iy\cdot\hat{k}k_0} d\hat{k},$$

so that

$$C_0(t,0,x,y) = \frac{2\pi k_0}{c_0} U(t,x,k_0) J_0(|y|k_0).$$

We then compute the spatial correlations $C_\varepsilon(t,0,x,y)$ given by (5). Let ϕ be a test function in the spatial variables. According to the discussion of section 2, we expect the following to hold formally: for all $t > 0$ and $y \in \mathbb{R}^d$,

$$\begin{aligned}\int_{\mathbb{R}^d} C_\varepsilon(t,0,x,y)\phi(x)dx &\simeq \int_{\mathbb{R}^d} \mathbb{E}\{C_\varepsilon\}(t,0,x,y)\phi(x)dx,\\ &\simeq \frac{2\pi k_0}{c_0} J_0(|y|k_0) \int_{\mathbb{R}^d} U(t,x,k_0)\phi(x)dx.\end{aligned}$$

For ϕ the characteristic function of a detector D, we finally compare

$$I(t,y) := \frac{\displaystyle\int_D C_\varepsilon(t,0,x,y)dx}{\displaystyle\int_D C_\varepsilon(t,0,x,0)dx}, \qquad \text{and} \qquad J(|y|) := J_0(|y|k_0) = J_0(2\pi|y|\nu_0 c_0^{-1}).$$

As previously noted in [7], the theoretical sound speed $c_0 = 1$ is not sufficiently accurate for discrete waves due to dispersion effects, and we thus set it to $c_0 = 0.95$ as in [7]. We compute the correlations in the vertical direction as shown in figure 1, for a square detector D of size a, with $a = 30\lambda_0, 10\lambda_0, 5\lambda_0$. Correlations calculated along the horizontal direction yield equivalent results. Results are displayed on figure 2, and we denote by Σ^{-1} the mean free time, approximately equal to 100 in dimensionless units.

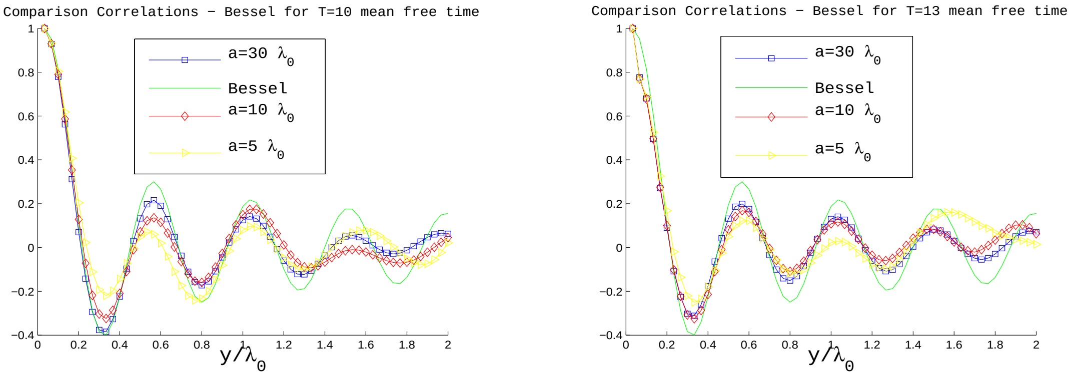

Figure 2: Comparison $I(t,y)$ and $J(y)$ for $t = 10\Sigma^{-1}$ (left) and $t = 13\Sigma^{-1}$ (right)

Results obtained for $t = 10\Sigma^{-1}$ and $t = 13\Sigma^{-1}$ are comparable since the diffusive regime is already a good approximation for wave propagation when $t = 10\Sigma^{-1}$. Moreover, correlations averaged over a larger domain better fit the theoretical prediction given by J. The qualitative agreement between numerics and theory obtained for $a = 5\lambda_0$ is satisfactory. The discrepancy observed between the exact solution and the computed solution can be explained by the following factors: the non-zero value of the wavelength, here $\varepsilon = \frac{1}{200}$, which introduces an error between the high frequency wave solution and its transport approximation; the error between the transport solution and its diffusion approximation; and the statistical instabilities that are not entirely smoothed out by the spatial averaging. As the ratio $\frac{y}{\lambda_0}$ increases, these error mechanisms become stronger because of the statistical instabilities as can clearly be seen in figure 3. The observed oscillations nevertheless corroborate the theoretical predictions of cancellations of the correlations.

