Convergence to Homogenized or Stochastic Partial Differential Equations

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Abstract

We consider the behavior of solutions to parabolic equations with large, highly oscillatory, possibly time dependent, random potential with Gaussian statistics. The Gaussian potential fluctuates in the spatial variables and possibly in the temporal variable. We seek the limit of the solution to the parabolic equation as the scale ε at which the random medium oscillates converges to zero. Depending on spatial dimension and on the decorrelation properties of the Gaussian potential, we show that the solution converges, as ε tends to 0, either to the solution of a deterministic, homogenized, equation with negative effective medium potential or to the solution of a stochastic partial differential equation with multiplicative noise that should be interpreted as a Stratonovich integral. The transition between the deterministic and stochastic limits depends on the elliptic operator in the parabolic equation and on the decorrelation properties of the random potential. In the setting of convergence to a deterministic solution, we characterize the random corrector, which asymptotically captures the stochasticity in the solution. Such models can be used to calibrate upscaling schemes that aim at understanding the influence of microscopic structures in macroscopic calculations.

1 Introduction

Small scale structures abound in all areas of applied science. Because their microscopic description is often unavailable, and when it is available, generates prohibitively expensive computations in practice, there is considerable interest in understanding the influence of the micro-scale structures at the macroscopic level. This allows us to "remove" the micro-structure in a consistent manner and handle macroscopic objects that are physically and computationally much more tractable. Mathematically, the consistent removal of the micro-scale is typically achieved by introducing a small scale $\varepsilon \ll 1$ for the micro-structure, and by obtaining a limit u to the solution u_ε of an equation describing the micro-structure as the scale $\varepsilon \to 0$.

Arguably the most useful method to obtain macroscopic equations for the limit u is the homogenization methodology. Under appropriate assumptions of stationarity and ergodicity of the micro-structure modeled as a random field, homogenization theory provides a description for u in the form of a deterministic, effective medium, equation; see e.g. [6, 8, 12, 16, 18, 22] for references on homogenization in random and periodic media. Another fruitful macroscopic description of micro-structure is the introduction of stochastic forcing in what results as a stochastic partial differential equation (SPDE) model for u; see e.g. [13, 17, 19, 21, 27] for a few references on the topic.

We are concerned here with the derivation of (deterministic) homogenized or stochastic models as the limit of solutions to parabolic equations perturbed by a large, highly oscillatory potential. Our model takes the form

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} + P(x,D)u_\varepsilon - \frac{1}{\varepsilon^\alpha} q\Big(\frac{t}{\varepsilon^\beta}, \frac{x}{\varepsilon}\Big) u_\varepsilon = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^d \\ &u_\varepsilon(0,x) = g(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{1}$$

where $d \geq 1$ is spatial dimension, $P(x,D)$ is an elliptic operator, and $q(t,x)$ is a stationary Gaussian field. The typical operator $P(x,D)$ we are interested in is the Laplacian $P(x,D) = -\Delta$ and powers of the Laplacian $P(x,D) = (-\Delta)^{\frac{\mathfrak{m}}{2}}$ for some $\mathfrak{m} > 0$. We assume that the initial condition $g(x)$ is sufficiently smooth (see (48) below for a sufficient condition of smoothness).

The potential $q(t,x)$ may or may not depend on time, and when it does depend on time, is assumed to oscillate at the scale ε^β with $0 < \beta \leq \mathfrak{m}$, where $\mathfrak{m}$ is the order of the principal symbol of $P(x,D)$. We are thus interested in the scaling where the fluctuations in space dominate the randomness in the potential. The potential is assumed to be large. The scaling factor α is chosen so that the potential in (1) has an order $O(1)$ effect. In other words, so that $u_\varepsilon \not\to u_{\rm un}$ as $\varepsilon \to 0$, where $u_{\rm un}$ is the unperturbed solution to (1) with q set to 0.

Once α is properly chosen, we aim at understanding the limiting *stochasticity* of u_ε as $\varepsilon \to 0$. More precisely, we wish to answer the following questions: (i) Does u_ε converge to the solution of a deterministic homogenized equation or to the solution of a stochastic PDE? (ii) When the former occurs, what is the structure of the random corrector to the homogenized solution? We shall see in section 2 that the answer depends on the relationship between dimension d, the strength $\mathfrak{m}$ of the elliptic operator, and the decorrelation properties of the Gaussian potential q.

For dimensions $d < d_{\rm cr}(\mathfrak{m}, q)$, where $d_{\rm cr}(\mathfrak{m}, q)$ is a critical dimension that depends on $\mathfrak{m}$ and q, we observe convergence to the solution of a SPDE with multiplicative noise. The multiplicative noise contribution should be understood as a Stratonovich integral with respect to an appropriate fractional Brownian motion. When $\mathfrak{m} = 2$ and q is time independent and has integrable correlation function, then $d_{\rm cr}(\mathfrak{m}, q) = 2$. In this setting, convergence to a stochastic equation occurs only when $d = 1$. This was confirmed for instance in [23] for not-necessarily-Gaussian, mixing potentials, and in [3] for Gaussian potentials.

When $d \geq d_{\rm cr}(\mathfrak{m}, q)$, we observe a transition to a deterministic limit, solution of a homogenized equation with negative effective medium potential. In some sense, it is easier for the random solution to visit the whole space of randomness in high spatial dimensions than in low spatial dimensions. As a consequence, averaging takes place more efficiently and a deterministic homogenized equation arises in the limit $\varepsilon \to 0$.

In the latter configuration, randomness “disappears” from the leading term in the macroscopic limit. In many applications, it is useful to quantify the uncertainty in the solution and thus control the size of the random fluctuations about the deterministic limit. Much less is known about the structure of the fluctuations in homogenization in random media than in homogenization in periodic media. Explicit expressions for the correctors are available in simplified configurations [2, 5, 7, 10] but no theory is available in the general setting for which homogenization theory applies. When $d \geq d_{\rm cr}(\mathfrak{m}, q)$, we can characterize for the solution to (1) the size of $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$ and obtain that after appropriate rescaling, it converges in distribution to a Gaussian random field. Such a convergence result is consistent with the central limit correction to the law of large numbers.

The above model (1) may be seen as a (generalization to arbitrary strength $\mathfrak{m}$ of the) continuous version of the parabolic Anderson model. The parabolic Anderson model is related to the analysis of localization of waves in random media. In one dimension of space, waves localize as soon as the underlying medium exhibits some disorder; this is known as Anderson localization. In higher spatial dimensions, no such phenomenon occurs, at least for sufficiently small disorder. Instead, we observe that the energy density of the waves converges to a homogenized, deterministic limit, solution of a radiative transport equation [1, 9, 11, 20, 24, 25, 26]. A similar behavior, albeit in a simpler context, is observed here. We obtain a stochastic limit, as is the case in Anderson localization, for low dimension, and a deterministic limit for large dimensions. One of the main results obtain below is that for potential with very long range correlations, the limit is stochastic even in large spatial dimensions and this for all values of $\mathfrak{m}$ including the practically interesting case $\mathfrak{m} = 2$ corresponding to the Laplacian.

Stochastic models may also be defined in the range of parameters where a homogenized limit arises. For instance, stochastic PDE with multiplicative white noise in space and space-time may be defined for the Laplacian $\mathfrak{m} = 2$ in dimensions $d \geq 2$. In such a framework however, the equations cannot be defined in the Stratonovich form but should rather be defined using a Wick-Skorohod integral (which may be seen as a generalization of the Itô integral defined for non-anticipative processes). The Skorohod integrals are the right tools to remove (renormalize) infinite terms that would arise otherwise. Their physical justification and interpretation is however somewhat more difficult and their solutions are singular distributions; see e.g. [13, 14, 15, 19]. The stochastic models presented here all have solutions that are square-integrable functions (in the probability space in which randomness is defined). Moreover, they display the feature that they can be obtained as limits of solutions of equations with equations with random coefficients that oscillate at large but finite frequencies.

The rest of the paper is structured as follows. The main assumptions and main results of the paper are stated in section 2. The derivation of the results relies on earlier work available in [3, 4]. The reader is referred to the latter references for many details of the derivation of the homogenized and stochastic equations. The derivation of the homogenized limit is presented in section 3. The analysis of the SPDE in Stratonovich form and the convergence of u_ε to its solution is carried out in section 4.

2 Main results

Potential and power spectrum. The potential $q(t,x)$ in (1) is a stationary Gaussian process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}q = 0$, where $\mathbb{E}$ is mathematical expectation with respect to the measure $\mathbb{P}$: $\mathbb{E}f = \int_\Omega f(\omega) d\mathbb{P}(\omega)$. The

Gaussian process is then uniquely characterized by its correlation function

$$R(t,x) = \mathbb{E}\{q(s,y)q(s+t,y+x)\}. \tag{2}$$

We denote by $\hat{R}(t,\xi)$ the Fourier transform of $R(t,x)$ with respect to the second variable and normalized such that

$$(2\pi)^d \hat{R}(t,\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} R(t,x) dx. \tag{3}$$

The limiting behavior of the solution in (1) depends on the long range correlations of the potential in both space and time. We consider two types of correlation functions. In the first model corresponding to long range correlations in time, we assume that

$$R(t,x) \sim \frac{\tilde{R}(x)}{t^{\mathfrak{b}}} \quad \text{as } |t| \to \infty \text{ and } \begin{cases} \text{either} & \tilde{R}(x) \sim \dfrac{\kappa}{|x|^{\mathfrak{p}}} \quad \text{as } |x| \to \infty \ 0 < \mathfrak{p} < d, \\ \text{or} & \displaystyle\int_{\mathbb{R}^d} \tilde{R}(x) dx < \infty, \end{cases} \tag{4}$$

This translates in the Fourier domain as

$$\hat{R}(t,\xi) = \frac{\hat{S}(t,\xi)}{t^{\mathfrak{b}}|\xi|^{\mathfrak{n}}}, \qquad \mathfrak{n} = (d-\mathfrak{p}) \wedge 0, \tag{5}$$

where $\hat{S}(t,\xi)$ is a bounded function integrable in the ξ variable and converging to a bounded and integrable function $\hat{S}_\infty(\xi)$ as $t \to \infty$. We use the notation $a \wedge b = \min(a,b)$. The case of time-independent potential can formally be modeled by choosing $\mathfrak{b} = 0$ and $\hat{S}(\xi)$ independent of time.

