Lecture Notes. Waves in Random Media

Guillaume Bal [1]

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[1]Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027; gb2030@columbia.edu

Contents

Introduction

This set of notes covers the material introduced in the course on Waves in Random Media taught in the fall of 2005 at Columbia University. The first chapter covers fundamental aspects of wave equations and includes a (partial) theory for first-order hyperbolic systems of equations. The second chapter concerns the **effective medium** theory of wave equations. This corresponds to *low frequency waves* propagating in *high frequency media*. Chapter three analyzes *high frequency waves* in *low frequency media* by the method of **geometric optics**.

Random media are introduced in Chapter four, which is devoted to perturbation analyses of *low frequency waves* in *low frequency random media*. The rest of the notes concerns the analysis of *high frequency waves* in *high frequency media*, which means that both the typical wavelength and the typical correlation length of the underlying media are small compared to the size of the system.

Chapter five introduces the main tool of (micro-local) analysis of **high frequency** waves used in these notes, namely the **Wigner transform**. The **semiclassical** analysis of quantum waves, an example of *high frequency waves* in *low frequency media*, is analyzed using the framework of Wigner transforms.

The Wigner transform, which offers a description of the wave energy density, at least asymptotically, in the *phase space*, is used in Chapter six to derive **radiative transfer equations** from a two-by-two system of acoustic wave equations. The radiative transfer equations provide a phase space description of the propagation of the acoustic energy density in random media characterized by a regime called the **weak-coupling regime**. Thought it has solid foundations, the derivation of radiative transfer is formal (i.e., is not justified rigorously mathematically).

Chapter seven addresses the derivation of radiative transfer equations for an approximation of acoustic wave propagation: the **paraxial (or parabolic) equation**. In this approximation, waves primarily propagate in privileged direction. By modeling randomness as a *Markov* process in that direction of propagation, the equations of radiative transfer can be justified rigorously in this simplified setting.

Chapter 1

Wave equations and First-order hyperbolic systems

1.1 Introduction

This chapter presents classical wave propagation models, including first-order hyperbolic systems of equations, the theory of which is briefly presented. References useful to this chapter include [15, 21, 26].

1.2 Wave equations

1.2.1 Acoustic waves

First-order symmetric hyperbolic system The linear system of acoustic wave equations for the pressure $p(t,\mathbf{x})$ and the velocity field $\mathbf{v}(t,\mathbf{x})$ takes the form of the following first-order hyperbolic system

$$\begin{aligned} \rho(\mathbf{x})\frac{\partial \mathbf{v}}{\partial t} + \nabla p &= 0, \qquad t>0, \quad \mathbf{x}\in\mathbb{R}^d, \\ \kappa(\mathbf{x})\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{v} &= 0, \qquad t>0, \quad \mathbf{x}\in\mathbb{R}^d, \\ p(0,\mathbf{x}) &= p_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d, \\ \mathbf{v}(0,\mathbf{x}) &= \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $\rho(\mathbf{x})$ is density and $\kappa(\mathbf{x})$ compressibility. Both quantities are assumed to be uniformly bounded from below by a positive constant. Our notation is that $\nabla = (\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_d})^t$ and that $\nabla\cdot$ is the negative of the adjoint operator for the usual L^2 inner product, i.e., $\nabla\cdot\mathbf{u} = \sum_{j=1}^d \frac{\partial u_j}{\partial x_j}$.

Acoustic energy conservation is characterized by the fact that

$$\mathcal{E}_B(t) = \frac{1}{2}\int_{\mathbb{R}^d} (\rho(\mathbf{x})|\mathbf{v}|^2(t,\mathbf{x}) + \kappa(\mathbf{x})p^2(t,\mathbf{x}))d\mathbf{x} = \mathcal{E}_B(0). \tag{1.2}$$

Exercise 1.2.1 Derive this for sufficiently smooth solutions.

We now know that total energy is conserved. The role of a kinetic model is to describe its spatial distribution (at least asymptotically). This course's main objective is precisely the derivation of such kinetic models.

Scalar wave equation The pressure $p(t,\mathbf{x})$ also solves following closed form scalar equation

$$\begin{array}{rcll} \dfrac{\partial^2 p}{\partial t^2} & = & \dfrac{1}{\kappa(\mathbf{x})}\nabla\cdot\dfrac{1}{\rho(\mathbf{x})}\nabla p, & t>0,\ \mathbf{x}\in\mathbb{R}^d, \\ p(0,\mathbf{x}) & = & g(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d \\ \dfrac{\partial p}{\partial t}(0,\mathbf{x}) & = & h(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d. \end{array} \tag{1.3}$$

We verify that energy conservation takes the form

$$\mathcal{E}_H(t)=\frac{1}{2}\int_{\mathbb{R}^d}\big(\kappa(\mathbf{x})(\frac{\partial p}{\partial t})^2(t,\mathbf{x})+\frac{|\nabla p|^2(t,\mathbf{x})}{\rho(\mathbf{x})}\big)d\mathbf{x}=\mathcal{E}_H(0). \tag{1.4}$$

Both conservation laws (1.2) and (1.4) are equivalent.

Exercise 1.2.2 Prove this statement. *Hint:* define the pressure potential $\phi(t,\mathbf{x})$ as a solution to (1.3) and then $(\mathbf{v},p)=(-\rho^{-1}\nabla\phi,\partial_t\phi)$; show that $(\mathbf{v},p)$ solves (1.1) and that $\mathcal{E}_H[\phi](t)=\mathcal{E}_B[\mathbf{v},p](t)$.

A non-symmetric two by two system. In this paragraph we assume to simplify that $\rho(\mathbf{x})=\rho_0$ is constant and define the sound speed

$$c(\mathbf{x})=\frac{1}{\sqrt{\rho_0\kappa(\mathbf{x})}}. \tag{1.5}$$

Let us consider

$$q(t,\mathbf{x})=c^{-2}(\mathbf{x})\frac{\partial p}{\partial t}(t,\mathbf{x}). \tag{1.6}$$

Then $\mathbf{u}=(p,q)^t$ solves the following 2×2 system

$$\begin{array}{rcll} \dfrac{\partial \mathbf{u}}{\partial t}+A\mathbf{u} & = & 0, & t>0,\ \mathbf{x}\in\mathbb{R}^d, \\ \mathbf{u}(0,\mathbf{x}) & = & (g(\mathbf{x}),c^{-2}(\mathbf{x})h(\mathbf{x}))^t, & \mathbf{x}\in\mathbb{R}^d. \end{array} \tag{1.7}$$

where

$$A=-\begin{pmatrix}0 & c^2(\mathbf{x})\\ \Delta & 0\end{pmatrix}=J\ \Lambda(\mathbf{x}),\quad J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \Lambda(\mathbf{x})=\begin{pmatrix}-\Delta & 0\\ 0 & c^2(\mathbf{x})\end{pmatrix}. \tag{1.8}$$

Note that J is a skew-symmetric matrix ($J^t=-J$) and that Λ is a symmetric matrix-valued operator for the usual L^2 scalar product:

$$(\mathbf{u},\mathbf{v})=\int_{\mathbb{R}^d}\mathbf{v}^*(\mathbf{x})\mathbf{u}(\mathbf{x})d\mathbf{x}, \tag{1.9}$$

where $\mathbf{v}^*$ is the complex-conjugated (for complex-valued vectors) transpose vector to $\mathbf{v}$. For real-valued vectors, we find that $\mathbf{v}^*(\mathbf{x})\mathbf{u}(\mathbf{x})=\mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})$.

Exercise 1.2.3 Show that Λ is a symmetric operator, i.e., show (by integrations by parts assuming sufficient regularity for $\mathbf{u}$ and $\mathbf{v}$) that

$$(\Lambda\mathbf{u},\mathbf{v})=(\mathbf{u},\Lambda\mathbf{v}).$$

Exercise 1.2.4 Write a two-by-two system for (p,p_t). Here and below we use alternatively p_t and $\frac{\partial p}{\partial t}$ to denote partial derivative with respect to the t variable.

Exercise 1.2.5 Show that the following quantity is conserved:

$$\mathcal{E}(t)=\frac{1}{2\rho_0}\int_{\mathbb{R}^d}\mathbf{u}\Lambda\mathbf{u}d\mathbf{x}=\mathcal{E}(0).$$

Relate this conserved quantity to those in (1.2) and (1.4).

1.2.2 Elastic and Electromagnetic waves

See [28].

1.2.3 Schrödinger equation

The Schrödinger equation takes the form

$$\begin{aligned} &i\frac{\partial \psi}{\partial t} + \frac{1}{2}\Delta\psi - V(\mathbf{x})\psi = 0, \qquad t > 0,\ \mathbf{x} \in \mathbb{R}^d \\ &\psi(0,\mathbf{x}) = \psi_0(\mathbf{x}). \end{aligned} \tag{1.10}$$

Here $V(\mathbf{x})$ is a real-valued potential, which we assume e.g. with compact support to simplify. We have the conservation of the number of particles:

$$N(t) = \int_{\mathbb{R}^d} |\psi(t,\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^d} |\psi_0(\mathbf{x})|^2 d\mathbf{x}. \tag{1.11}$$

Exercise 1.2.6 Verify this for sufficiently smooth solutions of (1.10).

1.3 First-order symmetric hyperbolic systems

The equations of acoustics, electromagnetism, and elasticity can all be put in the framework of first-order symmetric hyperbolic systems.

Let $\mathbf{u}$ be an $n-$vector-valued function in the sense that $\mathbf{u}(t,\mathbf{x})$ is a vector with n components for each $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^d$. We consider the equation

$$\begin{aligned} &A_0(\mathbf{x})\frac{\partial \mathbf{u}}{\partial t} + A_j(\mathbf{x})\frac{\partial \mathbf{u}}{\partial x_j} = 0, && t > 0,\ \mathbf{x} \in \mathbb{R}^d \\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), && \mathbf{x} \in \mathbb{R}^d, \end{aligned} \tag{1.12}$$

where we use the convention of summation over repeated indices (so that the above left-hand side is a summation over $1 \leq j \leq d$). We assume that $A_0(\mathbf{x})$ is a smooth (in $\mathbf{x}$) positive-definite symmetric matrix for all $\mathbf{x} \in \mathbb{R}^d$ and that the $A_j(\mathbf{x})$, $1 \leq j \leq n$ are smooth symmetric matrices for all $\mathbf{x} \in \mathbb{R}^d$.

Exercise 1.3.1 (i) Show that the equations of acoustics, electromagnetism, and elasticity can be put in the framework (1.14) with matrices A_j, $1 \leq j \leq d$, that are moreover constant.
(ii) Assuming that A_j, $1 \leq j \leq d$, are constant show that

$$\frac{d}{dt}(\mathbf{u}(t,\cdot),\mathbf{u}(t,\cdot))_{A_0} = 0, \qquad \text{where} \quad (\mathbf{u},\mathbf{u})_{A_0} = (A_0\mathbf{u},\mathbf{u}). \tag{1.13}$$

Show that there is at most one solution to (1.12). Relate this to the energy conservation (1.2).

Since $A_0(\mathbf{x})$ is symmetric positive definite, it admits a square root $A_0^{1/2}(\mathbf{x})$ and we define the quantity $\mathbf{v} = A_0^{1/2}\mathbf{u}$. We verify that $\mathbf{v}$ satisfies the system

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} + B_j(\mathbf{x})\frac{\partial \mathbf{v}}{\partial x_j} + B_0(\mathbf{x})\mathbf{v} = 0, && t > 0,\ \mathbf{x} \in \mathbb{R}^d \\ &\mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}), && \mathbf{x} \in \mathbb{R}^d, \end{aligned} \tag{1.14}$$

where we have defined the matrices

$$B_j = A_0^{-1/2}A_jA_0^{-1/2},\ 1 \leq j \leq n, \quad B_0 = \sum_{j=1}^{d} A_0^{-1/2}A_j\big(\frac{\partial A_0^{-1/2}}{\partial x_j}\big). \tag{1.15}$$

The matrices B_j for $1 \leq j \leq n$ are now symmetric matrices. Note that $B_0(\mathbf{x})\mathbf{v}$ is a bounded operator (for instance in the L^2-sense). We can thus recast (1.12) into a system of the form (1.14), which is more commonly analyzed in the mathematical literature.