The numerical simulations confirm there existence of an ensemble of offsets for which the incoherent correlations vanish. In the diffusion regime and when the offsets are not too large, there is a good agreement with the theory. For large offsets, statistical instabilities become stronger, and averaging on even larger domains would be necessary to match the theoretical predictions. This observation is very damaging in imaging of small inclusions as spatial averaging clearly lowers resolution.

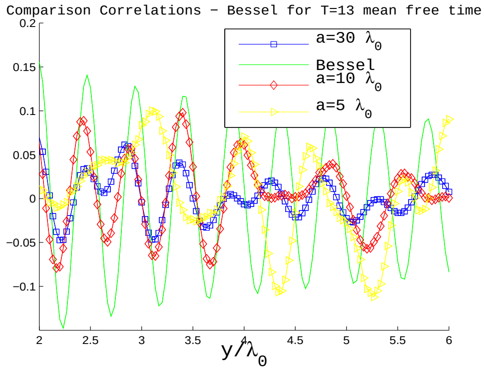

Figure 3: Comparison $I(t, y)$ and $J(y)$ for $t = 13\Sigma^{-1}$ and $y \geq 2\lambda_0$.

7 Conclusion

Field-field correlations offer an important tool in the imaging and reconstructions of inclusions buried in unknown heterogeneous environments. The reason is that these correlations are statistically stable, at least when they are averaged over sufficiently large arrays. Correlations-based imaging is the backbone of Coherent Interferometry (CINT) [1, 12, 13, 14] in weakly scattering environments and Transport-based imaging procedures [3, 7, 8, 9, 19].

In this work, we have shown that a class of such well-chosen correlations called coherent correlations propagated in a homogeneous medium. The coherent correlations are still attenuated during the propagation and are therefore affected by the underlying heterogeneities. However, they do not undergo any scattering. Such correlations are characterized by shifts (y, τ) that depend on the statistical properties of the underlying heterogeneous medium. Knowledge of the statistics of the random fluctuations is therefore necessary to construct the coherent correlations. In the diffusive regime of wave propagation, however, we have observed that the shifts depended only on the frequency of the propagating and not on the statistical properties of the underlying medium.

In an imaging context, such coherent correlations could therefore be used in models based on back-propagation of information in a homogeneous medium. When incoherent correlations are back-propagated, errors in the imaging functional are introduced unless scattering effects are accounted for (as is the case in the more complex transport-based imaging procedures recalled above). We have also recalled in a simple imaging setting that back-propagating selected correlations created some blurring in the reconstructions. Ideally, in the setting of back-propagation in a homogeneous medium, one would like to back-propagate as many correlations as possible that are not too strongly affected by scattering. Correlations undergoing multiple scattering generate some blurring when they are not back-propagated and some bias in the reconstruction when they are back-propagated. A more extensive characterization of the set of shifts (y, τ) would be useful to assess which correlations should be used in the back-propagation or not.

The studies in this paper assume that the correlations are *sufficiently* statistically stable so that $C_\varepsilon - \mathbb{E}\{C_\varepsilon\}$ is negligible. In the setting of the reconstruction of small inclusions, the main limiting factor to obtaining a good spatial resolution is precisely that statistical fluctuations of the form $C_\varepsilon - \mathbb{E}\{C_\varepsilon\}$ are no longer necessarily negligible;

see, e.g.,[1]. In addition to back-propagating coherent correlations and filtering out incoherent correlations strongly affected by scattering, optimal imaging strategies need to characterize statistical fluctuations of the form $C_\varepsilon - \mathbb{E}\{C_\varepsilon\}$, which is a difficult problem that was not addressed in this paper.

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