In the second model corresponding to short range correlations in time, we assume that

$$\begin{cases} \text{either} & R(t,x) \sim \dfrac{\kappa(t)}{|x|^{\mathfrak{p}}}, \quad \text{as } |x| \to \infty \text{ for } 0 < \mathfrak{p} < d \text{ with } \kappa(t) \text{ integrable}, \\ \text{or} & \displaystyle\int_{\mathbb{R}^{d+1}} R(t,x) dx dt < \infty. \end{cases} \tag{6}$$

This translates in the Fourier domain as

$$\hat{R}(t,\xi) = \frac{\hat{S}(t,\xi)}{|\xi|^{\mathfrak{n}}}, \qquad \mathfrak{n} = (d-\mathfrak{p}) \wedge 0, \tag{7}$$

where $\hat{S}(t,\xi)$ is a bounded function integrable in both the ξ and the t variables.

In both models, the case $0 < \mathfrak{p} < d$ and $0 < \mathfrak{n} = d - \mathfrak{p} < d$ corresponds to spatial long range correlations, whereas the case of an integrable correlation function in time is described by $\mathfrak{n} = 0$.

Duhamel expansion and solution to (1)**.** The analysis of equation (1) is rendered more complicated by the fact that the Gaussian potential $q_\varepsilon(t,x) := \frac{1}{\varepsilon^\alpha} q(\frac{t}{\varepsilon^\beta}, \frac{x}{\varepsilon})$ is unbounded. The theory of existence and uniqueness of a solution to (1) would be greatly simplified if the spatial domain were bounded and the equation augmented with, say, Dirichlet condition at the domain's boundary. The reason is that with probability one, the Gaussian potential would be bounded on the bounded domain for all times $0 \le t \le T$. Classical theories would then ensure existence of a unique solution to the equation with random potential. Here, we assume that the spatial domain is $\mathbb{R}^d$ to simplify the convergence analysis. We construct a solution $u_\varepsilon(t)$ by Duhamel expansion, which we prove is in $L^2(\Omega \times \mathbb{R}^d)$ for sufficiently small times. We do not address the question of uniqueness, which could be addressed using a setting similar to the one described in [3].

The Duhamel solution is constructed as follows. We formally replace (1) by

$$u_\varepsilon(t,x) = \mathcal{G}_t g(x) + \mathcal{H}_\varepsilon u_\varepsilon(t,x), \qquad \mathcal{H}_\varepsilon u_\varepsilon(t,x) := \int_0^t \mathcal{G}_{t-s} q_\varepsilon u_\varepsilon(s,x) ds, \tag{8}$$

where $\mathcal{G}_t = e^{-tP(x,D)}$ is the unperturbed propagator, solution operator of (1) with q set to 0. The Duhamel solution is then defined as

$$u_\varepsilon(t,x) = \sum_{n\ge 0} u_{\varepsilon,n}(t,x), \qquad u_{\varepsilon,n}(t,x) = \mathcal{H}_\varepsilon^n[\mathcal{G}_t g](t,x). \tag{9}$$

We show that $u_\varepsilon(t) \in L^2(\Omega \times \mathbb{R}^d)$, at least for sufficiently small times $0 \le t \le T$, where T depends on the statistics of the random potential $q(t,x)$; see (49) below.

Convergence to a homogenized solution. Convergence to a homogenized limit is obtained by analyzing (1) in the Fourier domain. We thus restrict ourselves to the case where $P(x,D) = (-\Delta)^{\frac{\mathfrak{m}}{2}}$.

In the cases where the limit u of u_ε as $\varepsilon \to 0$ is deterministic, it is the solution of an equation of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} + (-\Delta)^{\frac{\mathfrak{m}}{2}} u - \rho u = 0 \qquad t \ge 0, \quad x \in \mathbb{R}^d \\ &u(0,x) = g(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{10}$$

where $\rho = \rho(d,\mathfrak{m},\mathfrak{n},\mathfrak{b},\beta)$ is a positive constant that depends on the structure of the power spectrum $\hat{R}(t,\xi)$ and may be expressed as the following limit:

$$\rho = \lim_{\varepsilon\to 0} \varepsilon^{d-2\alpha} \int_0^T \int_{\mathbb{R}^d} e^{-t|\xi|^{\mathfrak{m}}} \hat{R}\Big(\frac{t}{\varepsilon^\beta}, \varepsilon\xi\Big) d\xi dt. \tag{11}$$

The random corrector $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$ typically significantly differs from $u_\varepsilon - u$. The reason is that u is the leading contribution to the deterministic component $\mathbb{E}\{u_\varepsilon\}$ of u_ε. There are however many deterministic corrections that can potentially be larger than $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$. It is the latter term we are interested in since it asymptotically captures the stochasticity part of u_ε. In the situations considered here, the corrector has the following structure. The size of the corrector is ε^γ for a properly chosen coefficient $\gamma = \gamma(d,\mathfrak{m},\mathfrak{n},\mathfrak{b},\beta)$. After rescaling, the corrector converges weakly in space and in distribution to the solution of the following stochastic equation with additive noise

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + (-\Delta)^{\frac{\mathfrak{m}}{2}} u_1 - \rho u_1 = \sigma u \dot{W} \qquad t \ge 0, \quad x \in \mathbb{R}^d \\ &u_1(0,x) = 0, \qquad x \in \mathbb{R}^d, \end{aligned} \tag{12}$$

for some $\sigma = \sigma(d, \mathfrak{m}, \mathfrak{n}, \mathfrak{b}, \beta)$ and some mean zero Gaussian field $\dot{W}(t,x)$ whose statistics depend on $(\mathfrak{m}, \mathfrak{n}, \mathfrak{b})$.

For the first model of power spectrum corresponding to long-range correlations in time, we have the following result:

Theorem 2.1 *Let u_ε be the solution given by* (9) *for a Gaussian potential with statistics given by* (5). *Let us assume that*

$$(1-\mathfrak{b})\mathfrak{m} + \mathfrak{n} < d \qquad \text{and} \qquad \alpha = \frac{\mathfrak{m}}{2} - \frac{\mathfrak{b}(\mathfrak{m}-\beta)}{2} = \frac{(1-\mathfrak{b})\mathfrak{m}}{2} + \frac{\beta\mathfrak{b}}{2}. \tag{13}$$

Then $u_\varepsilon(t)$ converges strongly in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in time for $0 \le t \le T < \frac{1}{4C_\eta}$ to $u(t,x)$ solution of (10) *with homogenized potential given by*

$$\rho(\mathfrak{m}, \mathfrak{n}, \mathfrak{b}, \beta) = \int_0^\infty \int_{\mathbb{R}^d} e^{-t|\xi|^{\mathfrak{m}}} \frac{\hat{S}_\beta(t,\xi)}{t^{\mathfrak{b}}|\xi|^{\mathfrak{n}}} d\xi dt, \quad \hat{S}_\beta(t,\xi) = \left\{ \begin{array}{ll} \hat{S}(t,\xi) & \beta = \mathfrak{m}. \\ \hat{S}(0,\xi) & 0 \le \beta < \mathfrak{m}. \end{array} \right. \tag{14}$$

The constant C_η depends on the power spectrum and is defined in (44) *below. When the potential $q(x)$ is independent of time, the above expression simplifies as*

$$\rho(\mathfrak{m}, \mathfrak{n}) = \int_{\mathbb{R}^d} \frac{\hat{S}(\xi)}{|\xi|^{\mathfrak{m}+\mathfrak{n}}} d\xi = \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^{\mathfrak{m}}} d\xi, \tag{15}$$

which has an integrable singularity at $\xi = 0$ only when $\mathfrak{m} + \mathfrak{n} < d$.

The corrector to homogenization is determined by

$$\frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\varepsilon^\gamma} \Longrightarrow u_1, \qquad \gamma = \frac{d - (1-\mathfrak{b})\mathfrak{m} - \mathfrak{n}}{2} > 0, \tag{16}$$

weakly in space and in distribution, where u_1 is the solution to (12) *with $\sigma(\mathfrak{m}, \mathfrak{n}, \mathfrak{b}, \beta)$ and the mean-zero, self-similar, Gaussian field $\dot{W}$ given by*

$$\sigma^2 = \hat{S}_\infty(0), \qquad \mathbb{E}\{\dot{W}(t,x)\dot{W}(t+s,x+y)\} = \frac{1}{|t|^{\mathfrak{b}}} \left\{ \begin{array}{ll} |y|^{-\mathfrak{p}}, & 0 < \mathfrak{p} < d \\ \delta(y), & \int_{\mathbb{R}^d} \tilde{R}(x)dx < \infty. \end{array} \right. \tag{17}$$

When the potential $q(x)$ is independent of time, the above mean zero Gaussian field $\dot{W}(x)$ has the statistics given in (17) *with $\mathfrak{b} = 0$.*

For the second model for the power spectrum corresponding to short memory in time, the results should be modified as follows.