1.3.1 Case of constant coefficients

In the case of constant matrices A_j $0 \leq j \leq d$, the above equation takes the form

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + B_j \frac{\partial \mathbf{u}}{\partial x_j} = 0, && t > 0,\, \mathbf{x} \in \mathbb{R}^d \\ &\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), && \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{1.16}$$

The result in Exercise 1.3.1 shows that the energy $(\mathbf{u}, \mathbf{u})$ is a conserved quantity in time. Let us define the Fourier transform as

$$\mathcal{F}_{\mathbf{x}\to\mathbf{k}} u(\mathbf{k}) \equiv \hat{u}(\mathbf{k}) = \int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{x}} u(\mathbf{x}) d\mathbf{x}, \tag{1.17}$$

with inverse transformation the inverse Fourier transform

$$\mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}} \hat{u}(\mathbf{x}) \equiv u(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{u}(\mathbf{k}) d\mathbf{k}. \tag{1.18}$$

The Fourier transform is defined on vectors component by component. Using the classical result

$$\mathcal{F}[\frac{\partial u}{\partial x_j}](\mathbf{k}) = i k_j \hat{u}(\mathbf{k}), \tag{1.19}$$

we deduce that (1.16) may formally be recast in the Fourier domain as

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + iA(\mathbf{k})\hat{\mathbf{u}} = 0, \quad \hat{\mathbf{u}}(0, \mathbf{k}) = \hat{\mathbf{u}}_0(\mathbf{k}), \tag{1.20}$$

where we have defined the *dispersion* matrix

$$A(\mathbf{k}) = \sum_{j=1}^{d} k_j B_j. \tag{1.21}$$

As usual, the Fourier transform recasts constant-coefficient partial differential equations into ordinary differential equations, which can be solved to yield

$$\hat{\mathbf{u}}(t, \mathbf{k}) = \exp\big(-iA(\mathbf{k})t\big)\hat{\mathbf{u}}_0(\mathbf{k}), \qquad \mathbf{k} \in \mathbb{R}^d. \tag{1.22}$$

We have thus constructed a solution to (1.16) of the form

$$\mathbf{u}(t, \mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k} - iA(\mathbf{k})t} \mathbf{u}_0(\mathbf{y}) d\mathbf{k} d\mathbf{y}. \tag{1.23}$$

It remains to show that the above integral makes sense, for instance in the L^2 setting, which we leave as an exercise:

Exercise 1.3.2 Show that $\mathbf{u}(t, \mathbf{x})$ is uniformly bounded in time in $(L^2(\mathbb{R}^d))^n$ for an initial condition $\mathbf{u}_0 \in (L^2(\mathbb{R}^d))^n$. *Hint*: Since $A(\mathbf{k})$ is symmetric, we can decompose it as

$$A(\mathbf{k}) = \sum_{m=1}^{n} \lambda_m(\mathbf{k}) \mathbf{b}_m(\mathbf{k}) \mathbf{b}_m^*(\mathbf{k}), \tag{1.24}$$

for some real-valued eigenvalues $\lambda_m(\mathbf{k})$ and for some eigenvectors $\mathbf{b}_m(\mathbf{k})$. Show then the decomposition

$$e^{iA(\mathbf{k})t} = \sum_{m=1}^{n} e^{i\lambda_m(\mathbf{k})t}\mathbf{b}_m(\mathbf{k})\mathbf{b}_m^*(\mathbf{k}), \tag{1.25}$$

and deduce that

$$|\hat{\mathbf{u}}(t,\mathbf{k})| = |\hat{\mathbf{u}}_0(\mathbf{k})|, \tag{1.26}$$

for all $\mathbf{k} \in \mathbb{R}^d$ (use the fact that the $\mathbf{b}_m$ form an orthonormal basis on $\mathbb{R}^n$). The latter equality shows that $e^{iA(\mathbf{k})t}$ is a *unitary* operator, i.e., is norm-preserving. Conclude by using the Parseval relation

$$(2\pi)^d \int_{\mathbb{R}^d} u(\mathbf{x})v^*(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^d} \hat{u}(\mathbf{k})\hat{v}^*(\mathbf{k})d\mathbf{k}. \tag{1.27}$$

A similar results holds when $A(\mathbf{k})$ has real-valued eigenvalues but is not necessarily symmetric. We refer to [15, §7.3.3] for the more involved derivation and for additional results on the regularity of the constructed solution $\mathbf{u}(t,\mathbf{x})$. Note that since $\mathbf{u}(t,\mathbf{x})$ is unique within the set of uniformly bounded functions in the L^2 sense, (1.23) provides the unique solution to (1.16).

1.3.2 Plane Wave solutions

Following the notation in [26], we denote by $\mathbf{y} = (y_0, \ldots, y_d) = (t, \mathbf{x})$ the variable in $\mathbb{R}^{d+1}$. Consider the slightly more general first-order hyperbolic system

$$\begin{aligned} &L(\mathbf{D})\mathbf{u}(\mathbf{y}) = 0, \qquad t > 0,\ \mathbf{x} \in \mathbb{R}^d \\ &L(\mathbf{D}) = \sum_{m=0}^{d} A_m \frac{\partial}{\partial y_m}. \end{aligned} \tag{1.28}$$

and look for solutions of the form

$$\mathbf{u}(\mathbf{y}) = \mathbf{f}(\mathbf{y}\cdot\boldsymbol{\eta}), \qquad \mathbf{f} : \mathbb{R} \to \mathbb{C}^n. \tag{1.29}$$

We compute that

$$L(\mathbf{D})\mathbf{f}(\mathbf{y}\cdot\boldsymbol{\eta}) = L(\boldsymbol{\eta})\mathbf{f}'(\mathbf{y}\cdot\boldsymbol{\eta}), \qquad \text{where} \qquad L(\boldsymbol{\eta}) = \sum_{m=0}^{d} \eta_m A_m.$$

Here, $L(\boldsymbol{\eta})$ is the *symbol* of the constant-coefficient operator $L(\mathbf{D})$. Note the following equality in terms of operators

$$L(\mathbf{D}) = \mathcal{F}^{-1}_{\boldsymbol{\eta}\to\mathbf{y}} L(i\boldsymbol{\eta})\mathcal{F}_{\mathbf{y}\to\boldsymbol{\eta}}.$$

For a function $\mathbf{f}$ whose derivative never vanishes (such as e.g. $\mathbf{f}(\mu) = e^{i\mu}\mathbf{a}$ for some vector $\mathbf{a} \in \mathbb{R}^{d+1}$), we see that $L(\mathbf{D})\mathbf{f}(\mathbf{y}\cdot\boldsymbol{\eta}) = 0$ implies that $L(\boldsymbol{\eta})$ is not invertible. This is equivalent to saying that

$$\det L(\boldsymbol{\eta}) = 0. \tag{1.30}$$

The above equation is called the *characteristic equation* of the operator $L(D)$. More specifically for $\boldsymbol{\eta} = (\tau, k_1, \ldots, k_d)$, we find that the characteristic equation is given by

$$\det\big(A_0\tau + \sum_{j=1}^{d} k_j A_j\big) = 0, \tag{1.31}$$

which is equivalent to saying that

$$-\tau \in \sigma\big(A_0^{-1}\sum_{j=1}^{d} k_j A_j\big) = \sigma\big(\sum_{j=1}^{d} A_0^{-1/2} k_j A_j A_0^{-1/2}\big). \tag{1.32}$$

Here $\sigma(A)$ stands for the spectrum of the matrix A. For plane wave solutions of the form $e^{i\boldsymbol{\eta}\cdot\mathbf{y}}\mathbf{a}$, we observe that $\mathbf{a}$ is then in the kernel of the operator $L(\boldsymbol{\eta})$, or equivalently that

$$A_0\tau\mathbf{a} + \sum_{j=1}^{d} k_j A_j \mathbf{a} = 0. \tag{1.33}$$

For each τ defined in (1.32) we thus obtain an *eigenvector* $A_0^{1/2}\mathbf{a}$ of the matrix $A_0^{-1/2} k_j A_j A_0^{-1/2}$. As we shall see later, the largest τ of these eigenvalues determines the fastest speed of propagation in the system. The associated plane waves are those propagating with the fastest speed.

More specifically, let us consider plane waves of the form $e^{i(\tau t+\mathbf{k}\cdot\mathbf{x})}\mathbf{a}$ with $\mathbf{k}$ real-valued (which corresponds to bounded "physical" plane waves). Then the characteristic equation takes the form

$$\det L(i\tau, i\mathbf{k}) = 0. \tag{1.34}$$

For fixed $\mathbf{k}$, the n solutions $\tau_j(\mathbf{k})$ define the *dispersion relations* of the system of equations. The associated eigenvectors $\mathbf{b}_j$ are the propagating plane waves. For conservative equations (i.e., when some energy is conserved), the roots $\tau_j(\mathbf{k})$ are real-valued so that the initial condition $e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{a}$ is simply translated:

$$\mathbf{u}(t,\mathbf{x}) = e^{i(\tau t+\mathbf{k}\cdot\mathbf{x})}\mathbf{a} = \mathbf{u}_0(\mathbf{x}+t\mathbf{v}), \qquad \text{where e.g.} \quad \mathbf{v} = \frac{\tau}{|\mathbf{k}|}\hat{\mathbf{k}}, \quad \hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}. \tag{1.35}$$

The *phase velocity* is defined up to addition of a vector orthogonal to $\mathbf{k}$ as is easily verified. When a root $\tau_j(\mathbf{k})$ is a multiple eigenvalue, the different eigenvectors are referred to as the *polarization* modes associated to that specific eigenvalue.

Exercise 1.3.3 Relate the above construction to the derivation in section 1.3.1, in particular (1.20) and (1.24).

Exercise 1.3.4 (i) Find the characteristic equation, the dispersion relations and the corresponding eigenvectors for the system of acoustic equations (1.1).
(ii) Do the same exercise for Maxwell's equations and show that there are two modes of polarization associated to each propagating mode (such that $\tau_j \neq 0$) in space dimension $d = 3$. (You may verify that there are $d(d-1)/2$ modes of polarization in each dimension $d \geq 2$; see e.g. [3, §8].)

1.3.3 Case of spatially varying coefficients

Still denoting by $\mathbf{y} = (y_0, \ldots, y_d) = (t, \mathbf{x})$ the variable in $\mathbb{R}^{d+1}$, we define the first-order operator

$$L(\mathbf{y}, \mathbf{D}) = \sum_{m=0}^{d} A_m(\mathbf{y})\frac{\partial}{\partial y_m} + B(\mathbf{y}) = A_0(\mathbf{y})\frac{\partial}{\partial t} + G(\mathbf{y}, \mathbf{D}). \tag{1.36}$$

The above operator is called *symmetric hyperbolic* if the matrices $A_m(\mathbf{y})$ are smooth symmetric matrices (which are allowed to depend on the temporal variable as well), $B(\mathbf{y})$ is a smooth

matrix, and $A_0(\mathbf{y})$ is in addition uniformly positive definite: $A_0(\mathbf{y}) \geq \alpha_0 I$ for some $\alpha_0 > 0$ independent of $\mathbf{y}$. The first-order system of equations associated to L is thus

$$\begin{aligned} &L(\mathbf{y}, \mathbf{D})\mathbf{u} = 0, \qquad && t > 0,\ \mathbf{x} \in \mathbb{R}^d \\ &\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \qquad && \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{1.37}$$

When A_m is constant for $m \geq 1$ and $B = 0$, we have seen that $(A_0\mathbf{u}, \mathbf{u})$ is conserved when $\mathbf{u}$ is a solution of $L\mathbf{u} = 0$.

Exercise 1.3.5 Show that the same energy is conserved when B is such that $B + B^* = 0$.

Energy conservation no longer necessarily holds in more general situations. However a very useful energy estimate still holds.

Let $G(t)$ be an operator acting on $(L^2(\mathbb{R}^d)^n)$ such that $G(t) + G^*(t)$ is uniformly bounded in time:

$$\|G(t) + G^*(t)\| \leq 2C. \tag{1.38}$$

We recall that $G^*(t)$ is defined as the operator such that for all $\mathbf{u}$ and $\mathbf{v}$ in $(L^2(\mathbb{R}^d))^n$ we have

$$(G\mathbf{u}, \mathbf{v}) = (\mathbf{u}, G^*\mathbf{v}). \tag{1.39}$$

Let us consider the problem

$$A_0(\mathbf{y})\frac{\partial \mathbf{u}}{\partial t} + G(t)\mathbf{u} = 0, \qquad t > 0, \tag{1.40}$$

with some initial conditions $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x})$. Then we observe that

$$\|\mathbf{u}(t)\|_{A_0} \leq e^{ct/\alpha_0}\|\mathbf{u}_0\|_{A_0} \tag{1.41}$$

for some constant c, where the above norm is defined such that

$$\|\mathbf{u}\|_{A_0}^2 = (A_0\mathbf{u}, \mathbf{u}) = (\mathbf{u}, \mathbf{u})_{A_0}$$

Proof. Let us first recast (1.40) as

$$\frac{\partial}{\partial t}(A_0(\mathbf{y})\mathbf{u}) + \tilde{G}(t)\mathbf{u} = 0, \qquad t > 0, \tag{1.42}$$

where $\tilde{G} = G - \partial_t A_0(\mathbf{y})$ is clearly a bounded operator with constant C in (1.38) replaced by another constant c.