Theorem 2.2 *Let u_ε be the solution given by* (9) *for a Gaussian potential with statistics given by* (7). *Then independent of dimension and for a choice of potential scaling*

$$\alpha = \Big(\frac{\mathfrak{m}-\beta}{\mathfrak{m}} \frac{d-\mathfrak{n}}{2} + \frac{\beta}{2}\Big) \wedge \frac{\mathfrak{m}}{2}, \tag{18}$$

we have that u_ε converges strongly in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in time for $0 \le t \le T$ to $u(t,x)$ solution of (10) *with homogenized potential given by*

$$\rho(\mathfrak{m}, \mathfrak{n}, \beta) = \left\{ \begin{array}{ll} \displaystyle\int_{\mathbb{R}^d} \frac{e^{-|\xi|^{\mathfrak{m}}}}{|\xi|^{\mathfrak{n}}} d\xi \int_0^\infty \frac{\hat{S}(t,0)}{t^{\frac{d-\mathfrak{n}}{\mathfrak{m}}}} dt, & d < \mathfrak{m} + \mathfrak{n} \ \ \& \ \ \beta < \mathfrak{m} \\ \displaystyle\int_0^\infty \int_{\mathbb{R}^d} e^{-t|\xi|^{\mathfrak{m}}} \hat{R}(t,\xi) d\xi dt & \beta = \mathfrak{m} \\ \displaystyle\int_{\mathbb{R}^d} \frac{\hat{R}(0,\xi)}{|\xi|^{\mathfrak{m}}} d\xi & d > \mathfrak{m} + \mathfrak{n}. \end{array} \right. \tag{19}$$

The corrector to homogenization is determined by

$$\frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\varepsilon^\gamma} \Longrightarrow u_1, \qquad \gamma = \frac{d-(1-\mathfrak{b})\mathfrak{m}-\mathfrak{n}}{2} > 0, \tag{20}$$

weakly in space and in distribution, where u_1 *is the solution to* (12) *with* $\sigma(\mathfrak{m},\mathfrak{n},\mathfrak{b},\beta)$ *and the mean-zero, self-similar, Gaussian field* $\dot W$ *given by*

$$\sigma^2 = \int_{\mathbb{R}} \hat S(t,0)dt, \qquad \mathbb{E}\{\dot W(t,x)\dot W(t+s,x+y)\} = \delta(t)\left\{\begin{array}{ll} |y|^{-\mathfrak{p}} & 0<\mathfrak{p}<d \\ \delta(y) & \int_{\mathbb{R}^{d+1}} R(t,x)dtdx < \infty. \end{array}\right. \tag{21}$$

We thus observe that potentials with short memory in time always give rise to a homogenized limit independent of dimension. The corrector is then the solution of a stochastic equation with additive noise that is white in time and white or colored in space.

The behavior is different for potentials with long memory in time (which includes time-independent potentials). Only when the dimension d is sufficiently large as in (13) do we obtain a homogenized limit. The critical dimension $d_{\rm cr} = (1-\mathfrak{b})\mathfrak{m}+\mathfrak{n}$ also gives rise to a homogenized limit for an appropriate scaling of ε^α. We do not treat this critical case here and refer the reader to [3] where the case $\mathfrak{b}=0$ is treated. When the dimension $d<d_{\rm cr}$, we observe a totally different behavior. The "corrector" given in (20) then becomes as large as the "leading" term and a different regime arises.

Convergence to a stochastic limit. The convergence of u_ε to a stochastic limit is analyzed in the physical domain. Let $G(t,x;y)$ be the Green's function of the unperturbed operator, i.e.,

$$e^{-tP(x,D)}g(t,x) = \int_{\mathbb{R}^d} G(t,x;y)g(y)dy. \tag{22}$$

The stochastic partial differential equation we obtain for the limit of u_ε is of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} + P(x,D)u = \sigma u \circ \dot W(t,x), \quad t\geq 0, \quad x\in\mathbb{R}^d \\ &u(0,x) = g(x), \qquad x\in\mathbb{R}^d, \end{aligned} \tag{23}$$

where $\sigma>0$ is a constant and $\dot W(t,x)$ is a centered Gaussian field with correlation function given by

$$\mathbb{E}\{\dot W(t,x)\dot W(t+s,x+y)\} = \frac{1}{|s|^{\mathfrak{b}}}\left\{\begin{array}{ll} |y|^{-\mathfrak{p}} & 0<\mathfrak{p}<d \\ \delta(y) & \mathfrak{p}>d \end{array}\right. , \tag{24}$$

and where $u\circ \dot W(t,x)$ means that the product is understood as a Stratonovich integral. More precisely, this means that we look for mild solutions to (23) of the form

$$u(t,x) = e^{-tP(x,D)}g(t,x) + \sigma\int_0^t\int_{\mathbb{R}^d} G(t-s,x;y)u(s,y)\circ W(ds,dy), \tag{25}$$

with $\dot W(t,x)$ formally defined such that $W(dy,ds) = \dot W(t,x)ds\,dy$. The solution $u(s,y)$ in (25) is not deterministic and the above integral needs to be defined carefully. This is done as in [3] by means of iterated Stratonovich integrals. More precisely, let us define

$$\mathcal{H}u(t,x) = \sigma\int_0^t\int_{\mathbb{R}^d} G(t-s,x;y)u(s,y)\circ W(ds,dy), \tag{26}$$

and iteratively

$$u_0(t,x) = e^{-tP(x,D)}g(t,x), \qquad u_{n+1}(t,x) = \mathcal{H}u_n(t,x) = \mathcal{H}^{n+1}g(t,x). \tag{27}$$

We observe that u_n is defined as n iterated Stratonovich integrals that can be defined as in [3]; see below. We then have the following result.

Theorem 2.3 *Let us define $u(t,x) = \sum_{n\geq 0} u_n(t,x)$. Then $u(t) \in L^2(\Omega\times\mathbb{R}^d)$ uniformly on compact intervals in time is a solution to (25). Moreover, there exists a dense subspace $\mathfrak{M} = \mathfrak{M}(T)$ of $L^2((0,T)\times\mathbb{R}^d\times\Omega)$ in which we can show that $u(t,x)$ above is the unique solution to (25).*

The main ideas of the proof of the theorem will be presented in section 4. The solution to the $\varepsilon-$dependent problem is likewise constructed by a Duhamel expansion as in (8)-(9). For the regimes that are not covered in Theorem 2.1, namely when dimension is sufficiently small so that $(1-\mathfrak{b})\mathfrak{m}+\mathfrak{n} > d$, it turns out that the operators $\mathcal{H}$ and $\mathcal{H}_\varepsilon$ are very "close" in an appropriate metric. More precisely, we have the following result.

Theorem 2.4 *Let u_ε be the solution given by (9) for a Gaussian potential with statistics given by (5). Let us assume that $P(x,D) = (-\Delta)^{\frac{\mathfrak{m}}{2}}$ as in Theorem 2.1 and that*

$$(1-\mathfrak{b})\mathfrak{m}+\mathfrak{n} > d, \qquad \alpha = \frac{d-\mathfrak{n}+\beta\mathfrak{b}}{2} = \frac{\mathfrak{p}+\beta\mathfrak{b}}{2}. \tag{28}$$

Then u_ε converges in distribution to the solution u of (23) described in Theorem 2.3. The multiplicative noise has statistics given by (24) and the constant σ is given by

$$\sigma^2 = \begin{cases} \kappa & 0 < \mathfrak{p} < d \\ \displaystyle\int_{\mathbb{R}^d} \tilde{R}(x)dx & \textit{integrable } \tilde{R}, \end{cases} \tag{29}$$

where κ and $\tilde{R}$ are defined in (4).

Some remarks. Before addressing the derivation of the results presented above, we give some examples of application and remarks.

Remark 2.5 *The above theorem was stated for $P(x,D) = (-\Delta)^{\frac{\mathfrak{m}}{2}}$. More generally, let the Green's function $G(t,x;y)$ of $P(x,D)$ defined in (22) satisfy the following integrability constraint*

$$\int_0^T \int_{\mathbb{R}^d} \frac{G(t,x;x+y)}{|t|^{\mathfrak{b}}|y|^{\mathfrak{p}}} dtdy < \infty, \tag{30}$$

uniformly in x and some regularity condition outlined in (71) below. Then the above existence and convergence results still hold; see [4].

Remark 2.6 Consider the practically interesting case $P(x,D) = -\Delta$ with thus $\mathfrak{m} = 2$. Then we have the following results. Consider a Gaussian potential $q(t,x)$ with correlation function satisfying (4). The behavior of the correlation function at infinity dictates the type of convergence. When $\mathfrak{p}+2\mathfrak{b} < 2$, i.e., for long (spatial and/or temporal) memory effects, we obtain a stochastic limit whose multiplicative noise term $\dot{W}(t,x)$ is self-similar and has the same asymptotic behavior as $q(t,x)$ for large t and

x. For time independent potentials $q(x)$, the constraint becomes $R(x) \sim \kappa|x|^{-\mathfrak{p}}$ with $\mathfrak{p} < 2$ and this independent of spatial dimension. Stochastic models of the form (23) for the heat equation are therefore possible in all dimensions provided that they have sufficiently slow decorrelations.

When $p + 2\mathfrak{b} \geq 2$, we observe a sharp transition to a homogenization regime. The critical case $\mathfrak{p} + 2\mathfrak{b} = 2$ was not treated in this paper. We refer the reader to [4] for an example of a critical case. When $\mathfrak{p} + 2\mathfrak{b} \geq 2$, the decorrelation is sufficiently fast that the solution u_ε "sees" enough of the randomness in space-time to efficiently average over it and converge to a deterministic limit by ergodicity. The effects of randomness then appear as a corrector to homogenization. In the simplified setting of a Gaussian potential, the corrector to homogenization takes a very explicit form as the solution to a stochastic equation with additive noise (12).

Remark 2.7 We have considered two types of stochastic models. The first model in (12) is a stochastic PDE with additive noise. Since u is deterministic in the regime of homogenization, it is not difficult to make sense of the product on the right hand side of (12) and obtain that

$$u_1(t,x) = \int_0^t \int_{\mathbb{R}^d} G_\rho(t-s,x,y) u(s,y) W(dy,ds), \tag{31}$$

where $G_\rho(t-s,x,y)$ is the Green's function of the operator $e^{-t[(-\Delta)^{\frac{\mathfrak{m}}{2}}-\rho]}$. The above solution is well-defined for a wide class of driving noises $\dot{W}(t,x)$ independent of the spatial dimension.

The second model in (23) is a stochastic PDE with multiplicative noise. Its analysis is significantly more complicated. The reason is that the solution u, as in the model with additive noise (12), can be rather singular. The product $u\dot{W}$ of singular objects may therefore not make sense. What Theorem 2.3 shows is that the product indeed makes sense when a condition essentially of the form (30) holds, which is equivalent to the constraint on dimension in (28) for $P(x,D) = (-\Delta)^{\frac{m}{2}}$ as may be verified in the Fourier domain.

When (30) does not hold, we verify that $\mathbb{E}\{u_2(t,x)\}$ is unbounded in the Duhamel expansion (27). There are no reasonable way to define solutions to (23) with Stratonovich product. Other stochastic models are then possible, for instance models that interpret $u(t,x) \diamond W(dx,dt)$ as a Skorohod integral, which generalizes the Itô integral to anticipative processes. The physical interpretation of such models is however quite different and they cannot be derived as limits of solutions to equations with random coefficients of the form (1). We refer the reader to e.g. [13, 17, 19, 27] for references on such models.

Remark 2.8 The different expressions for the effective medium potential ρ in Theorems 2.1 and 2.2 all come from analyzing the limit as $\varepsilon \to 0$ of

$$\rho_\varepsilon = \varepsilon^{d-2\alpha} \int_0^T \int_{\mathbb{R}^d} e^{-t|\xi|^{\mathfrak{m}}} \hat{R}\Big(\frac{t}{\varepsilon^\beta}, \varepsilon\xi\Big) d\xi dt. \tag{32}$$

Boundedness of ρ_ε hinges on the technical fact that $\hat{R}(\tau,\xi)$ is bounded for small values of τ and that its singular behavior for large values of τ is compatible with its singular behavior for small values of ξ. This result is in sharp contrast with the analysis of (30), which for $G(x,D)=(-\Delta)^{\frac{\mathfrak{m}}{2}}$ is equivalent to

$$s=\int_0^T\int_{\mathbb{R}^d}\frac{e^{t|\xi|^{\mathfrak{m}}}}{|\xi|^{\mathfrak{n}}t^{\mathfrak{b}}}=\int_0^T\frac{dt}{t^{\mathfrak{b}+\frac{d-\mathfrak{n}}{\mathfrak{m}}}}\Big(\int_{\mathbb{R}^d}\frac{e^{-|\xi|^{\mathfrak{m}}}}{|\xi|^{\mathfrak{n}}}d\xi\Big)<\infty.$$

The latter bound implies that $d<\mathfrak{n}+(1-\mathfrak{b})\mathfrak{m}$ while ρ_ε is bounded as soon as $d>\mathfrak{n}+(1-\mathfrak{b})\mathfrak{m}$ (Theorem 2.1) and independent of d when $\hat{R}(\tau,\xi)$ is integrable in the time variable (Theorem 2.2).

Remark 2.9 We have assumed in (4) and (6) that the correlation was isotropic in the spatial variables. More generally, we can replace $|x|^{-\mathfrak{p}}$ in (4) and (6) by any homogeneous function $h(x)$ of degree $-\mathfrak{p}$, i.e., a function such that $h(\lambda x)=|\lambda|^{-\mathfrak{p}}h(x)$. All the results in the above theorems hold with $|\xi|^{-\mathfrak{n}}$ replaced by $C_h\hat{h}(\xi)$, where C_h is an appropriate constant and $\hat{h}(\xi)$ is homogeneous of degree $-\mathfrak{n}$.

3 Convergence to a homogenized equation

This section is devoted to a derivation of the results presented in Theorems 2.1 and 2.2. The proofs are based on modifications of similar proofs in [4]. We focus on the differences between the derivations and refer the reader to [4] for more details. The methodology followed in [4] consists of recasting (1) in the Fourier domain

$$\big(\frac{\partial}{\partial t}+\xi^{\mathfrak{m}}\big)\hat{u}_\varepsilon=\hat{q}_\varepsilon*\hat{u}_\varepsilon, \tag{33}$$

with $\hat{u}_\varepsilon(0,\xi)=\hat{g}(\xi)$, where

$$\hat{q}_\varepsilon*\hat{u}_\varepsilon(t,\xi)=\int_{\mathbb{R}^d}\hat{u}_\varepsilon(t,\xi-\zeta)\hat{q}_\varepsilon(\zeta)d\zeta.$$

Here and below, we use the notation $\xi^{\mathfrak{m}}=|\xi|^{\mathfrak{m}}$. Since $q(x)$ is a stationary mean zero Gaussian random field, it admits the following spectral representation

$$q(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{i\xi\cdot x}\hat{q}(\xi)d\xi, \tag{34}$$

where $\hat{q}(\xi)d\xi$ is the complex spectral process such that

$$\mathbb{E}\Big\{\int_{\mathbb{R}^d}f(\xi)\hat{q}(\xi)d\xi\overline{\int_{\mathbb{R}^d}g(\xi)\hat{q}(\xi)d\xi}\Big\}=\int_{\mathbb{R}^d}f(\xi)\bar{g}(\xi)(2\pi)^d\hat{R}(\xi)d\xi,$$

for all f and g in $L^2(\mathbb{R}^d;\hat{R}(\xi)d\xi)$. Note that $\mathbb{E}\{\hat{q}(\xi)\hat{q}(\zeta)\}=\hat{R}(\xi)\delta(\xi+\zeta)$ and $\mathbb{E}\{\hat{q}(\xi)\overline{\hat{q}(\zeta)}\}=\hat{R}(\xi)\delta(\xi-\zeta)$.

Duhamel expansion. The equation (33) is then recast as

$$\hat{u}_\varepsilon(t,\xi)=e^{-t\xi^{\mathfrak{m}}}\hat{g}(\xi)+\int_0^t e^{-s\xi^{\mathfrak{m}}}\int_{\mathbb{R}^d}\hat{q}_\varepsilon(t-s,\xi-\xi_1)\hat{u}_\varepsilon(t-s,\xi_1)d\xi_1ds. \tag{35}$$

This allows us to write the formal Duhamel expansion

$$\hat{u}_\varepsilon(t,\xi) = \sum_{n\in\mathbb{N}} \hat{u}_{n,\varepsilon}(t,\xi), \tag{36}$$

$$\hat{u}_{n,\varepsilon}(t,\xi_0) = \int_{\mathbb{R}^d} \prod_{k=0}^{n-1} \int_0^{t_k} e^{-\xi_k^{\mathfrak{m}} s_k} e^{-t_n \xi_n^{\mathfrak{m}}} \prod_{k=0}^{n-1} \hat{q}_\varepsilon(t_{k+1}, \xi_k - \xi_{k+1}) \hat{g}(\xi_n) d\mathbf{s} d\boldsymbol{\xi}. \tag{37}$$

Here, we have introduced the following notation:

$$\mathbf{s} = (s_0, \ldots, s_{n-1}),\ t_k(\mathbf{s}) = t - s_0 - \ldots - s_{k-1},\ t_0(\mathbf{s}) = t,\ d\mathbf{s} = \prod_{k=0}^{n-1} ds_k,\ d\boldsymbol{\xi} = \prod_{k=1}^{n} d\xi_k.$$

We want to show that for sufficiently small times, the expansion (36) converges (uniformly for all ε sufficiently small) in the $L^2(\Omega\times\mathbb{R}^d)$ sense and that the L^2 norm of $u_\varepsilon(t)$ is bounded by the $L^2(\mathbb{R}^d)$ norm of $\hat{g}$.

Estimates of moments. The convergence results are based on the analysis of the following moments

$$U_\varepsilon^{n,m}(t,\xi,\zeta) = \mathbb{E}\{\hat{u}_{\varepsilon,n}(t,\xi)\overline{\hat{u}_{\varepsilon,m}}(t,\zeta)\}, \tag{38}$$

which are given by

$$\int_{\mathbb{R}^{d(n+m)}} \prod_{k=0}^{n-1} \int_0^{t_k(\mathbf{s})} \prod_{l=0}^{m-1} \int_0^{t_l(\boldsymbol{\tau})} e^{-s_k \xi_k^{\mathfrak{m}}} e^{-(t-\sum_{k=0}^{n-1} s_k)\xi_n^{\mathfrak{m}}} e^{-\tau_l \zeta_l^{\mathfrak{m}}} e^{-(t-\sum_{l=0}^{m-1}\tau_l)\zeta_m^{\mathfrak{m}}}$$
$$\times \mathbb{E}\Big\{ \prod_{k=0}^{n-1}\prod_{l=0}^{m-1} \hat{q}_\varepsilon(t_{k+1}, \xi_k - \xi_{k+1}) \bar{\hat{q}}_\varepsilon(l_{k+1}, \zeta_l - \zeta_{l+1}) \Big\} \hat{g}(\xi_n) \bar{\hat{u}}_0(\zeta_m)\, d\mathbf{s} d\boldsymbol{\tau} d\boldsymbol{\xi} d\boldsymbol{\zeta}.$$

We also need to define the moments

$$U_\varepsilon^n(t,\xi) = \mathbb{E}\{\hat{u}_{\varepsilon,n}(t,\xi)\}, \tag{39}$$

and the covariances

$$V_\varepsilon^{n,m}(t,\xi,\zeta) = \mathrm{cov}(\hat{u}_{\varepsilon,n}(t,\xi), \hat{u}_{\varepsilon,m}(t,\zeta)) = U_\varepsilon^{n,m}(t,\xi,\zeta) - U_\varepsilon^n(t,\xi)\overline{U_\varepsilon^m(t,\zeta)}. \tag{40}$$

We introduce the notation $s_n(\mathbf{s}) = t_n(\mathbf{s}) = t - \sum_{k=0}^{n-1} s_k$ and $\tau_m(\boldsymbol{\tau}) = t_m(\boldsymbol{\tau}) = t - \sum_{l=0}^{m-1}\tau_l$. We also define $\xi_{n+k+1} = \zeta_{m-k}$ and $s_{n+k+1} = \tau_{m-k}$ for $0\le k\le m$. Then, we observe that $U_\varepsilon^{n,m}(t,\xi_0,\xi_{n+m+1})$ may be recast as

$$\int \prod_{k=0}^{n+m+1} e^{-s_k \xi_k^{\mathfrak{m}}} \mathbb{E}\Big\{ \prod_{k=1, k\neq n+1}^{n+m+1} \hat{q}_\varepsilon(t_k, \xi_{k-1} - \xi_k) \Big\} \hat{g}(\xi_n) \bar{\hat{u}}_0(\xi_{n+1}) d\mathbf{s} d\boldsymbol{\xi} \tag{41}$$

where the domain of integration in the s and ξ variables is inherited from the previous expression. The potential is of the form

$$q_\varepsilon(t,x) = \frac{1}{\varepsilon^\alpha} q\Big(\frac{t}{\varepsilon^\beta}, \frac{x}{\varepsilon}\Big).$$

We observe that the analysis of (38) requires that we calculate the expectation of a finite product of potentials. The expectation in $U_\varepsilon^{n,m}$ vanishes unless there is $\bar n \in \mathbb{N}$ such that $n+m=2\bar n$ is even.