Upon multiplying (1.42) by $\mathbf{u}(t, \mathbf{x})$, integrating over $\mathbb{R}^d$ and performing a few integrations by parts, we get

$$0 = \frac{d}{dt}(A_0\mathbf{u}, \mathbf{u}) + 2(\tilde{G}(t)\mathbf{u}, \mathbf{u}) = \frac{d}{dt}(A_0\mathbf{u}, \mathbf{u}) + ((\tilde{G}(t) + \tilde{G}^*(t))\mathbf{u}, \mathbf{u}),$$

so that

$$\frac{d}{dt}(\mathbf{u}, \mathbf{u})_{A_0} \leq 2c(\mathbf{u}, \mathbf{u}) \leq \frac{2c}{\alpha_0}(\mathbf{u}, \mathbf{u})_{A_0}.$$

Now we deduce from $u' \leq hu$ with $u(t) \geq 0$ that $u(t) \leq e^{ht}u(0)$. This concludes the proof. □

This result may be generalized to more complicated equations [26]. The a priori bound (1.41) shows that the solution to (1.37) is unique since L is a linear operator. Indeed we verify that $G + G^*$ is bounded for G defined in (1.36).

Exercise 1.3.6 Calculate G^* using (1.39) and verify that $G + G^*$ is indeed bounded for G defined in (1.36). Note that both G and G^* are first-order differential operators and thus are not bounded on $(L^2(\mathbb{R}^d))^n$.

The same type of proof, using Gronwall's inequality, provides the following a priori estimate:

Theorem 1.3.1 *For every $s \in \mathbb{R}$, there is a constant C such that for all $\mathbf{u}(t, \mathbf{x})$ a sufficiently smooth function with compact support in space uniformly on $0 \leq t \leq T$ for all $T > 0$, we have the a priori estimate*

$$\|\mathbf{u}(t)\|_{H^s(\mathbb{R}^d)} \leq Ce^{Ct}\|\mathbf{u}(0)\|_{H^s(\mathbb{R}^d)} + \int_0^t Ce^{C(t-s)}\|L\mathbf{u}(\sigma)\|_{H^s(\mathbb{R}^d)}d\sigma. \tag{1.43}$$

Exercise 1.3.7 Prove the above theorem for $s = 0$ using a modification of the above result and Gronwall's inequality (see below). Next, prove the result for all integer s. The result then holds for all $s \in \mathbb{R}$ by interpolation.

We recall that Gronwall's inequality states the following:

Lemma 1.3.2 (Gronwall's inequality) *(i) Let $u(t)$ be a non-negative, continuous function on $[0, T]$ and $\varphi(t)$ and $\phi(t)$ non-negative integrable functions on $[0, T]$ such that the following inequality holds*

$$\dot{u}(t) \leq \varphi(t)u(t) + \phi(t). \tag{1.44}$$

Then

$$u(t) \leq e^{\int_0^t \varphi(s)ds}\big(u(0) + \int_0^t \phi(s)ds\big), \quad \text{for all } 0 \leq t \leq T. \tag{1.45}$$

(ii) In integral form, let $\xi(t)$ be a non-negative, integrable function on $[0, T]$ such that for constants C_1 and C_2,

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2. \tag{1.46}$$

Then for $0 \leq t \leq T$, we have

$$\xi(t) \leq C_2(1 + c_1 te^{C_1 t}). \tag{1.47}$$

The same type of bounds is also useful in showing existence of a solution to (1.37). Since the explicit construction in the Fourier domain is no longer available, the existence result (due to Friedrichs for general first-order hyperbolic systems) is more involved technically. The main idea is to construct approximate solutions and then use a priori estimates of the form (1.41) to pass to the limit and obtain a solution to (1.37). We refer to [15, §7.3.2] for a construction based on the vanishing viscosity method and the existence of solutions for the heat equation; to [26, §2.2] for a construction based on finite discretizations of the hyperbolic system; and to [21] for a construction based on the density of functions of the form $Lu = 0$ (which uses in a crucial way the property of finite speed of propagation that we now consider).

1.3.4 Finite speed of propagation

A very important property of first-order hyperbolic systems is that information propagates at finite speed. Let us recast the first-order hyperbolic system as

$$\begin{aligned} 0 &= \sum_{m=0}^{d} A_m \frac{\partial \mathbf{u}}{\partial y_m} + B\mathbf{u} \\ 0 &= \sum_{m=0}^{d} \frac{\partial \mathbf{u}^*}{\partial y_m} A_m + \mathbf{u}^* B^*, \end{aligned}$$

using the symmetry of A_m, $m \geq 0$. Upon multiplying the first equation on the left by $\mathbf{u}^*$ and the second equation on the right by $\mathbf{u}$ and summing the results, we obtain that

$$0 = \sum_{m=0}^{d} \frac{\partial}{\partial y_m}(\mathbf{u}^* A_m \mathbf{u}) + \mathbf{u}^* Z \mathbf{u}, \qquad Z = \sum_{m=0}^{d} \frac{\partial A_m}{\partial y_m} - B - B^*. \tag{1.48}$$

Note that Z is a bounded operator and that the first term is written in divergence form.

Let $\mathbf{y} \in \mathbb{R}^{d+1}$ and $\mathbf{k} \in \mathbb{R}^d$ such that $|\mathbf{k}| = 1$. We define

$$r(\mathbf{y}, \mathbf{k}) = \rho(A_0^{-1}(\mathbf{y}) A_j(\mathbf{y}) k_j) = \rho(A_0^{-1/2}(\mathbf{y}) A_j(\mathbf{y}) k_j A_0^{-1/2}(\mathbf{y})), \tag{1.49}$$

where $\rho(A)$ denotes spectral radius (here the largest eigenvalue in modulus). Verify that both spectral radii indeed agree. Now the maximal speed of propagation c is defined as

$$c = \sup_{\mathbf{y} \in \mathbb{R}^{d+1} \text{ and } |\mathbf{k}|=1} r(\mathbf{y}, \mathbf{k}). \tag{1.50}$$

The above definition implies that $k_j A_j \leq c A_0$ as symmetric matrices for $|\mathbf{k}| = 1$ so that for arbitrary vectors $\boldsymbol{\eta} \in \mathbb{R}^d$, we have $|\boldsymbol{\eta}| c A_0 + \eta_j A_j \geq 0$. This implies that c is the smallest number with the property that

$$\eta_0 \geq |\beta| \quad \Rightarrow \quad \sum_{m=0}^{d} \eta_m A_m \geq 0.$$

You may want to check this assertion carefully. This shows that if "enough" of the positive operator A_0 is added to the operator $\sum_{j=1}^{d} k_j A_j$, then we obtain a positive operator as well. This has the following interesting consequence.

Let $\mathbf{a} \in \mathbb{R}^d$ and $R > 0$. We define the (truncated) cone $\Omega(\mathbf{a}, R)$ as

$$\Omega(\mathbf{a}, R) = \{(t, \mathbf{x}), \text{ s.t. } 0 \leq t \leq R/c \text{ and } |\mathbf{x} - \mathbf{a}| < R - ct\}. \tag{1.51}$$

Here c is the maximal speed defined in (1.50). We define the sections $\Omega(\mathbf{a}, R, t)$ as

$$\Omega(t) \equiv \Omega(\mathbf{a}, R, t) = \{\mathbf{x}, \text{ s.t. } (\mathbf{x}, t) \in \Omega(\mathbf{a}, R)\}. \tag{1.52}$$

Then we have the result first proved by Friedrichs:

Theorem 1.3.3 *Suppose that* $\mathbf{u}$ *is a smooth solution of the equation*

$$L\mathbf{u}(\mathbf{y}) = 0, \tag{1.53}$$

such that $\mathbf{u}(0, \mathbf{x}) = 0$ *in the ball* $\{|\mathbf{x} - \mathbf{a}| < R\}$. *Then* $\mathbf{u} = 0$ *in* $\Omega(\mathbf{a}, R)$.

Proof. For $0 \leq t \leq R/c$, define

$$\varphi(t) = \int_{\Omega(t)} \mathbf{u}^*(\mathbf{y}) A_0(\mathbf{y}) \mathbf{u}(\mathbf{y}) d\mathbf{x}.$$

In particular, $\varphi(0) = 0$ thanks to the initial condition. Recall that

$$0 = \frac{\partial}{\partial y_m}(\mathbf{u}^* A_m \mathbf{u}) + \mathbf{u}^* Z \mathbf{u}$$

Let us integrate the above equation spatially over $\Omega_t = \Omega(\mathbf{a}, R) \cap (0, t)$. We thus integrate over a truncated cone. We find that because A_0 is uniformly positive and Z is a bounded operator that

$$\varphi(t) - \varphi(0) + \int_0^t \int_{|\mathbf{x}-\mathbf{a}|=R-cs} \mathbf{u}^* \sum_{m=0}^{d} \eta_m A_m \mathbf{u} d\sigma(\mathbf{y}) \leq C \int_0^t \varphi(s) ds.$$

Here, $(\eta_0, \ldots, \eta_d) \in \mathbb{R}^{d+1}$ is the outward normal to the truncated cone at times $0 < s < t$.

Exercise 1.3.8 Verify the above formula in detail. In particular, show that the boundary of Ω_t is composed of the lateral surface $\{|\mathbf{x}-\mathbf{a}| = R - cs,\ 0 < s < t\}$ and of $\Omega(t)$ and $\Omega(0)$; then apply the divergence theorem on Ω_t to find the result.

The speed c is defined exactly so that $\sum_{m=0}^{d} \eta_m A_m$ is a positive operator. We thus obtain that

$$\varphi(t) \leq C \int_0^t \varphi(s) ds.$$

We conclude by using Gronwall's lemma that $\varphi(t) \equiv 0$, which implies that $\mathbf{u} = 0$ on $\Omega(\mathbf{a}, R)$. □

The same type of proof provides the following important local estimate:

Theorem 1.3.4 *For all $\mathbf{a} \in \mathbb{R}^d$, $R > 0$, $T > 0$ such that $0 \leq T \leq R/c$, there is a constant $C = C(L)$ such that for all sufficiently smooth functions $\mathbf{u}(t, \mathbf{x})$, we have*

$$\|\mathbf{u}(t)\|_{L^2(\Omega(t))} \leq C\big(\|\mathbf{u}(0)\|_{L^2(\Omega(0))} + \int_0^t \|L\mathbf{u}(s)\|_{L^2(\Omega(s))} ds\big). \tag{1.54}$$

Exercise 1.3.9 Prove the above theorem and relate it to Theorem 1.3.1.

1.3.5 Characteristic equation and dispersion relation

The characteristic equation of first-order operators with constant coefficients was introduced in (1.30). We now generalize this notion to spatially varying operators. The main idea is the following. Consider operators that vary smoothly and "not-too-fast" in space and let them act on sufficiently highly-oscillatory functions. Then at the scale of the fast oscillations, one may as a first approximation assume that the spatial coefficients of the differential operator are frozen (i.e., constant). This is the whole purpose of geometric optic to make this statement more precise. In any event this justifies the introduction of the following notion.

Let L be a first-order (to simplify) operator of the form

$$L(\mathbf{y}, \mathbf{D}) = \sum_{m=0}^{d} A_m(\mathbf{y}) \frac{\partial}{\partial y_m} + B(\mathbf{y}) = L_1(\mathbf{y}, \mathbf{D}) + B(\mathbf{y}). \tag{1.55}$$

Here L_1 is thus the "leading" order operator in L, which accounts for all highest-order derivatives (of order one here).

Definition 1.3.5 *The characteristic variety of L, denoted by Char L is the set of pairs $(\mathbf{y}, \boldsymbol{\eta}) \in \mathbb{R}^{d+1} \times (\mathbb{R}^{d+1} \backslash \mathbf{0})$ such that*

$$\det L_1(\mathbf{y}, \boldsymbol{\eta}) = 0. \tag{1.56}$$

Here again, $L_1(\mathbf{y}, \boldsymbol{\eta})$ is the symbol of $L_1(\mathbf{y}, \mathbf{D})$, where each derivative in y_m, $0 \leq m \leq d$ is replaced by $i\eta_m$. For sufficiently high frequencies, $L_1(\mathbf{y}, \boldsymbol{\eta})$ has a much greater effect than $B(\mathbf{y}, \boldsymbol{\eta})$ (the symbol of B), since the former is linear in $i\eta_m$ whereas the latter is bounded independent of $i\eta_m$. It is therefore useful to consider the dispersion relation associated to $L_1(\mathbf{y}, \boldsymbol{\eta})$.