The variables ξ_k are gathered in pairs as follows. For each $1\leq k\leq n+m$ with $k\neq n+1$, we define a pair (ξ_k,ξ_l) with $k<l$ for the contribution such that

$$\begin{aligned}&\mathbb{E}\{\hat q_\varepsilon(t_k,\xi_{k-1}-\xi_k)\hat q_\varepsilon(t_l,\xi_{l-1}-\xi_l)\}\\&=\varepsilon^{d-2\alpha}\hat R(\tfrac{t_k-t_l}{\varepsilon^\beta},\varepsilon(\xi_k-\xi_{k-1}))\delta(\xi_k-\xi_{k-1}+\xi_l-\xi_{l-1}).\end{aligned}$$

The number of pairings in a product of $n+m=2\bar n$ terms (i.e., the number of allocations of the set $\{1,\dots,2\bar n\}$ into $\bar n$ unordered pairs) is equal to

$$\frac{(2\bar n-1)!}{2^{\bar n-1}(\bar n-1)!}=\frac{(2\bar n)!}{\bar n!2^{\bar n}}=(2\bar n-1)!!.$$

In each instance of the pairings, we have $\bar n$ terms k and $\bar n$ terms $l\equiv l(k)$. We denote by **simple pairs** the pairs such that $l(k)=k+1$, which thus involve a delta function of the form $\delta(\xi_{k+1}-\xi_{k-1})$. The collection of pairs $(\xi_k,\xi_{l(k)})$ for $\bar n$ values of k and $\bar n$ values of $l(k)$ constitutes a graph $\mathfrak{g}\in\mathfrak{G}$, the collection of all possible $|\mathfrak{G}|=\frac{(2\bar n-1)!}{2^{\bar n-1}(\bar n-1)!}$ graphs that can be constructed for a given value of $\bar n$. We denote by $A_0=A_0(\mathfrak{g})$ the collection of the $\bar n$ values of k and by $B_0=B_0(\mathfrak{g})$ the collection of the $\bar n$ values of $l(k)$. Then we find that the product of random terms in $U_\varepsilon^{n,m}$ is given by

$$\begin{aligned}&\mathbb{E}\Big\{\prod_{k=1,k\neq n+1}^{n+m+1}\hat q_\varepsilon(t_k,\xi_{k-1}-\xi_k)\Big\}\\&=\sum_{\mathfrak{g}\in\mathfrak{G}}\prod_{k\in A_0(\mathfrak{g})}\varepsilon^{d-2\alpha}\hat R\Big(\frac{t_k-t_{l(k)}}{\varepsilon^\beta},\varepsilon(\xi_k-\xi_{k-1})\Big)\delta\big(\xi_k-\xi_{k-1}+\xi_{l(k)}-\xi_{l(k)-1}\big).\end{aligned}$$

This allows us to summarize the main results:

$$\begin{aligned}U_\varepsilon^{n,m}(t,\xi_0,\xi_{n+m+1})=&\int\prod_{k=0}^{n+m+1}e^{-s_k\xi_k^{\mathfrak{m}}}\hat g(\xi_n)\bar{\hat u}_0(\xi_{n+1})\sum_{\mathfrak{g}\in\mathfrak{G}}\\&\prod_{k\in A_0(\mathfrak{g})}\varepsilon^{d-2\alpha}\hat R\Big(\frac{t_k-t_{l(k)}}{\varepsilon^\beta},\varepsilon(\xi_k-\xi_{k-1})\Big)\delta\big(\xi_k-\xi_{k-1}+\xi_{l(k)}-\xi_{l(k)-1}\big)d\mathbf{s}d\boldsymbol{\xi}.\end{aligned}\tag{42}$$

Similarly,

$$\begin{aligned}U_\varepsilon^{n}(t,\xi_0)=\hat g(\xi_0)&\int\prod_{k=0}^{n}e^{-s_k\xi_k^{\mathfrak{m}}}\sum_{\mathfrak{g}\in\mathfrak{G}}\\&\prod_{k\in A_0(\mathfrak{g})}\varepsilon^{d-2\alpha}\hat R\Big(\frac{t_k-t_{l(k)}}{\varepsilon^\beta},\varepsilon(\xi_k-\xi_{k-1})\Big)\delta\big(\xi_k-\xi_{k-1}+\xi_{l(k)}-\xi_{l(k)-1}\big)d\mathbf{s}d\boldsymbol{\xi}.\end{aligned}\tag{43}$$

Analysis of crossing and non-simple graphs. The analysis of the sum over the graphs $\mathfrak{g}$ is handled as in [4]. The graphs in $\mathfrak{G}$ can be grouped into several categories. We denote by $\mathfrak{G}_c$ the set of crossing graphs, which admit at least one value of $k\leq n$ such that $l(k)\geq n+2$. We denote the non-crossing graphs by $\mathfrak{G}_{\rm nc}=\mathfrak{G}\backslash\mathfrak{G}_c$. The unique graph composed solely of simple pairs is called $\mathfrak{g}_s$. The crossing graphs with a single crossing are called $\mathfrak{G}_{cs}$ while $\mathfrak{G}_{cns} = \mathfrak{G}_c \backslash \mathbf{G}_{cs}$ is the set of graphs with at least two crossings.

With these definitions, $V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$ is the sum over the crossing graphs and $U_\varepsilon^n(t, \xi_0)\overline{U_\varepsilon^m(t, \xi_{n+m+1})}$ is the sum over the non-crossing graphs in $U_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$.

The single graph $\mathfrak{g}_s$ is responsible for the homogenized limit. The non-crossing graphs are responsible for the ensemble average $\mathbb{E}\{u_\varepsilon\}$. The crossing graphs with a single crossing $\mathfrak{G}_{cs}$ are responsible for the corrector to homogenization. All the other graphs are shown collectively to contribute less than the aforementioned terms.

The estimates for the crossing and non-crossing graphs are based on the following generalization of [4, Lemma 2.1]:

Lemma 3.1 *Let $k = 0$ or $k = 1$. Assume that $\hat{R}(t, \xi)$ is of the form given in (5). Then for each $\eta > 0$, there exists a constant C_η independent of $\zeta_0 \in \mathbb{R}^d$, $\zeta_1 \in \mathbb{R}^d$, and $u \in \mathbb{R}$ such that*

$$\varepsilon^{-2\alpha} \int_{\mathbb{R}^d} \int_0^{\tau \wedge t} e^{-s\varepsilon^{-\mathfrak{m}}\xi^{\mathfrak{m}}} \Big(\frac{\varepsilon^{\mathfrak{m}}}{|\xi - \zeta_0|^{\mathfrak{m}}} \wedge t \Big)^k \hat{R}\Big(\frac{s-u}{\varepsilon^\beta}, \xi - \zeta_1 \Big) ds d\xi \le C_\eta \varepsilon^{k\gamma_\eta}, \tag{44}$$

where $\gamma_\eta = (d - \mathfrak{n} - (1-\mathfrak{b})\mathfrak{m} - \eta) \wedge \mathfrak{m}$. Assume that $\hat{R}(t, \xi)$ is of the form given in (6). Then the above holds with $\mathfrak{b} = 0$. In both cases, when $d - \mathfrak{n} - (1-\mathfrak{b})\mathfrak{m} - \eta \neq \mathfrak{m}$, we can set $\eta = 0$ with $C_{\eta=0} = C_0 < \infty$.

Proof. Let us first replace $\tau \wedge t$ by τ for some $\tau < T$. We then observe that we need to bound the following term

$$\varepsilon^{-2\alpha+\beta\mathfrak{b}} \int_{\mathbb{R}^d} \int_0^{\tau \wedge t} e^{-s\varepsilon^{-\mathfrak{m}}\xi^{\mathfrak{m}}} \Big(\frac{\varepsilon^{\mathfrak{m}}}{|\xi - \zeta_0|^{\mathfrak{m}}} \wedge t \Big)^k \frac{1}{|s-u|^\beta |\xi - \zeta_1|^{\mathfrak{n}}} \hat{S}\Big(\frac{s-u}{\varepsilon^\beta}, \xi - \zeta_1 \Big) ds d\xi.$$

Recall that $2\alpha = (1-\mathfrak{b})\mathfrak{m} + \beta\mathfrak{b}$. We first consider the integral in ξ over the domain $|\xi - \zeta_1| > 1$. Since $\hat{S}$ is bounded in the first variable, we observe that

$$\int_0^\tau \frac{e^{-s\varepsilon^{-\mathfrak{m}}\xi^{-\mathfrak{m}}}}{|s-u|^{\mathfrak{b}}} ds \le \frac{\varepsilon^{\mathfrak{m}(1-\mathfrak{b})}}{\xi^{\mathfrak{m}}}.$$

On the ball of radius one centered at ξ_0, we obtain that

$$\int_{B(\xi_0,1)} \Big(\frac{\varepsilon^{\mathfrak{m}}}{|\xi - \zeta_0|^{\mathfrak{m}}} \wedge t \Big)^k \frac{\varepsilon^{\mathfrak{m}(1-\mathfrak{b})}}{\xi^{\mathfrak{m}}} d\xi \le C \varepsilon^{k((d-\mathfrak{m})\wedge\mathfrak{m}) + \mathfrak{m}(1-\mathfrak{b}) - \eta}.$$

On the domain $\mathbb{R}\backslash(B(\xi_0, 1) \cup B(\xi_1, 1))$, the integral is yet smaller by integrability of $\hat{S}$ with respect to the second variable. Both contributions are smaller than the result announced in (44), whose leading contribution comes from the integral over $B(\xi_1, 1)$. The contribution to (44) on $B(\xi_1, 1)$ is bounded by

$$\varepsilon^{-(1-\mathfrak{b})\mathfrak{m}} \int_{B(\xi_1,1)} \int_0^\tau \frac{e^{-s\varepsilon^{-\mathfrak{m}}\xi^{\mathfrak{m}}}}{|s-u|^{\mathfrak{b}} |\xi - \zeta_1|^{\mathfrak{n}}} \Big(\frac{\varepsilon^{\mathfrak{m}}}{|\xi - \zeta_0|^{\mathfrak{m}}} \wedge t \Big)^k ds d\xi.$$

We verify that the maximum for such a contribution is attained when $\zeta_0 = \zeta_1 = 0$ and $u = 0$ (see e.g. [4, Lemma 2.2]). We then separate the integrals $|\xi| < \varepsilon$ and $|\xi| > \varepsilon$. The latter contribution is bounded by

$$C\varepsilon^{k\mathfrak{m}} \int_\varepsilon^1 r^{d-1-\mathfrak{n}-k\mathfrak{m}-(1-\mathfrak{b})\mathfrak{m}} dr \int_0^\infty \frac{e^{-s}}{s^{\mathfrak{b}}} ds \le C\varepsilon^{k((d-\mathfrak{n}-(1-\mathfrak{b})\mathfrak{m})\wedge\mathfrak{m})},$$

when $d-\mathfrak{n}-k\mathfrak{m}-(1-\mathfrak{b})\mathfrak{m}\neq 0$ and with a logarithmic singularity otherwise (and hence the presence of $\eta>0$ in (44)). We verify that the contribution $|\xi|<\varepsilon$ is at best of the same order and this concludes the proof of the lemma. The proof is also quite similar when the power spectrum is integrable in time. □

In order to exhibit those crossing graphs that contribute to the random corrector to homogenization, we need to generalize the estimate [4, Eq.(50)]. This is based on replacing [4, Eq.(48)] in [4, Eq.(47)] by

$$\varepsilon^{d-2\alpha}\int_0^\tau \hat{R}\Big(\frac{s-u}{\varepsilon^\beta},\varepsilon\xi\Big)ds \leq C_\infty \frac{\varepsilon^{d-2\alpha-\mathfrak{n}+\beta\mathfrak{b}}}{|\xi|^{\mathfrak{n}}}, \tag{45}$$

where C_∞ is a constant independent of u. This inequality is a direct consequence of the assumptions on the power spectrum both in the integrable and non-integrable cases.

Equipped with these inequalities, the derivation of Theorems 2.1 and 2.2 is dealt with as in [4]. We merely highlight the differences here. The above lemma with $k=0$ allows one to generalize [4, Eq.(39)] to the setting of Theorems 2.1 and 2.2. The lemma with $k=1$ allows one to generalize [4, Eq.(40)] and show that crossing graphs have negligible energy as in [4, Eq.(44)]. Using (45) in lieu of [4, Eq.(48)], we find that [4, (50)] is replaced by

$$|V_\varepsilon^{n,m}(t,\xi_0,\xi_{n+m+1})| \leq CC_\infty \varepsilon^{d-2\alpha-\mathfrak{n}+\beta\mathfrak{b}}|\mathfrak{G}_{\rm c}|\Big(\int d\tilde{s}\Big)C_\eta^{\bar{n}-1}\|\hat{g}\|_{\mathfrak{n}}^2, \tag{46}$$

where we have defined $|\mathfrak{G}_{\rm c}|$ as the number of crossing graphs, where

$$|\mathfrak{G}_{\rm c}|C_\eta^{\bar{n}}\Big(\int d\tilde{s}\Big) \leq \frac{\bar{n}}{T}\big(4C_\eta T\big)^{\bar{n}}. \tag{47}$$

as in [4, (42)], and where we have assumed that g is bounded for the norm (in the Fourier domain)

$$\|\hat{g}\|_{\mathfrak{n}}^2 = \sup_{\zeta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{|\xi-\zeta|^{\mathfrak{n}}}|\hat{g}(\xi)|^2d\xi. \tag{48}$$

This shows that all crossing graphs contribute at most a contribution of size $\varepsilon^{d-2\alpha-\mathfrak{n}+\beta\mathfrak{b}}=\varepsilon^{d-\mathfrak{n}-(1-b)\mathfrak{m}}\ll 1$. Moreover, the estimate (47) shows that the summation over all graphs is converging as soon as

$$4C_\eta T<1. \tag{49}$$

This is the smallness condition we need to impose on final time in order to obtain our convergence result.

Estimating the various contributions. Now, each graph with a non-crossing pair $(\xi_k,\xi_{l(k)})$ such that $l(k)>k+1$ (non simple pair) generates a contribution that is $\varepsilon^{\gamma_\eta}$ smaller than when such a pair cannot be found. This shows that the only contribution of order $O(1)$ is the simple graph $\mathfrak{g}_s$. All graphs in $\mathfrak{G}_{\rm ncs}$ are therefore of order $\varepsilon^{d-\mathfrak{n}-(1-b)\mathfrak{m}}\varepsilon^{\gamma_\eta}$. This implies that the graphs of order exactly $\varepsilon^{d-\mathfrak{n}-(1-b)\mathfrak{m}}$ are the crossing graphs with simple pairs, i.e., graphs in $\mathfrak{G}_{\rm cs}$.

These estimates plus the control (47)-(49) also allow one to show that $u_\varepsilon(t)$ is indeed a function in $L^2(\mathbb{R}^d\times\Omega)$ uniformly in time for $0<t<T$, that the graph $\mathfrak{g}_s$ provides the only order $O(1)$ contribution, and that the corrector is given by the graphs in $\mathfrak{G}_{\rm cs}$. The analysis of the corrector is then performed as in [4, Section 3.2].

Analysis of the homogenization contribution. The analysis of the leading contribution is performed as in [4, Section 3.1]. The simple graph contribution is given by

$$U_{\varepsilon,s}(t,\xi_0)=\sum_{n\in 2\mathbb{N}} U^n_{\varepsilon,s}(t,\xi_0):=\mathcal{U}_\varepsilon(t,\xi_0)\hat{g}(\xi_0), \tag{50}$$

where, using the notation $\bar{n}=2n$ (with $U^{2n+1}_{\varepsilon,s}\equiv 0$)

$$\begin{aligned} U^n_{\varepsilon,s}(t,\xi_0) &= \mathcal{U}^n_\varepsilon(t,\xi_0)\hat{g}(\xi_0) \qquad (51)\\ \mathcal{U}^n_\varepsilon(t,\xi_0) &= \int \prod_{k=0}^{n} e^{-s_k\xi_k^{\mathfrak{m}}}\prod_{k=0}^{\bar{n}-1}\varepsilon^{d-2\alpha}\hat{R}\Big(\frac{t_{2k+1}-t_{2k+2}}{\varepsilon^\beta},\varepsilon(\xi_{2k+1}-\xi_{2k})\Big)\delta(\xi_{2k+2}-\xi_{2k})d\mathbf{s}d\boldsymbol{\xi}. \end{aligned}$$

We can sum these contributions and verify that $U_{\varepsilon,s}(t,\xi_0)$ solves the following integral equation in the time variable

$$\begin{aligned} U_{\varepsilon,s}(t,\xi) &= e^{-t\xi^{\mathfrak{m}}}\hat{g}(\xi)\\ &+\int_0^t e^{-\xi^{\mathfrak{m}}s}\int_0^{t-s}\int_{\mathbb{R}^d} e^{-\xi_1^{\mathfrak{m}}s_1}\varepsilon^{d-2\alpha}\hat{R}\Big(\frac{s_1}{\varepsilon^\beta},\varepsilon(\xi_1-\xi)\Big)U_{\varepsilon,s}(t-s-s_1,\xi)d\xi_1 ds_1 ds. \end{aligned} \tag{52}$$

The last term may be recast as

$$\int_0^t\Big(\int_0^v e^{-\xi^{\mathfrak{m}}(v-s_1)}e^{-\xi_1^{\mathfrak{m}}s_1}\varepsilon^{d-2\alpha}\int_{\mathbb{R}^d}\hat{R}\Big(\frac{s_1}{\varepsilon^\beta},\varepsilon\xi_1-\varepsilon\xi\Big)d\xi_1 ds_1\Big)U_{\varepsilon,s}(t-v,\xi)dv.$$

The same analysis as in [4, Section 3.1] shows that the above term may be well approximated by $\int_0^t e^{-\xi^m v}\rho_\varepsilon U_{\varepsilon,s}(t-v,\xi)dv$, where

$$\rho_\varepsilon=\int_0^v\int_{\mathbb{R}^d} e^{-\xi_1^{\mathfrak{m}}s_1}\varepsilon^{d-2\alpha}\hat{R}\Big(\frac{s_1}{\varepsilon^\beta},\varepsilon\xi_1\Big)d\xi_1 ds_1, \tag{53}$$

which is independent of v for $v>0$. This is the expression given in (32). It remains to analyze the limit of (53) as $\varepsilon\to 0$ to obtain the results stated in Theorems 2.1 and 2.2. The technical but straightforward details are left to the reader. This concludes the proof of Theorems 2.1 and 2.2.