For a "macroscopic" $\mathbf{y}$ frozen, we look for plane wave solutions of the form $e^{i(\tau t + \mathbf{k}\cdot\mathbf{x})}\mathbf{a}$ with $\mathbf{k}$ real-valued. Neglecting variations in $\mathbf{y}$ of the coefficients in L and zero-th order terms, such solutions approximately satisfy the equation

$$\det L(\mathbf{y}, i\tau, i\mathbf{k}) = 0. \tag{1.57}$$

The roots $\tau_j(\mathbf{y},\mathbf{k})$ define the dispersion relations of the equation $L\mathbf{u}=0$. They are associated with eigenmodes $\mathbf{b}_j(\mathbf{y},\mathbf{k})$, which generalize the construction obtained in section 1.3.1 for constant coefficients.

A useful normalization adopted in [28] for the eigenvectors is as follows. The dispersion relation may be recast as

$$\big(\tau + A_0^{-1}(\mathbf{y})k_j A_j(\mathbf{y})\big)\mathbf{b}(\mathbf{y},\mathbf{k}) = 0, \tag{1.58}$$

so that $-\tau$ is an eigenvalue of the *dispersion matrix*

$$M(\mathbf{y},\mathbf{k}) = A_0^{-1}(\mathbf{y})k_j A_j(\mathbf{y}). \tag{1.59}$$

Since M is not symmetric, we define by $\mathbf{b}_j(\mathbf{y},\mathbf{k})$ its right-eigenvector associated to the eigenvalue $-\tau_j(\mathbf{y},\mathbf{k})$ and by $\mathbf{c}_j(\mathbf{y},\mathbf{k})$ the corresponding left-eigenvectors so that $\mathbf{c}_j^* M = -\tau_j \mathbf{c}_j^*$. We normalize the eigenvectors as follows:

$$\mathbf{c}_j(\mathbf{y},\mathbf{k}) = A_0\mathbf{b}_j(\mathbf{y},\mathbf{k}), \qquad \mathbf{c}_i^*(\mathbf{y},\mathbf{k})\mathbf{b}_j(\mathbf{y},\mathbf{k}) = \delta_{ij}. \tag{1.60}$$

This allows us to recast the dispersion matrix as

$$M(\mathbf{y},\mathbf{k}) = -\sum_{j=1}^{n} \tau_j(\mathbf{y},\mathbf{k})\mathbf{b}_j(\mathbf{y},\mathbf{k})\mathbf{c}_j^*(\mathbf{y},\mathbf{k}). \tag{1.61}$$

Note that $\mathbf{b}_j(\mathbf{y},\mathbf{k})\mathbf{c}_j^*(\mathbf{y},\mathbf{k})$ is a $n\times n$ matrix whereas $\mathbf{c}_i^*(\mathbf{y},\mathbf{k})\mathbf{b}_j(\mathbf{y},\mathbf{k})$ is a real number.

Exercise 1.3.10 (i) Work out the dispersion relation for the system of acoustic equations (1.1), the dispersion matrix, and calculate the corresponding eigenvalues and eigenvectors.
(ii) Same problem for Maxwell's equations. The solution can be found in [28].

Chapter 2

Homogenization Theory for the wave equation

This chapter comes from earlier notes and has a lot more material than what will be covered in the course.

2.1 Effective medium theory in periodic media

We are interested in approximate solutions of the wave equation when the physical coefficients vary on a fast spatial scale. A natural question is whether we can replace the rapidly varying coefficients by homogeneous coefficients. One case where this can be done is when the wavelength is large compared to the spatial oscillations of the physical coefficients. The approximation is then given by the *effective medium*, or *homogenization*, theory.

This theory is valid for distances of propagation on the order of the wavelength, and is therefore very useful in the analysis of standing wave problems in confined regions but cannot account for the radiative transport that will be taken on in subsequent chapters.

To present the homogenization theory of waves, we start with the simple problem of acoustic waves in layered media. We introduce a small adimensionalized parameter $\varepsilon > 0$, which is the ratio between the *characteristic* length scale of the physical coefficient variations and the wavelength. The acoustics equations take then the form

$$\rho\big(\frac{\mathbf{x}}{\varepsilon}\big)\frac{\partial \mathbf{u}_\varepsilon(t,\mathbf{x})}{\partial t} + \nabla p_\varepsilon(t,\mathbf{x}) = \mathbf{F}(t,\mathbf{x}) \tag{2.1}$$

$$\kappa\big(\frac{\mathbf{x}}{\varepsilon}\big)\frac{\partial p_\varepsilon(t,\mathbf{x})}{\partial t} + \nabla\cdot\mathbf{u}_\varepsilon(t,\mathbf{x}) = 0, \tag{2.2}$$

with vanishing initial conditions and with a volume source term $\mathbf{F}$ that we assume is smooth and independent of ε. We have defined the coordinates $\mathbf{x} = (x_1, x_2, z)$ and the differential operator $\nabla = (\partial/\partial_{x_1}, \partial/\partial_{x_2}, \partial/\partial_z)^t$. In layered media, the coefficients ρ and κ only depend on the third variable $\rho(\mathbf{x}/\varepsilon) = \rho(z/\varepsilon)$ and $\kappa(\mathbf{x}/\varepsilon) = \kappa(z/\varepsilon)$.

We want to derive the asymptotic behavior of $\mathbf{u}$ and p as $\varepsilon \to 0$. It will be simpler to analyze the second-order wave equation for the pressure p

$$\mathcal{L}_\varepsilon p_\varepsilon(t,\mathbf{x}) \equiv \kappa\big(\frac{\mathbf{x}}{\varepsilon}\big)\frac{\partial^2 p_\varepsilon(t,\mathbf{x})}{\partial t^2} - \nabla\cdot\frac{\nabla p_\varepsilon(t,\mathbf{x})}{\rho\big(\frac{\mathbf{x}}{\varepsilon}\big)} = -\nabla\cdot\frac{\mathbf{F}(t,\mathbf{x})}{\rho\big(\frac{\mathbf{x}}{\varepsilon}\big)}, \tag{2.3}$$

with vanishing initial conditions, obtained by differentiating (2.2) in time and (2.1) in space, and eliminating $\mathbf{u}_\varepsilon$.

2.1.1 Multiple scale expansion

The asymptotic behavior of p_ε is now obtained by using the theory of *multiple scale expansions*. Let us assume to simplify that ρ and κ are periodic functions of period 1. The solution of the wave equation will then sample these periodic oscillations and itself have periodic variations at the characteristic length ε. At the same time, the source term $\mathbf{F}$, which is a non-periodic smooth function in $\mathbf{x}$, will generate variations of the velocity and pressure at the larger scale of order $O(1)$.

The first basic assumption justifying the multiple scale expansion is that these two scales separate in the following sense. We suppose that p has approximately the form

$$p_\varepsilon(t,\mathbf{x}) \approx p_\varepsilon\big(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\big), \tag{2.4}$$

where $p_\varepsilon(t,\mathbf{x},\mathbf{y})$ is periodic (of period 1) with respect to the last variable $\mathbf{y}\in Y=(0,1)^3$ and a smooth function with respect to *all* variables. We define the fast variable $\mathbf{y}=\mathbf{x}/\varepsilon$ and denote by y its third component. In this multiple scale form, the spatial gradient acts on both the slow and fast variables. If we define $p_\varepsilon(\mathbf{x})=p(\mathbf{x},\mathbf{x}/\varepsilon)$, then we have by the chain rule that

$$\nabla p_\varepsilon(\mathbf{x}) = \nabla_{\mathbf{x}} p(\mathbf{x},\mathbf{y}) + \frac{1}{\varepsilon}\nabla_{\mathbf{y}} p(\mathbf{x},\mathbf{y})\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}}. \tag{2.5}$$

Assuming that the fast scale $\mathbf{y}=\mathbf{x}/\varepsilon$ and the slow scale $\mathbf{x}$ separate, we recast (2.3) in the two-scale framework as

$$\begin{aligned}\kappa(\mathbf{y})\frac{\partial^2 p_\varepsilon}{\partial t^2} - \frac{1}{\varepsilon^2}\nabla_{\mathbf{y}}\cdot\frac{\nabla_{\mathbf{y}}p_\varepsilon}{\rho(\mathbf{y})} - \frac{1}{\varepsilon}\big(\nabla_{\mathbf{x}}\cdot\frac{\nabla_{\mathbf{y}}p_\varepsilon}{\rho(\mathbf{y})} - \nabla_{\mathbf{y}}\cdot\frac{\nabla_{\mathbf{x}}p_\varepsilon}{\rho(\mathbf{y})}\big) - \nabla_{\mathbf{x}}\cdot\frac{\nabla_{\mathbf{x}}p_\varepsilon}{\rho(\mathbf{y})} \\ = -\frac{1}{\varepsilon}\nabla_{\mathbf{y}}\cdot\frac{\mathbf{F}(t,\mathbf{x})}{\rho(\mathbf{y})} - \nabla_{\mathbf{x}}\cdot\frac{\mathbf{F}(t,\mathbf{x})}{\rho(\mathbf{y})}.\end{aligned} \tag{2.6}$$

Multiplying through by ε^2, we recast the above expansion as

$$L_\varepsilon p_\varepsilon \equiv (L_0+\varepsilon L_1+\varepsilon^2 L_2)p_\varepsilon = \varepsilon S_0 + \varepsilon^2 S_1, \tag{2.7}$$

where we have defined the operators

$$L_0 = -\nabla_{\mathbf{y}}\cdot\frac{1}{\rho(\mathbf{y})}\nabla_{\mathbf{y}} \tag{2.8}$$

$$L_1 = -\big(\nabla_{\mathbf{y}}\frac{1}{\rho(\mathbf{y})}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\frac{1}{\rho(\mathbf{y})}\nabla_{\mathbf{y}}\big) \tag{2.9}$$

$$L_2 = \kappa(\mathbf{y})\frac{\partial^2}{\partial t^2} - \frac{1}{\rho(\mathbf{y})}\Delta_{\mathbf{x}}. \tag{2.10}$$

The source terms are $S_0(t,\mathbf{x},\mathbf{y}) = -\nabla_{\mathbf{y}}\cdot(\rho^{-1}(\mathbf{y})\mathbf{F}(t,\mathbf{x}))$ and $S_1(t,\mathbf{x},\mathbf{y}) = -\nabla_{\mathbf{x}}\cdot(\rho^{-1}(\mathbf{y})\mathbf{F}(t,\mathbf{x}))$.

The first assumption concerned the two-scale separation of the wave operator $\mathcal{L}_\varepsilon$ and the expansion of the two-scale operator L_ε. The second assumption is that the two-scale quantity $p_\varepsilon(t,\mathbf{x},\mathbf{y})$ can also be expanded in power series of ε, so that

$$p_\varepsilon(t,\mathbf{x}) = p_0\big(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\big) + \varepsilon p_1\big(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\big) + \varepsilon^2 p_2\big(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\big) + \ldots, \tag{2.11}$$

where the p_i are $1-$periodic with respect to the third variable. The equations for the successive terms p_0, p_1, and p_2 are obtained by plugging (2.11) into the wave equation (2.3) and equating like powers of ε. The term of order ε^{-2} yields

$$L_0 p_0(t,\mathbf{x},\mathbf{y}) \equiv -\nabla_{\mathbf{y}}\cdot\frac{1}{\rho(\mathbf{y})}\nabla_{\mathbf{y}} p_0 = 0. \tag{2.12}$$

Since p_0 is periodic in $\mathbf{y}$, the above equation implies that p_0 is independent of $\mathbf{y}$: $p_0 = p_0(t, \mathbf{x})$. The next-order equation is

$$(L_0 p_1 + L_1 p_0 - S_0)(t, \mathbf{x}, \mathbf{y}) \equiv -\nabla_{\mathbf{y}} \cdot \frac{1}{\rho(\mathbf{y})} \big(\nabla_{\mathbf{y}} p_1 + \nabla_{\mathbf{x}} p_0 - \mathbf{F}\big) = 0. \tag{2.13}$$

By linearity, the solution p_1 satisfies

$$p_1(t, \mathbf{x}, \mathbf{y}) = \boldsymbol{\theta}(\mathbf{y}) \cdot (\nabla_{\mathbf{x}} p_0 - \mathbf{F})(t, \mathbf{x}), \tag{2.14}$$

where the vector function $\boldsymbol{\theta}(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \theta_3(\mathbf{y}))$ solves

$$\begin{aligned} &-\nabla_{\mathbf{y}} \cdot \frac{1}{\rho(\mathbf{y})} \big(\nabla_{\mathbf{y}} \theta_i + \mathbf{e}_i\big) = 0, \qquad i = 1, 2, 3 \\ &\mathbf{y} \mapsto \theta_i(\mathbf{y}) \quad \text{ is 1-periodic.} \end{aligned} \tag{2.15}$$

It is a classical result in elliptic partial differential equations that the above equations admit unique solutions up to the addition of constants.