4 Convergence to a stochastic equation

The derivation of Theorems 2.3 and 2.4 closely follows the presentation in [3]. Lengthy calculations very similar to those in [3] are not reproduced here. Rather, we merely describe what needs to be modified in the proofs in [3].

The first item is the construction of iterated Stratonovich integrals that allow us to make sense of the Duhamel solution defined in Theorem 2.3. We use estimates that are similar to those obtained in the preceding section. The construction of a space in which we can obtain uniqueness of mild solutions to the limiting stochastic equation (23) may be done as in [3] and is not considered here. It finally remains to address the perturbed problem (1) by showing that the Duhamel solution is well-defined and that it converges in law to the solution to (23) as $\varepsilon\to 0$.

Construction of iterated Stratonovich integrals. Let $z=(t,x)$ denote the spatio-temporal variables. Let $f(z_1,\dots,z_n)$ be a function of n variables in $\mathbb{R}^{d+1}$. We want to define the iterated Stratonovich integral $\mathcal{I}_n(f)$ and first assume that f separates as $f(z_1,\dots,z_n)=\prod_{k=1}^n f_k(z_k)$. Then we define

$$\mathcal{I}_n\big(\prod_{k=1}^n f_k(z_k)\big)=\prod_{k=1}^n \mathcal{I}_1(f_k(z_k)), \tag{54}$$

where $\mathcal{I}_1(f)=\int_{\mathbb{R}^{d+1}} f(z)dW(z)$ is defined as the usual multi-parameter Wiener integral for the Gaussian measure $dW(z)$. We recall that the Gaussian measure with "density" $\dot W(z)$ is defined by its correlation function in (24), which we denote by

$$\varphi(z)=\mathbb{E}\{\dot W(\zeta)\dot W(\zeta+z)\}. \tag{55}$$

Note that $\mathcal{I}_1(f)$ is then the centered Gaussian random variable with variance given by $\int_{\mathbb{R}^{2d+2}} f(z)f(\zeta)\varphi(z-\zeta)dzd\zeta$.

It remains to generalize the above definition to a larger class of functions by density. We define the symmetrized function

$$f_{\mathfrak{s}}(z_1,\dots,z_n)=\frac{1}{n!}\sum_{\mathfrak{s}\in\mathfrak{S}_n} f(z_{\mathfrak{s}(1)},\dots,z_{\mathfrak{s}(n)}), \tag{56}$$

where the sum is taken over the $n!$ permutations of the variables $z_1,\dots,z_n$. We then define $\mathcal{I}_n(f)=\mathcal{I}_n(f_{\mathfrak{s}})$ and thus now consider functions that are symmetric in their arguments. As in [3], we drop the $\circ$ in the definition of the Stratonovich integral and write the Itô-Skorohod integral as δW. The iterated Itô integral is defined as

$$I_n(g_n)=\int_{\mathbb{R}^{n(d+1)}} g_n(z_1,\dots,z_n)\delta W(dz_1)\dots\delta W(dz_n). \tag{57}$$

The above iterated integral is well-known to exist for all functions g_n such that

$$\|g_n\|_\varphi^2=\int_{\mathbb{R}^{2n(d+1)}} g_n(z)g_n(z')\varphi^{\otimes n}(z-z')dzdz'<\infty. \tag{58}$$

Here, $\varphi^{\otimes n}(z)=\varphi(z_1)\dots\varphi(z_n)$. Let us pretend that iterated Stratonovich integrals are defined and that we want to write them in terms of iterated Itô integrals. More precisely, let us assume that a square integrable random variable $f(\omega)$ may be decomposed as

$$f=\sum_{n\ge 0}\mathcal{I}_n(f_n)=\sum_{m\ge 0} I_m(g_m).$$

Then we may project Stratonovich integrals onto the orthogonal basis of Itô integrals as follows

$$\mathbb{E}\{\mathcal{I}_n(f_n)I_m(\phi_m)\}=\mathbb{E}\{I_m(g_m)I_m(\phi_m)\}=m!\int_{\mathbb{R}^{2m(d+1)}} g_m(z)\phi_m(z')\varphi^{\otimes m}(z-z')dzdz',$$

where ϕ_m is a test function. We find that $\mathbb{E}\{\mathcal{I}_n(f_n)I_m(\phi_m)\}$ is equal to

$$\int_{\mathbb{R}^{(n+m)(d+1)}} f_n(z_1,\dots,z_n)\phi_m(\zeta_1,\dots,\zeta_m)\mathbb{E}\{dW(z_1)\dots dW(z_n)\delta W(\zeta_1)\dots\delta W(\zeta_m)\}.$$

We have again to look at a product of Gaussian measures with the additional rule that $\mathbb{E}\{\delta W(y_k)\delta W(y_l)\} = 0$ for $k \neq l$ by renormalization of the Itô-Skorohod integral. The functions f_n and ϕ_m are symmetric in their arguments (i.e., invariant by permutation of their variables). We observe that the variables y need be paired with m variables x so that we need $n \geq m$ for the above expression not to vanish. There are $\binom{n}{m}$ ways of pairing the y variables. There remain $n - m = 2k$ variables that need be paired, for a possible number of pairings equal to

$$\frac{(2k-1)!}{(k-1)!2^{k-1}}.$$

The above term is thus given by

$$\binom{m+2k}{m}\frac{(2k-1)!}{(k-1)!2^{k-1}}\int f_{m+2k}(\zeta,\xi,\xi')\varphi^{\otimes k}(\xi-\xi')d\xi d\xi'\phi_m(\zeta)d\zeta.$$

This shows that

$$g_m(\zeta) = \frac{(m+2k)!}{m!k!2^k}\int f_{m+2k}(\zeta,\xi,\xi')\varphi^{\otimes k}(\xi-\xi')d\xi d\xi'. \tag{59}$$

As a consequence, we have shown by projection onto the basis of iterated Itô-Skorohod integrals that

$$\mathcal{I}_n(f_n) = \sum_{k=0}^{[\frac{n}{2}]}\frac{n!}{(n-2k)!k!2^k}I_{n-2k}\Big(\int_{\mathbb{R}^{2k(d+1)}} f_n(x_{n-2k},\xi,\xi')\varphi^{\otimes k}(\xi-\xi')d\xi d\xi'\Big). \tag{60}$$

This serves as a possible definition for the iterated Stratonovich integral. We observe that the above integral is well-defined provided that the right-hand side is well-defined. This imposes $[\frac{n}{2}]+1$ constraints on $f_n(z_n)$ in order for the integral to be well-defined since each g_m has to satisfy (58). We easily verify that (60) is a generalization of (54) obtained for simple functions.

Duhamel solution. It remains to show that the iterated Stratonovich integrals indeed define the terms $u_n(t,x) = \mathcal{I}_n(f_n(t,x,\cdot)$ for appropriate kernels $f_n(t,x,\cdot)$, in the Duhamel expansion and that the sum over n is a square integrable function $u \in L^2((0,T)\times\mathbb{R}^d\times\Omega)$ for all $0 < T < \infty$.

For $z = (t,x)$, we define $\mathbb{R}^d_z = (0,t)\times\mathbb{R}^d$ and $\mathbb{R}^d_T = (0,T)\times\mathbb{R}^d$. We also define $G(z,z') = G(t-t',x;x')$ and $u_0(z) = \big(e^{-tP(x,D)}g\big)(z)$. Then we observe that

$$u_n(z_0) = \sigma^n\int_{\mathbb{R}^d_{z_0}}\cdots\int_{\mathbb{R}^d_{z_{n-1}}} G(z_0,z_1)\ldots G(z_{n-1},z_n)u_0(z_n)W(dz_1)\ldots W(dz_n),$$

where the above is defined as an iterated Stratonovich integral. That $u \in L^2((0,T)\times\mathbb{R}\times\Omega)$ is obtained by estimating the following correlations

$$I_{n,m} = \mathbb{E}\Big\{\int_{\mathbb{R}^d_T} u_n(z)u_m(z)dz\Big\}.$$

Derivation of the estimates. The hypotheses on the Green's function guaranties the existence of $\gamma < 1$ such that the L^2 norm of the Green's function in space is bounded by $t^{-\gamma}$. Moreover, using the explicit expression $e^{-t|\xi|^{\mathfrak{m}}}$ for the Green's function in the Fourier domain, we obtain that

$$\int_{\mathbb{R}^{2d}} G(t-\tau,x,y)G(u-v,z,\zeta)\varphi(x-z,t-u)dxdz \leq \frac{C}{((t-\tau)+(u-v))^{\frac{\mathfrak{p}}{\mathfrak{m}}}|t-u|^{\mathfrak{b}}}. \quad (61)$$

Using the above estimate in a straightforward manner, lengthy calculations similar to those in [3] show that I_n in [3, Eq. (18)] should be replaced by

$$C^n \int_0^{t_0} t_1^{-\gamma} \int_0^{t_1} \cdots \int_0^{t_{n-1}} \prod_{k=0}^{n-1} \frac{dt_{k+1}}{(t_k - t_{k+1})^{\frac{\mathfrak{p}}{2\mathfrak{m}}+\frac{\mathfrak{b}}{2}}}.$$

The calculation below [3, Eq. (18)] shows that the above term is bounded by

$$C_\rho C^n n^{-\rho n}, \qquad 1 - \frac{\mathfrak{p}}{\mathfrak{m}} - \mathfrak{b} > \rho > 0.$$

As a consequence, we have

$$\sum_{m,n} I_{n,m} \leq 2 \sum_{m\leq n} I_{n,m} \leq 2 \sum_n n C_\rho C^n n^{-\rho n} < \infty.$$

This shows the L^2 bound. It is then straightforward to generalize [3, Theorem 2] and obtain that the Duhamel solution is the unique solution to the stochastic PDE problem in the space $\mathcal{M}$ constructed in [3, Eq. (23)] with the coefficients g_m defined in (59).

Convergence result. The proof of convergence of u_ε to u is also similar to the corresponding result in [3]. The main difference is that the potential and the limiting Gaussian measure are now allowed to depend on time.

Let us define

$$\mathcal{H}_\varepsilon u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)u(s,y)q_\varepsilon(s,y)dyds, \quad (62)$$

where we have defined $q_\varepsilon(s,y) = \varepsilon^{-\alpha} q(\frac{s}{\varepsilon^\beta}, \frac{y}{\varepsilon})$. The Duhamel solution is defined formally as

$$u_\varepsilon(t,x) = \sum_{n=0}^\infty u_{n,\varepsilon}(t,x), \;\; u_{n+1,\varepsilon}(t,x) = \mathcal{H}_\varepsilon u_{n,\varepsilon}(t,x), \;\; u_0(t,x) = e^{-tP(x,D)}[g(x)], \quad (63)$$

where g is the initial condition of the stochastic equation. The same proof that leads to the square integrability of $u(t,x)$ solution of the SPDE also shows that $u_\varepsilon(t,x)$ is well defined as an element in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in time. It remains to analyze the convergence of u_ε.

We consider the case of long range memory effects both in time and space for concreteness. The other cases may be handled similarly. We thus consider the setting where

$$R(t,x) \sim \kappa\varphi(t,x), \quad (t,x) \to \infty, \qquad \varphi(t,x) = \frac{1}{t^{\mathfrak{b}}}\frac{1}{|x|^{\mathfrak{p}}} := \mathbb{E}\{\dot{W}(t-s,x-y)\dot{W}(s,y)\}. \quad (64)$$

Let $\check{R}(\omega,\xi)$ be the power spectrum of $q(t,x)$ in all variables and $\check{\varphi}(\omega,\xi)$ the Fourier transform of $\varphi(t,x)$ in all variables, which is proportional to $|\omega|^{\mathfrak{b}-1}|\xi|^{\mathfrak{p}-d}$. By Bochner's theorem, $\check{R}(\omega,\xi)$ is non-negative and we may define

$$\hat{\rho}(\omega,\xi)=\Big(\frac{\hat{R}(\omega,\xi)}{\check{\varphi}(\omega,\xi)}\Big)^{\frac{1}{2}}. \tag{65}$$

Moreover, by normalization, we find that $\hat{\rho}(0,0)=\sqrt{\kappa}=\sigma$. Let $\rho(t,x)$ be the inverse Fourier transform of $\hat{\rho}(\omega,\xi)$. The construction of ρ, which is real-valued, is such that

$$R(t,x)=\int_{\mathbb{R}^{2d+2}}\rho(t-s,x-y)\varphi(s-\sigma,y-z)\rho(\sigma,z)dsd\sigma dydz. \tag{66}$$

The reason for these calculations is that we are now in a position to define a mollification of $\dot{W}(s,x)$ as follows:

$$\tilde{q}_\varepsilon(t,x)=\int_{\mathbb{R}^{d+1}}\frac{1}{\varepsilon^{d+\beta}}\rho\Big(\frac{t-s}{\varepsilon^\beta},\frac{x-y}{\varepsilon}\Big)\dot{W}(s,y)dsdy. \tag{67}$$

This defines a mean-zero Gaussian process whose covariance function is found, using (66) and (64), to be:

$$\mathbb{E}\{\tilde{q}_\varepsilon(t,x)\tilde{q}_\varepsilon(0,0)\}=\frac{1}{\varepsilon^{\beta\mathfrak{b}+\mathfrak{p}}}R\Big(\frac{t}{\varepsilon^\beta},\frac{x}{\varepsilon}\Big). \tag{68}$$

In other words, $\tilde{q}_\varepsilon$ and q_ε have the same correlation function and hence have the same probability distribution. The solution $\tilde{u}_\varepsilon$ constructed as in (63) with q_ε replaced by $\tilde{q}_\varepsilon$ thus has the same distribution as u_ε. Convergence in distribution of $\tilde{u}_\varepsilon$ to u thus implies convergence in distribution of u_ε to u. It turns out that $\tilde{u}_\varepsilon$ converges strongly to u.

We now drop the ˜ in $\tilde{u}_\varepsilon$ and look at the corrector $u-u_\varepsilon$, where u is the solution of the limiting stochastic PDE. We can show as in [3] that $u_\varepsilon-u$ converges to 0 in the $L^2(\Omega\times\mathbb{R}^d)$ sense uniformly in time. We refer the reader to that paper for additional details and stress here the main differences. Convergence is obtained by analyzing terms of the form

$$\delta I_{\varepsilon,n,m}(t)=\int_{\mathbb{R}^d}\mathbb{E}\{(u_n(t)-u_{n,\varepsilon}(t))(u_m(t)-u_{m,\varepsilon}(t))\}dx,$$

with

$$\begin{aligned}&\delta I_{\varepsilon,n,m}(t)=\\&\int_{\mathbb{R}^d}\prod_{k=0}^{n-1}\int_0^{t_k}\int_{\mathbb{R}^{dn}}\prod_{k=0}^{n-1}G(t_k-t_{k+1},x_k;x_{k+1})\int_{\mathbb{R}^d}G(t_n,x_n;\xi)u_0(\xi)d\xi\\&\prod_{l=0}^{m-1}\int_0^{s_l}\int_{\mathbb{R}^{dm}}\prod_{l=0}^{m-1}G(s_l-s_{l+1},y_l;y_{l+1})\int_{\mathbb{R}^d}G(s_m,y_m;\zeta)u_0(\zeta)d\zeta\delta(x_0-x)\delta(y_0-x)\\&\mathbb{E}\Big\{\Big(\prod_{k=1}^{n}\sigma dW(t_k,x_k)-\prod_{k=1}^{n}q_\varepsilon(t_k,x_k)dt_kdx_k\Big)\Big(\prod_{l=1}^{m}\sigma dW(s_l,y_l)-\prod_{l=1}^{m}q_\varepsilon(s_l,y_l)ds_ldy_l\Big)\Big\}dx.\end{aligned}$$

The above statistical moments involve sums over quantities of the form $dW(t,x)-q_\varepsilon(t,x)dtdx:=(\dot{W}(t,x)-q_\varepsilon(t,x))dtdx$. That such quantities are small in an appropriate sense is based on the following calculations, generalizing [3, Eq. (32)] (where each instance of dW should read σdW. We recall that $\sigma^2=\kappa$.):

$$\begin{aligned}
\mathbb{E}\{\sigma\dot W(t,x)\sigma\dot W(0,0)\} &= \sigma^2\varphi(t,x)\\
\mathbb{E}\{\sigma\dot W(t,x)q_\varepsilon(0,0)\} &= \sigma\int_{\mathbb{R}^{d+1}}\rho\Big(\frac{t-s}{\varepsilon^\beta},\frac{x-y}{\varepsilon}\Big)\frac{1}{\varepsilon^{d+\beta}}\varphi(s,y)dsdy\\
\mathbb{E}\{q_\varepsilon(t,x)q_\varepsilon(0,0)\} &= \frac{1}{\varepsilon^{\beta\mathfrak{b}+\mathfrak{p}}}R\Big(\frac{t}{\varepsilon^\beta},\frac{x}{\varepsilon}\Big).
\end{aligned} \tag{69}$$

Because $\check\rho(0,0)=1$, we verify that $\mathbb{E}\{\sigma\dot W(t,x)q_\varepsilon(0,0)\}$ converges to $\sigma\varphi(t,x)$ weakly (as a distribution) as $\varepsilon\to 0$ and that $\mathbb{E}\{q_\varepsilon(t,x)q_\varepsilon(0,0)\}$ converges to $\sigma^2\varphi(t,x)$ in the same sense. This shows that $\mathbb{E}\{(\sigma\dot W(t,x)-q_\varepsilon(t,x))\sigma\dot W(0,0)\}$ and $\mathbb{E}\{(\sigma\dot W(t,x)-q_\varepsilon(t,x))q_\varepsilon(0,0)\}$ converge to 0 in the same sense. We can now follow the proof of [3, Theorem 4] and obtain that $\delta I_{\varepsilon,n,m}(t)$ converges to 0 uniformly over compact intervals. The only difference with respect to the proof of [3, Theorem 4] is that [3, Eq. (33)] should now be replaced by

$$\int_{\mathbb{R}^{2d}}G(s-s_0,x;\zeta)G(\tau-\tau_0,y;\xi)h_\varepsilon(s-\tau,x-y)dxdy, \tag{70}$$

with $h_\varepsilon(t,x)$ of the form $\mathbb{E}\{(\sigma\dot W(t,x)-q_\varepsilon(t,x))\sigma\dot W(0,0)\}$ or $\mathbb{E}\{(\sigma\dot W(t,x)-q_\varepsilon(t,x))q_\varepsilon(0,0)\}$. Using the explicit expression of $\hat G(t,\xi)=e^{-t|\xi|^{\mathfrak{m}}}$ for the Fourier transform of $G(t,x;y)\equiv G(t,x-y)$, we find for the possible expressions of h_ε that

$$\Big|\int_{\mathbb{R}^{2d}}G(s-s_0,x;\zeta)G(\tau-\tau_0,y;\xi)h_\varepsilon(s-\tau,x-y)dxdy\Big|\leq\frac{C\varepsilon^\gamma}{((t-\tau)+(u-v))^{\frac{\mathfrak{p}}{\mathfrak{m}}}|t-u|^{\mathfrak{b}+\eta}} \tag{71}$$

for some positive values of γ and η with η arbitrary small (with then γ small as well). This shows that the contribution in (70) is small and as in [3] that $\delta I_{\varepsilon,n,m}(t)$ converges to 0 uniformly over compact intervals. This concludes the derivation of the convergence result.

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