In the case of layered media, these equations can be solved exactly. Indeed, $\rho(\mathbf{y}) = \rho(y)$, hence $\nabla_{\mathbf{y}} \rho^{-1}(y) \mathbf{e}_i = (\rho^{-1})' \delta_{i3}$, where the Kronecker symbol $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. We then readily deduce that θ_1 and θ_2 are constant on the periodicity cell $(0, 1)^3$. Moreover, θ_3 depends only on y by symmetry and solves

$$\frac{d}{dy} \frac{1}{\rho(y)} \big(\frac{d\theta_3(y)}{dy} + 1\big) = 0.$$

Upon integrating this equation, we get $\frac{d\theta_3}{dy} + 1 = C\rho(y)$, where C is a constant. Since θ_3 is periodic, we deduce that $C = \langle \rho \rangle^{-1}$, where $\langle \cdot \rangle$ denotes averaging over the cell Y. Hence we have

$$\frac{1}{\rho(y)} \big(\frac{d\theta_3}{dy} + 1\big) = \frac{1}{\langle \rho \rangle}, \qquad \nabla_{\mathbf{y}} \theta_1(\mathbf{y}) = \nabla_{\mathbf{y}} \theta_2(\mathbf{y}) = 0. \tag{2.16}$$

These equations determine $\boldsymbol{\theta}$ explicitly up to an additive constant vector, which we may choose to vanish.

The equation of order 0 is given by

$$(L_0 p_2 + L_1 p_1 + L_2 p_0 - S_1)(t, \mathbf{x}, \mathbf{y}) = 0, \tag{2.17}$$

which may be recast more explicitly as:

$$\begin{aligned} \kappa(\mathbf{y}) \frac{\partial^2 p_0}{\partial t^2} - \nabla_{\mathbf{y}} \cdot \frac{1}{\rho} \nabla_{\mathbf{y}} p_2 - \nabla_{\mathbf{x}} \cdot \frac{1}{\rho} \nabla_{\mathbf{y}} p_1 - \nabla_{\mathbf{y}} \cdot \frac{1}{\rho} \nabla_{\mathbf{x}} p_1 \\ - \nabla_{\mathbf{x}} \cdot \frac{1}{\rho} \nabla_{\mathbf{x}} p_0 = -\nabla_{\mathbf{x}} \cdot \frac{\mathbf{F}}{\rho}. \end{aligned} \tag{2.18}$$

Let us integrate this equation in $\mathbf{y}$ over $Y = (0, 1)^3$. All terms in divergence form vanish by periodicity and we obtain

$$\langle \kappa \rangle \frac{\partial^2 p_0}{\partial t^2} - \nabla_{\mathbf{x}} \cdot \big\langle \frac{1}{\rho} \big(\nabla_{\mathbf{y}} \boldsymbol{\theta} + \mathbf{I_3}\big) \big\rangle (\nabla_{\mathbf{x}} p_0 - \mathbf{F}) = 0, \tag{2.19}$$

where $\mathbf{I_3}$ is the 3×3 identity matrix. This is the **compatibility** condition ensuring the existence of a unique solution $p_2(t, \mathbf{x}, \mathbf{y})$ to (2.18) defined up to a constant function $p_{20}(t, \mathbf{x})$ in the $\mathbf{y}$ variable (we choose $p_{20} = 0$ for instance).

2.1.2 Homogenized equations

We recast the above equation for p_0 as

$$\kappa^* \frac{\partial^2 p_0}{\partial t^2} - \nabla_{\mathbf{x}} \cdot (\boldsymbol{\rho}^*)^{-1} (\nabla_{\mathbf{x}} p_0 - \mathbf{F}) = 0, \tag{2.20}$$

where the homogeneous density tensor $\boldsymbol{\rho}^*$ and the homogeneous compressibility coefficient κ^* are given by

$$\begin{aligned} \boldsymbol{\rho}^* &= \big\langle \frac{1}{\rho} \big(\nabla_{\mathbf{y}} \boldsymbol{\theta} + \mathbf{I_3} \big) \big\rangle^{-1}, & (2.21) \\ \kappa^* &= \langle \kappa \rangle. & (2.22) \end{aligned}$$

We augment the above equation with vanishing initial conditions, which is compatible with the expansion (2.11).

By virtue of (2.16), the density tensor $\boldsymbol{\rho}^*$ simplifies in layered geometry and is given by

$$\boldsymbol{\rho}^* = \mathrm{Diag}\big(\big\langle \frac{1}{\rho} \big\rangle^{-1}, \big\langle \frac{1}{\rho} \big\rangle^{-1}, \langle \rho \rangle \big). \tag{2.23}$$

To physically interpret the homogenized equation (2.20), we recast it as the following first-order system

$$\boldsymbol{\rho}^* \frac{\partial \mathbf{v}}{\partial t} + \nabla_{\mathbf{x}} p_0 = \mathbf{F}, \tag{2.24}$$

$$\kappa^* \frac{\partial p_0}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{v} = 0, \tag{2.25}$$

where the homogeneous velocity $\mathbf{v}$ is defined by

$$\mathbf{v}(t, \mathbf{x}) = (\boldsymbol{\rho}^*)^{-1} \int_0^t (\mathbf{F} - \nabla_{\mathbf{x}} p_0)(\tau, \mathbf{x}) d\tau. \tag{2.26}$$

Owing to the form of the density tensor (2.23), we deduce that

$$\begin{aligned} \big\langle \frac{1}{\rho} \big\rangle^{-1} \frac{\partial v_1}{\partial t} + \frac{\partial p_0}{\partial x_1} &= F_1(t, \mathbf{x}), \\ \big\langle \frac{1}{\rho} \big\rangle^{-1} \frac{\partial v_2}{\partial t} + \frac{\partial p_0}{\partial x_2} &= F_2(t, \mathbf{x}), \\ \langle \rho \rangle \frac{\partial v_3}{\partial t} + \frac{\partial p_0}{\partial z} &= F_3(x, \mathbf{x}), \\ \langle \kappa \rangle \frac{\partial p_0}{\partial t} + \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial z} &= 0. \end{aligned} \tag{2.27}$$

These are anisotropic versions of the original acoustic equations (2.1). The horizontal and vertical sound speeds are different and given by

$$c_v^2 = \frac{1}{\langle \kappa \rangle \langle \rho \rangle} \qquad \text{and} \qquad c_h^2 = \frac{1}{\langle \kappa \rangle} \big\langle \frac{1}{\rho} \big\rangle. \tag{2.28}$$

We have seen that the solution p of (2.3), or equivalently of (2.1)-(2.2), converges to the homogeneous function $p_0(t, \mathbf{x})$ that solves (2.20). Now, what can we say about the convergence

of the velocity field $\mathbf{u}$ and how does it relate to the homogeneous velocity field $\mathbf{v}$? Let us write $\mathbf{u} = \mathbf{u}_0 + O(\varepsilon)$. From (2.1), we then deduce that

$$\partial_t \mathbf{u}_0 = \frac{\mathbf{F} - (\nabla p)_0}{\rho(y)} = \frac{\mathbf{F} - \nabla_{\mathbf{x}} p_0 - \nabla_{\mathbf{y}} p_1}{\rho(y)} = \frac{\mathbf{I_3} + \nabla_y \boldsymbol{\theta}}{\rho(y)} \boldsymbol{\rho}^* \partial_t \mathbf{v}.$$

Therefore, since both $\mathbf{u}_0$ and $\mathbf{v}$ vanish at $t = 0$,

$$\mathbf{u}_0(t, \mathbf{x}, \mathbf{y}) = \frac{\mathbf{I_3} + \nabla_y \boldsymbol{\theta}(\mathbf{y})}{\rho(y)} \boldsymbol{\rho}^* \mathbf{v}. \tag{2.29}$$

In layered media, by virtue of (2.16) and (2.23), the relation (2.29) simplifies to

$$\mathbf{u}_0(t, \mathbf{x}, \mathbf{y}) = \begin{pmatrix} \frac{1}{\rho(y)} \langle \frac{1}{\rho} \rangle^{-1} & 0 & 0 \\ 0 & \frac{1}{\rho(y)} \langle \frac{1}{\rho} \rangle^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{v}(t, \mathbf{x}). \tag{2.30}$$

It is interesting to observe that the third component of the asymptotic velocity $\mathbf{u}_0$ does not depend on the fast scale. However, the two other components, describing the propagation of the waves perpendicularly to the direction of the layering, feel the heterogeneities and are inversely proportional to $\rho(y)$.

2.1.3 Energy estimates

We have replaced the heterogeneous equation (2.3) by the homogenized equation (2.20). It remains to estimate the error between the two solutions. The formal expansion (2.11) provides the starting point. Let us define

$$p_\varepsilon(t, \mathbf{x}) = p_0\big(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\big) + \varepsilon p_1\big(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\big) + \varepsilon^2 p_2\big(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\big) + \zeta_\varepsilon(t, \mathbf{x}), \tag{2.31}$$

where the terms p_j, $0 \leq j \leq 2$ are defined as in the preceding section and p_ε is the solution to (2.6). This uniquely defines ζ_ε. It remains to show that the latter term is small in some sense. In order to do this, we use an energy estimate for the wave equation with coefficients $\kappa(\mathbf{x}/\varepsilon)$ and $\rho(\mathbf{x}/\varepsilon)$, both assumed to be smooth and uniformly bounded from below by a positive constant.

The first objective is to write an equation for ζ_ε. Obviously the wave equation looks quite natural and we write:

$$\mathcal{L}_\varepsilon \zeta_\varepsilon = \mathcal{L}_\varepsilon\big(p_\varepsilon - p_0 - \varepsilon p_1 - \varepsilon^2 p_2\big). \tag{2.32}$$

We now use the multiple-scale rule:

$$\mathcal{L}_\varepsilon p(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) = L_\varepsilon p(t, \mathbf{x}, \mathbf{y})\Big|_{\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}}.$$

and the wave equation (2.6) to deduce that

$$\varepsilon^2 \mathcal{L}_\varepsilon \zeta_\varepsilon = \big(\frac{1}{\varepsilon} S_1(t, \mathbf{x}, \mathbf{y}) + S_0(t, \mathbf{x}, \mathbf{y})\big)\Big|_{\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}} - (L_0 + \varepsilon L_1 + \varepsilon^2 L_2)(p_0 + \varepsilon p_1 + \varepsilon^2 p_2)\Big|_{\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}}.$$

The asymptotic expansions developed in the preceding section are precisely tailored so that all the "high-order" terms in the above expression cancel. Using the equations (2.13) for p_1 and (2.17) for p_2, we deduce that

$$\mathcal{L}_\varepsilon \zeta_\varepsilon = -\varepsilon\big(L_2(p_1 + \varepsilon p_2) + \varepsilon L_1 p_2\big)\Big|_{\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}} \equiv \varepsilon S_\varepsilon(t, \mathbf{x}). \tag{2.33}$$

The source term $S_\varepsilon(t,\mathbf{x})$ involves applying the differential operators L_1 and L_2 to the correctors $p_1(t,\mathbf{x},\mathbf{y})$ and $p_2(t,\mathbf{x},\mathbf{y})$. The latter terms thus need to be sufficiently regular. We do not dwell on these details. However classical results in the theory of elliptic equations and the wave equation show that $p_1(t,\mathbf{x},\mathbf{y})$ and $p_2(t,\mathbf{x},\mathbf{y})$ are indeed sufficiently smooth provided that the coefficients ρ and κ and the source term $\mathbf{F}(t,\mathbf{x})$ are sufficiently smooth.

This allows us to conclude that $S_\varepsilon(t,\mathbf{x})$ is bounded in $\mathcal{C}(0,T;L^2(\mathbb{R}^d))$ independent of ε. It remains to find an estimate for ζ_ε. We have the following result.

Theorem 2.1.1 *Let $p_\varepsilon(t,\mathbf{x})$ be the solution of the heterogeneous wave equation* (2.3) *and $p_0(t,\mathbf{x})$ the solution of the homogenized wave equation* (2.20)*. Assuming that the coefficients κ and ρ and that the source term $\mathbf{F}(t,\mathbf{x})$ are sufficiently smooth we find that there exists a constant independent of ε such that*

$$\|p_\varepsilon(t)-p_0(t)\|_{L^2(\mathbb{R}^d)}\leq C\varepsilon, \tag{2.34}$$

uniformly on compact sets $0<t<T$.

Proof. It remains to show a comparable estimate for ζ_ε, since $\varepsilon p_1+\varepsilon^2 p_2$ are indeed of order ε in the above sense. Consider the equation $\mathcal{L}_\varepsilon\zeta_\varepsilon=\varepsilon S_\varepsilon$ and the energy

$$\mathcal{E}(t)=\frac{1}{2}\int_{\mathbb{R}^d}\big(\kappa(\frac{\mathbf{x}}{\varepsilon})(\frac{\partial\zeta_\varepsilon}{\partial t})^2(t,\mathbf{x})+\frac{1}{\rho(\frac{\mathbf{x}}{\varepsilon})}|\nabla\zeta_\varepsilon|^2(t,\mathbf{x})\big)d\mathbf{x}.$$

We find that

$$\dot{\mathcal{E}}(t)=\int_{\mathbb{R}^d}\big(\kappa(\frac{\mathbf{x}}{\varepsilon})\frac{\partial\zeta_\varepsilon}{\partial t}\frac{\partial^2\zeta_\varepsilon}{\partial t^2}+\frac{1}{\rho(\frac{\mathbf{x}}{\varepsilon})}\nabla\zeta_\varepsilon\cdot\nabla\frac{\partial}{\partial t}\zeta_\varepsilon\big)d\mathbf{x}=\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\zeta_\varepsilon\varepsilon S_\varepsilon d\mathbf{x},$$

after the usual integration by parts. Upon integrating the above equality over $t\in(0,T)$ we find using the definition of $\mathcal{E}(t)$ that

$$\big\|\frac{\partial\zeta_\varepsilon}{\partial t}(t)\big\|^2_{L^2(\mathbb{R}^d)}\leq\int_0^t\int_{\mathbb{R}^d}\frac{\partial\zeta_\varepsilon}{\partial t}(s)\varepsilon S_\varepsilon(s)d\mathbf{x}ds\leq\frac{\varepsilon^2}{2}\int_0^t\|S_\varepsilon(s)\|^2_{L^2(\mathbb{R}^d)}ds+\frac{1}{2}\int_0^t\big\|\frac{\partial\zeta_\varepsilon}{\partial t}(s)\big\|^2_{L^2(\mathbb{R}^d)}ds,$$

by Cauchy-Schwarz inequality (on $\mathbb{R}^d$) and the fact that $2ab\leq a^2+b^2$. The integral form of the Gronwall lemma allows us to conclude that

$$\big\|\frac{\partial\zeta_\varepsilon}{\partial t}(t)\big\|_{L^2(\mathbb{R}^d)}\leq\varepsilon C_T\|S_\varepsilon\|_{\mathcal{C}(0,T;L^2(\mathbb{R}^d))},$$

on $0\leq t\leq T$ for some constant C_T independent of ε and S_ε. Since ζ_ε vanishes initially, this yields that $\|\zeta_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\leq C\varepsilon$ uniformly on $0<t<T$ and the result. □

Exercise 2.1.1 Consider the homogenization of the pressure field in (2.3) obtained from (2.1)-(2.2) with the right-hand-side of (2.2) replaced by the smooth term $g(t,\mathbf{x})$.

Exercise 2.1.2 The above theorem shows an approximation of order $O(\varepsilon)$ even though the expansion in (2.11) was pushed to second-order. The reason is that $p_1(t,\mathbf{x},\mathbf{y})$ was defined up to the addition of a function p_{10} that depends only on $(t,\mathbf{x})$. Push the asymptotic expansion (2.11) to higher order and generalize the above theorem to obtain an asymptotic expansion of order $O(\varepsilon^N)$ for $N\in\mathbb{N}$.

Exercise 2.1.3 Consider the Schrödinger equation

$$-\Delta u_\varepsilon + V(\frac{\mathbf{x}}{\varepsilon})u_\varepsilon = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{2.35}$$

for $V(\frac{\mathbf{x}}{\varepsilon}) \geq V_0 > 0$ sufficiently smooth (the absorbing case) and periodic of period 1, and $f(\mathbf{x})$ a square integrable and sufficiently smooth term. Show that u_ε converges strongly to u as $\varepsilon \to 0$ in the $L^2(\mathbb{R}^d)$ topology (i.e. show that $\|u_\varepsilon - u\|_{L^2(\mathbb{R}^d)}$ goes to 0; in fact you can show that it is of order ε), where u is the solution of the same equation with $v(\frac{\mathbf{x}}{\varepsilon})$ replaced by its average over the cell Y.

Exercise 2.1.4 For u_ε defined as in the preceding exercise, find the limit of its gradient ∇u_ε as $\varepsilon \to 0$.

Exercise 2.1.5 *Find an error estimate for the homogenization of the first-order hyperbolic system* (2.1)-(2.2).

Exercise 2.1.3 shows that oscillations in the potential V have no influence on the solution u_ε at the leading order. However its derivatives feel the influence of the fluctuations. Compare this to the effects of $\rho(\mathbf{x})$ and $\kappa(\mathbf{x})$ in theorem 2.1.1.

2.2 Multidimensional case and estimates of the effective propagation speed

Most of the analysis carried out in the previous section holds in the multidimensional case, where the coefficients ρ and κ are allowed to depend also on the variables x_1 and x_2. The homogenization of the wave equation (2.3) is obtained by the multiple scale expansion (2.11), which yields the three equations (2.12), (2.13), and (2.18). We deduce from (2.12) that the pressure field $p_0(t, \mathbf{x})$ is independent of the fast variable $\mathbf{y}$ and that p_1 satisfies (2.14). The homogenized equation (2.20) still holds and the homogenized coefficients are given by (2.21) and (2.22).

However, no general analytic solution to (2.15) can be obtained in general. There is no equivalent in the multidimensional case to the layered media formulas (2.16) and the homogeneous density tensor in *not* given by (2.23) in general.

2.2.1 Effective density tensor in the case of small volume inclusions

Since (2.23) is not available to us, we must have recourse to approximate methods to estimate (2.21). Apart from numerical solutions of (2.15), several asymptotic methods have been devised to approximately calculate (2.21). An elegant method for doing so in the case of small volume inclusions in based on results of *potential theory* [20]. Let $Y = (-1/2, 1/2)^3$ be the unit periodicity cell and B_δ the ball of center the origin and radius δ, of volume $\beta = \frac{4\pi}{3}\delta^3$. We assume that the density $\rho = \rho_1$ is constant inside the unit cell except within the ball B_δ, where it takes the value $\rho = \rho_2$.

Let $\mathbf{x} = (x_1, x_2, x_3)$ in the unit cell Y. By symmetry of ρ on the cell, we easily verify that the effective tensor $\boldsymbol{\rho}^*$ in (2.21) is given by

$$\boldsymbol{\rho}^* = \big\langle \frac{1}{\rho}\big(\frac{\partial \theta_1}{\partial x_1} + 1\big)\big\rangle^{-1}\mathbf{I_3} \equiv \rho^*\mathbf{I_3}. \tag{2.36}$$

It is convenient to introduce the coefficients

$$D(\mathbf{x}) = (\rho(\mathbf{x}))^{-1} \quad i = 1, 2 \qquad \text{and} \qquad D^* = (\rho^*)^{-1}. \tag{2.37}$$

These coefficients are analogous to the *diffusion* coefficient and *effective diffusion* coefficient arising in heat conduction, although the physical units are different here. The equation (2.15) for θ_1 is equivalent to

$$\begin{aligned}
&\Delta\theta_1(\mathbf{x}) = 0 \quad \text{ except on } \mathbf{x} \in \partial B_\delta = \{|\mathbf{x}| = \delta\} \\
&\theta_1 \quad \text{ is continuous across } \partial B_\delta \\
&D(\mathbf{x})\nu(\mathbf{x}) \cdot (\nabla_{\mathbf{x}}\theta_1 + \mathbf{e}_1) \quad \text{ is continuous across } \partial B_\delta \\
&\mathbf{x} \mapsto \theta_1(\mathbf{x}) \quad \text{ is } 1-\text{periodic.}
\end{aligned} \tag{2.38}$$

Here $\nu(\mathbf{x})$ is the outward unit normal to B_δ at $\mathbf{x} \in \partial B_\delta$. Let us introduce the periodic unit Green's function that solves

$$\begin{aligned}
&-\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) - 1, \\
&\mathbf{x} \mapsto G(\mathbf{x}, \mathbf{y}) \quad \text{ is 1-periodic for all } \mathbf{y} \in \mathbb{R}^3.
\end{aligned} \tag{2.39}$$

This function is given by

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi^2} \sum_{\mathbf{m} \in \mathbf{Z}^3}{}' \frac{e^{2\pi i \mathbf{m} \cdot (\mathbf{x} - \mathbf{y})}}{|\mathbf{m}|^2}, \tag{2.40}$$

where the conditionally convergent sum runs over all integer values except $\mathbf{m} = \mathbf{0}$. This expression for G is called a *lattice sum*, familiar in the context of solid state physics [32]. The function G has zero mean over the unit cell for every $\mathbf{y}$ and satisfies

$$G(\mathbf{x}, \mathbf{y}) \sim \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \quad \text{ as } \quad |\mathbf{x} - \mathbf{y}| \to 0. \tag{2.41}$$

In other words, the periodic Green's function has the same behavior as the whole space Green's function when $|\mathbf{x} - \mathbf{y}| \to 0$, which is to be expected since the periodic boundary conditions are negligible when the source and observation points are very near each other.

We now use the above Green's function and some results of *potential* theory [20] to derive an asymptotic approximation of θ_1 as the volume of the inclusion $\beta \to 0$. We now write θ_1 in the form

$$\theta_1(\mathbf{x}) = \int_{\partial B_\delta} G(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) dS(\mathbf{y}), \tag{2.42}$$

where dS is the surface measure on ∂B_δ and σ is a surface distribution to be determined. The periodicity of θ_1 follows from that of $\mathbf{x} \mapsto G(\mathbf{x}, \mathbf{y})$. We deduce from the first line in (2.38) the constraint

$$\int_{\partial B_\delta} \sigma(\mathbf{y}) dS(\mathbf{y}) = 0. \tag{2.43}$$

Continuity of θ_1 is a consequence of the limit (2.41) and of the continuity of the single layer potential

$$U(\mathbf{x}) = \int_{\partial B_\delta} \frac{\sigma(\mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|} dS(\mathbf{y}) \tag{2.44}$$

across ∂B_δ. It is a classical result in potential theory [20, Theorem VI, Chapter VI] that

$$\begin{aligned}
\Big(\frac{\partial U}{\partial \nu}\Big)_+ &= -\frac{\sigma}{2} + \int_{\partial B_\delta} \frac{\partial}{\partial \nu}\Big(\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}\Big)\sigma(\mathbf{y}) dS(\mathbf{y}), \\
\Big(\frac{\partial U}{\partial \nu}\Big)_- &= \frac{\sigma}{2} + \int_{\partial B_\delta} \frac{\partial}{\partial \nu}\Big(\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}\Big)\sigma(\mathbf{y}) dS(\mathbf{y}),
\end{aligned} \tag{2.45}$$

where the subscripts $+/-$ stand for the normal derivative outside and inside B_δ, respectively. Using (2.45) and again the limit (2.41), we obtain from the third line in (2.38) the following integral equation for σ

$$D_1\big(-\frac{\sigma}{2}+\int_{\partial B_\delta}\frac{\partial G}{\partial \nu}\sigma dS\big)-D_2\big(\frac{\sigma}{2}+\int_{\partial B_\delta}\frac{\partial G}{\partial \nu}\sigma dS\big)=(D_2-D_1)\nu\cdot\mathbf{e}_1, \tag{2.46}$$

This relation is equivalent to

$$\int_{\partial B_\delta}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \nu}\sigma(\mathbf{y})dS(\mathbf{y})+\frac{D_2+D_1}{D_2-D_1}\frac{\sigma(\mathbf{x})}{2}=-\nu_1(\mathbf{x})\quad\text{ on }\partial B_\delta, \tag{2.47}$$

with $\nu_1(\mathbf{x})=\nu(\mathbf{x})\cdot\mathbf{e}_1$. Introduce now the rescaled variables $\mathbf{x}\to\mathbf{x}/\delta$ and $\mathbf{y}\to\mathbf{y}/\delta$ and functions

$$\sigma^\delta(\mathbf{x})=\sigma(\delta\mathbf{x})\quad\text{ and }\quad G^\delta(\mathbf{x},\mathbf{y})=\delta\,G(\delta\mathbf{x},\delta\mathbf{y}). \tag{2.48}$$

We can then recast (2.47) as

$$\int_{\partial B_1}\frac{\partial G^\delta(\mathbf{x},\mathbf{y})}{\partial \nu}\sigma^\delta(\mathbf{y})dS(\mathbf{y})+\frac{D_2+D_1}{D_2-D_1}\frac{\sigma^\delta(\mathbf{x})}{2}=-\nu_1(\mathbf{x})\quad\text{ on }\partial B_1. \tag{2.49}$$

Because of (2.41), we have

$$G^\delta(\mathbf{x},\mathbf{y})\to\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\equiv G^0(\mathbf{x},\mathbf{y})\quad\text{ as }\delta\to 0. \tag{2.50}$$

Defining then $\sigma_0(\mathbf{x})=\lim\limits_{\delta\to 0}\sigma^\delta(\mathbf{x})$, we pass to the limit in (2.49) and obtain

$$\int_{\partial B_1}\frac{\partial G^0(\mathbf{x},\mathbf{y})}{\partial \nu}\sigma^0(\mathbf{y})dS(\mathbf{y})+\frac{D_2+D_1}{D_2-D_1}\frac{\sigma^0(\mathbf{x})}{2}=-\nu_1(\mathbf{x})\quad\text{ on }\partial B_1. \tag{2.51}$$

Notice that the specific geometry of ∂B_1 did not play any role so far. The above equation for σ^0 is therefore valid for more general geometries of discontinuity than spheres. However, (2.51) can be solved exactly when ∂B_1 is the unit sphere.

Let us calculate the potential $U(\mathbf{x})$ in (2.44) for $\sigma(\mathbf{x})=\nu_1(\mathbf{x})$. Defining $\mathbf{e_x}=\mathbf{x}/|\mathbf{x}|$ and decomposing $\boldsymbol{\nu}(\mathbf{y})=(\boldsymbol{\nu}(\mathbf{y})\cdot\mathbf{e_x})\mathbf{e_x}+\boldsymbol{\nu}^\perp(\mathbf{y})$, with $\boldsymbol{\nu}^\perp(\mathbf{y})$ uniquely defined by $\boldsymbol{\nu}^\perp(\mathbf{y})\cdot\mathbf{e_x}=0$, we deduce from the symmetries of the unit sphere and of G^0 that

$$U(\mathbf{x})=\nu_1(\mathbf{x})\int_{\partial B_1}\frac{\nu(\mathbf{y})\cdot\mathbf{e_x}}{4\pi|\mathbf{x}-\mathbf{y}|}dS(\mathbf{y})=\frac{\nu_1(\mathbf{x})}{2}\int_0^\pi\frac{\cos\theta\sin\theta d\theta}{\sqrt{\sin^2\theta+(|\mathbf{x}|-\cos\theta)^2}}.$$

A simple calculation shows that

$$U(\mathbf{x})=\frac{\nu_1(\mathbf{x})}{3}\begin{cases}|\mathbf{x}|^{-2} & \text{for}\quad |\mathbf{x}|\geq 1,\\ |\mathbf{x}| & \text{for}\quad |\mathbf{x}|\leq 1.\end{cases} \tag{2.52}$$

As expected from (2.45), U has a discontinuous normal derivative at ∂B_1. The second equality in (2.45) now yields

$$\int_{\partial B_1}\frac{\partial G^0(\mathbf{x},\mathbf{y})}{\partial \nu}\nu_1(\mathbf{y})dS(\mathbf{y})=\big(\frac{\partial U}{\partial \nu}\big)_-\,-\frac{\nu_1(\mathbf{x})}{2}=-\frac{\nu_1(\mathbf{x})}{6}\quad\text{ for }\mathbf{x}\in\partial B_1. \tag{2.53}$$

This shows that $\nu_1(\mathbf{x})$, up to a proportionality constant, solves (2.51). This constant is given by

$$\sigma^0(\mathbf{x}) = 3\frac{D_1 - D_2}{2D_1 + D_2}\nu_1(\mathbf{x}). \tag{2.54}$$

Notice that σ^0 in (2.54) also satisfies the constraint (2.43).

Let us now return to the calculation of θ_1 and D^*. From (2.42) we obtain that

$$\theta_1(\delta\mathbf{x}) = \delta \int_{\partial B_1} G^\delta(\mathbf{x},\mathbf{y})\sigma^\delta(\mathbf{y})dS(\mathbf{y}) \sim 3\frac{D_1 - D_2}{2D_1 + D_2}\delta\frac{\nu_1(\mathbf{x})}{|\mathbf{x}|^2},$$

as $\delta \to 0$ for $|\mathbf{x}| \geq 1$. This shows that

$$\theta_1(\mathbf{x}) \sim 3\frac{D_1 - D_2}{2D_1 + D_2}\delta^3\frac{x_1}{|\mathbf{x}|^3} = -3\frac{D_1 - D_2}{2D_1 + D_2}4\pi\delta^3\frac{\partial G^0(\mathbf{x},\mathbf{0})}{\partial x_1} \quad \text{for } |\mathbf{x}| \geq \delta. \tag{2.55}$$

Since θ_1 is also of order $O(\delta^3)$ for $|\mathbf{x}| < \delta$, it is not necessary to carry out its expression as it will be negligible in the computation of D^* given by (2.36)-(2.37). Keeping this in mind, we obtain that

$$D^* \sim D_1\big(1 + 9\beta\frac{D_1 - D_2}{2D_1 + D_2}\int_Y \frac{\partial^2 G^0(\mathbf{x},\mathbf{0})}{\partial x_1^2}d\mathbf{x}\big)$$

where we recall that $\beta = 4\pi\delta^3/3$. Since $-\Delta G(\mathbf{x},0) = \delta(\mathbf{x})$ and $G(\mathbf{x},0)$ only depends on $|\mathbf{x}|$, we deduce that $\int_Y \frac{\partial^2 G^0(\mathbf{x},\mathbf{0})}{\partial x_1^2}d\mathbf{x} = -1/3$, hence

$$D^* = D_1\big(1 - 3\frac{D_1 - D_2}{2D_1 + D_2}\beta\big) + o(\beta), \tag{2.56}$$

or equivalently,

$$\rho^* = \rho_1\big(1 + 3\frac{\rho_2 - \rho_1}{2\rho_2 + \rho_1}\beta\big) + o(\beta). \tag{2.57}$$

This equation is an approximation of the effective coefficient formulas obtained by Rayleigh in 1892 [27]. An elegant method to derive higher order terms, based on a more accurate expansion of the lattice sum (2.40), was first used by Hasimoto [18].

2.2.2 Effective density tensor in the case of small contrast

The asymptotic calculation of the previous section was based on the assumption that the density was constant except in a small volume where it was allowed to have an $O(1)$ fluctuation. We now assume that the density has only small fluctuations about a constant mean, but not necessarily in a small volume region. Let us assume that

$$\frac{1}{\rho(\mathbf{x})} = \frac{1}{\rho} + \delta\frac{1}{\rho_1(\mathbf{x})}, \tag{2.58}$$

or using the more convenient notation in (2.37) that $D(\mathbf{x}) = D_0 + \delta D_1(\mathbf{x})$, where δ is a small parameter. We assume that $\langle D_1\rangle = 0$. We now want to obtain an approximation of the effective tensor (2.21) as $\delta \to 0$. To do so, we expand the field $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \delta\boldsymbol{\theta}_1 + \delta^2\boldsymbol{\theta}_2, \tag{2.59}$$

where all terms θ_i are $1-$periodic. Here, $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ are independent of δ and $\boldsymbol{\theta}_2$ is bounded uniformly in δ. Plugging this expression into (2.15) and equating like powers of δ yields several equations that we now analyze. The first one simply is

$$\Delta\boldsymbol{\theta}_0 = 0, \tag{2.60}$$

which yields that $\boldsymbol{\theta}_0$ is constant. Since only $\nabla_{\mathbf{y}}\boldsymbol{\theta}_0$ matters, we choose $\boldsymbol{\theta}_0 = \mathbf{0}$. The second equation is then

$$-D_0\Delta\boldsymbol{\theta}_1 - \nabla_{\mathbf{y}}D_1 = 0. \tag{2.61}$$

Using the Fourier decomposition of D_1 and $\boldsymbol{\theta}_1$ for $\mathbf{k} \in \mathbf{Z}^3$, $\mathbf{k} \neq \mathbf{0}$, we obtain that

$$\hat{\boldsymbol{\theta}}_1(\mathbf{k}) = -\frac{\hat{D}_1(\mathbf{k})}{D_0}\frac{\mathbf{k}}{|\mathbf{k}|^2}. \tag{2.62}$$

Because $\langle\nabla_{\mathbf{y}}\boldsymbol{\theta}_i\rangle = 0$ by periodicity, the effective diffusion tensor D^* is given by

$$\mathbf{D}^* = \langle D(\nabla_{\mathbf{y}}\boldsymbol{\theta} + \mathbf{I_3})\rangle = \langle D\rangle\mathbf{I_3} + \delta^2\langle D_1\nabla_{\mathbf{y}}\boldsymbol{\theta}_1\rangle + O(\delta^3).$$

Using the Parseval relation, this yields

$$\mathbf{D}^* = \langle D\rangle\mathbf{I_3} - \frac{\delta^2}{D_0}\sum_{\mathbf{k}\in\mathbf{Z}^3}|\hat{D}_1(\mathbf{k})|^2\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}. \tag{2.63}$$

To the first order in δ, the effective tensor can be equated with the average of D over the periodicity cell. The anisotropy of D only appears as a contribution of order δ^2. This property holds in a much more general context in homogenization [23, Chapter 14].

2.3 Case of random media

In this section, we extend the homogenization carried out in the periodic case in the preceding sections to the random case. We will illustrate the theory by looking at the propagation of acoustic waves in a bubbly liquid. The location, size, and number of bubbles in the liquid are unknown to the observer. This justifies the use of a probabilistic theory to estimate the role of these bubbles in the sound propagation.

Let $\omega \in \Omega$ be a realization of the randomness in the configuration space and $P(\omega)$ its probability density. The physical coefficients κ and ρ in the acoustic equation (2.3) now depend on the realization: $\kappa = \kappa(\mathbf{x}, \omega)$ and $\rho = \rho(\mathbf{x}, \omega)$. Assuming that $F(t, \mathbf{x})$ is deterministic, the solution p of (2.3) will also depend on the realization: $p = p(t, \mathbf{x}, \omega)$. The type of question homogenization aims at answering is: can one obtain some information about the average over realizations $\langle p\rangle(t, \mathbf{x}) = \int p(t, \mathbf{x}, \omega)P(d\omega)$ knowing that the number of bubbles increases to infinity while keeping a constant volume fraction?

We have already seen how to answer this question in the periodic case. We will see that the basic results, the existence of effective coefficients and the convergence in a suitable sense of the heterogeneous solution to the homogeneous solution, still hold in the random case. However, formal asymptotic expansions of the form (2.11) can no longer be justified and the analysis of the *cell* problem (2.15) that leads to the definition of the homogenized coefficients (2.21) is more involved.

Yet the periodic case shows the way to homogenization. One crucial point in periodic homogenization is a certain invariance by translation. The analog of periodicity in the random case is statistical homogeneity or stationarity. Let us again denote by $D = \rho^{-1}$. Stationarity means that for any set of points $\mathbf{x}_1, \ldots, \mathbf{x}_m$ in $\mathbb{R}^d$ and any vector $\mathbf{h} \in \mathbb{R}^d$, the joint distributions of

$$D(\mathbf{x}_1, \omega), \ldots, D(\mathbf{x}_m, \omega) \quad \text{and} \quad D(\mathbf{x}_1 + \mathbf{h}, \omega), \ldots, D(\mathbf{x}_m + \mathbf{h}, \omega)$$

are the same, and similarly for κ.

The periodic case can actually be seen as a special random case. Let the configuration space Ω be the unit torus in $\mathbb{R}^d$ and P the uniform measure ($dP = d\mathbf{x}$). Then ω is a point on the unit torus and we set

$$D(\mathbf{x}, \omega) = \tilde{D}(\omega - \mathbf{x}),$$

where $\tilde{D}$ is a function on the unit torus, i.e. a periodic function. The random acoustics equation is now (2.3) with the origin of coordinates relative to the period cell chosen randomly. Since the analysis is valid for every ω and independent of ω, nothing is gained by this randomization.

A more interesting example is the Poisson point process case, which models the location of the centers of the air bubbles. Here, Ω is the set of infinite sequences of points in $\mathbb{R}^d$, $\omega = (\boldsymbol{\xi}_1, \ldots)$ in which the order is irrelevant, which can be denoted as $\Omega = (\mathbb{R}^d)^\infty-$symmetric. For each Ω, let us define the point measure

$$\nu = \sum_j \delta_{\boldsymbol{\xi}_j}(d\mathbf{x}), \tag{2.64}$$

where $\delta_{\mathbf{y}}(d\mathbf{x})$ is the Dirac delta measure at $\mathbf{y}$. In other words, if A is a subset of $\mathbb{R}^d$, then $\nu(A)$ counts the number of points of ω that fall inside A. Let $\lambda > 0$ be a parameter called the intensity of the Poisson distribution. The probability density P can be defined via its Laplace functional. Introducing for every positive test function ϕ of compact support

$$(\phi, \nu) = \int_{\mathbb{R}^d} \phi(\mathbf{x})\nu(d\mathbf{x}) = \sum_j \phi(\boldsymbol{\xi}_j),$$

the Laplace functional of P is defined by

$$E^P\big[e^{-(\phi,\nu)}\big] = \langle e^{-(\phi,\nu)}\rangle = \exp\big[\lambda \int_{\mathbb{R}^d} (e^{-\phi(\boldsymbol{\xi})} - 1)d\boldsymbol{\xi}\big]. \tag{2.65}$$

An important property of the Poisson process is that

$$P(\nu(A) = n) = e^{-\lambda|A|}\big(\frac{\lambda|A|}{n!}\big)^n, \tag{2.66}$$

where $|A|$ is the volume of A. So λ gives the average number of points per unit volume. Given Ω and P, we can now model random distributions of bubbles of radius δ by

$$D(\mathbf{x}, \omega) = \begin{cases} D_2, & |\mathbf{x} - \boldsymbol{\xi}_j| \le \delta \quad \text{for some } j, \\ D_1, & |\mathbf{x} - \boldsymbol{\xi}_j| > \delta \quad \text{for all } j, \end{cases} \tag{2.67}$$

and a similar expression for κ. We can show that the Poisson point process is stationary: $\nu(d\mathbf{x})$ and $\nu(\mathbf{h} + d\mathbf{x})$ have the same probability P. Therefore, since D is a functional of the Poisson process, it is itself stationary. Notice that the bubbles are allowed to overlap with this model, which is not realistic physically.

Another interesting property is that the random functional $D^\varepsilon(\mathbf{x}, \omega)$ defined as in (2.67) with λ and δ replaced by $\varepsilon^{-d}\lambda$ and $\varepsilon\delta$ is statistically equivalent to the functional $D(\mathbf{x}/\varepsilon, \omega)$, where D is defined by (2.67). Therefore, the scaling $\mathbf{x} \to \mathbf{x}/\varepsilon$ and the resulting acoustic equation (2.3) indeed correspond to sending the number of bubbles per unit volume to infinity while keeping the volume fraction of air bubbles constant.

Let us now analyze the asymptotic behavior of the acoustic equation

$$\kappa\big(\frac{\mathbf{x}}{\varepsilon}, \omega\big)\frac{\partial^2 p(t, \mathbf{x}, \omega)}{\partial t^2} - \nabla \cdot D\big(\frac{\mathbf{x}}{\varepsilon}, \omega\big)\nabla p(t, \mathbf{x}, \omega) = -\nabla \cdot D\big(\frac{\mathbf{x}}{\varepsilon}, \omega\big)\mathbf{F}(t, \mathbf{x}), \tag{2.68}$$

with vanishing initial conditions. We now formally extend the analysis of section 2.1 to the random case. Using the two-scale expansion (2.11), we obtain the same sequence of equations for the terms p_i.

The first equation (2.12) shows that p_0 does not depend on the fast variable $\mathbf{y}$, i.e. in the random case is independent of the realization ω. We deduce from (2.14) that p_1 has a non-trivial dependence in the fast variable $\mathbf{y}$, i.e. will depend on the realization ω. The vector function $\boldsymbol{\theta}$ satisfies now

$$-\nabla \cdot D(\mathbf{y},\omega)(\nabla \boldsymbol{\theta}(\mathbf{y},\omega) + \mathbf{I_3}) = 0. \tag{2.69}$$

It turns out that there is no stationary solution $\boldsymbol{\theta}(\mathbf{y},\omega)$ to this equation. It is however possible to find a stationary solution $\nabla\boldsymbol{\theta}$ to the above equation, which suggests to slightly modify the equation (2.69). Let $\mathbf{e}$ be a unit vector in $\mathbb{R}^3$. We consider the following infinite medium problem: Find two vector fields $\mathbf{H}(\mathbf{x},\omega)$ and $\mathbf{G}(\mathbf{x},\omega)$ such that

$$\begin{aligned} &\mathbf{H}(\mathbf{x},\omega) = D(\mathbf{x},\omega)\mathbf{G}(\mathbf{x},\omega),\\ &\nabla \times \mathbf{G} = \mathbf{0},\\ &\nabla \cdot \mathbf{H} = 0,\\ &\langle \mathbf{G} \rangle = \mathbf{e}. \end{aligned} \tag{2.70}$$

When $\mathbf{e} = \mathbf{e}_i$, we find that $\mathbf{G} = \nabla\theta_i + \mathbf{e}_i$, where θ_i is the ith component of $\boldsymbol{\theta}$ solving (2.69). With this definition, we easily get the third equation in (2.70) from (2.69) and the first equation in (2.70). Also, since $\mathbf{G}$ can be written as a gradient, we clearly have the second equation in (2.69). Since $\mathbf{G}$ and $\mathbf{H}$ are sought among stationary vector fields only, we get that $\langle \mathbf{G} \rangle$ is a constant vector. The new infinite medium problem (2.70) admits a unique solution among stationary fields. The effective tensor is then given by

$$\mathbf{D}^*(\mathbf{e},\mathbf{l}) = (\boldsymbol{\rho}^*)^{-1}(\mathbf{e},\mathbf{l}) = \langle \mathbf{H_e} \cdot \mathbf{l} \rangle = \langle D\mathbf{G_e} \cdot \mathbf{l} \rangle = \langle D\mathbf{G_e} \cdot \mathbf{G_l} \rangle. \tag{2.71}$$

Here, we have denoted by $H_\mathbf{l}$ and $G_\mathbf{l}$ the solutions of (2.70) with $\mathbf{e} = \mathbf{l}$. The last relation follows from the ergodicity of the random process. Let us indeed write $\mathbf{G_l} = \mathbf{l} + \tilde{\mathbf{G}}_\mathbf{l}$ with $\langle \tilde{\mathbf{G}}_\mathbf{l} \rangle = 0$. Then $\nabla \times \tilde{\mathbf{G}}_\mathbf{l} = 0$, hence $\tilde{\mathbf{G}}_\mathbf{l}$ is a gradient $\nabla\chi$. The last relation in (2.71) is equivalent to $\langle \mathbf{H_e} \cdot \tilde{\mathbf{G}}_\mathbf{l} \rangle = 0$. However, by ergodicity, which means that spatial averaging for every realization ω corresponds to ensemble averaging (which is independent of position), we get that

$$\langle \mathbf{H_e} \cdot \tilde{\mathbf{G}}_\mathbf{l} \rangle = \int_{\mathbb{R}^d} \mathbf{H_e}(\mathbf{x},\omega)\tilde{\mathbf{G}}_\mathbf{l}(\mathbf{x},\omega)d\mathbf{x} = -\int_{\mathbb{R}^d} \nabla \cdot \mathbf{H_e}(\mathbf{x},\omega)\chi(\mathbf{x},\omega)d\mathbf{x} = 0.$$

This relation shows that the effective tensors $\mathbf{D}^*$ and $\boldsymbol{\rho}^*$ are positive definite since D_1 and D_2 are positive constants. The effective compressibility κ^* is still given by (2.22) and the homogeneous acoustics equation by (2.20).

To summarize, we see that the homogenized equations obtained in the periodic case (2.20), (2.21), and (2.22) carry over to the random case, with the slight exception that $\nabla\theta_i + \mathbf{e}_i$ is replaced by $\mathbf{G}_{\mathbf{e}_i}$. However, the asymptotic expansion (2.11) does not hold. All that can be said is that

$$\int_{\mathbb{R}^d} \left\langle (p(t,\mathbf{x},\omega) - p_0(t,\mathbf{x}))^2 \right\rangle d\mathbf{x} \to 0 \tag{2.72}$$

as $\varepsilon \to 0$. We will not present a proof of this result here. Moreover, the cell problem (2.15) that was tractable numerically because posed on the unit cell Y only is now replaced by (2.70), which is posed in all $\mathbb{R}^3$ and needs to be solved for each realization of the random process

to have access to (2.71). Estimating the effective tensor $\mathbf{D}^*$ numerically remains therefore a formidable task.

The asymptotic approximations obtained in the previous section are consequently all the more important. In the small volume fraction case, we can assume as a first approximation that the bubbles are far away from each other. The analysis taken up in section 2.2.1 then applies and (2.56) holds.

2.4 Variational formulation and effective parameter estimates

The homogenization procedures of the two previous sections were fraught with the same difficulty. Once the homogenized equation is found, how can one estimate the effective density tensor $\boldsymbol{\rho}^*$? Since brunt force calculation seems often out of range, approximations are in order. We have seen asymptotic expansion techniques in section 2.2.1 already, which are interesting when some physical quantity is small. Because the cell and infinite medium equations satisfy a natural variational interpretation, variational methods can be used to estimate the effective coefficients. The goal of this section is to present such methods.

2.4.1 Classical variational formulation

Let us return to the definition of (2.71). We have seen that an alternative expression is $\mathbf{D}^*(\mathbf{e},\mathbf{l}) = \langle D\mathbf{G}_\mathbf{e}\cdot\mathbf{G}_\mathbf{l}\rangle$, which shows that $\mathbf{D}^*$ is self-adjoint and positive definite. The knowledge of the quadratic form

$$W(\mathbf{e}) = \frac{1}{2}\mathbf{e}\cdot\mathbf{D}^*\mathbf{e}, \tag{2.73}$$

for all values of $\mathbf{e}$ then uniquely defines $\mathbf{D}^*$. We now claim that

$$\mathbf{e}\cdot\mathbf{D}^*\mathbf{e} = \min_{\substack{\nabla\times\mathbf{G}=\mathbf{0}\\ \langle\mathbf{G}\rangle=\mathbf{e}}} \langle D\mathbf{G}\cdot\mathbf{G}\rangle. \tag{2.74}$$

We therefore obtain that the effective tensor can be constructed as a minimization procedure over curl-free fields of constant ensemble averaging. This result obviously offers a very powerful tool to generate upper bounds to the not-easily calculable effective tensor.

Let us derive this result. We denote by $\mathbf{G}_\mathbf{e}$ the solution of (2.70). Let $\mathbf{G}$ be a field satisfying $\nabla\times\mathbf{G}=0$ and $\langle\mathbf{G}\rangle=\mathbf{e}$. Because D is positive, we have that

$$\langle D(\mathbf{G}-\mathbf{G}_\mathbf{e})\cdot(\mathbf{G}-\mathbf{G}_\mathbf{e})\rangle \geq 0,$$

which is equivalent to

$$\langle D\mathbf{G}\cdot\mathbf{G}\rangle - 2\langle D\mathbf{G}\cdot\mathbf{G}_\mathbf{e}\rangle \geq \langle D\mathbf{G}_\mathbf{e}\cdot\mathbf{G}_\mathbf{e}\rangle.$$

Since $\mathbf{G}$ is curl-free, it is a gradient $\mathbf{G}=\nabla\chi$. Hence by ergodicity,

$$\langle D\mathbf{G}\cdot\mathbf{G}_\mathbf{e}\rangle = \int_{\mathbb{R}^d} D(\mathbf{x})\nabla\chi(\mathbf{x})\mathbf{G}_\mathbf{e}\,d\mathbf{x} = -\int_{\mathbb{R}^d}\chi(\mathbf{x})\nabla\cdot\mathbf{H}_\mathbf{e}(\mathbf{x})\,d\mathbf{x} = 0.$$

Since the minimum is attained for $\mathbf{G}=\mathbf{G}_\mathbf{e}$, which is admissible, (2.74) follows.

We now have a means to constructing upper bounds for $\mathbf{D}^*$. What about lower bounds? A dual variational principle actually shows that

$$\mathbf{e}\cdot(\mathbf{D}^*)^{-1}\mathbf{e} = \min_{\substack{\nabla\cdot\mathbf{H}=0\\ \langle\mathbf{H}\rangle=\mathbf{e}}} \langle D^{-1}\mathbf{H}\cdot\mathbf{H}\rangle. \tag{2.75}$$

The derivation is similar to that of (2.74) by swapping the roles of $\mathbf{F}$ and $\mathbf{H}$.

The simplest consequence of these relations is the arithmetic and harmonic mean bounds first obtained by Hill [19]

$$\langle D^{-1}\rangle^{-1} \leq \mathbf{D}^* \leq \langle D\rangle, \tag{2.76}$$

in the sense that all eigenvalues of the tensor $\mathbf{D}^*$ satisfy these two inequalities. Notice that these two bounds are precisely the values taken by the density tensor in layered media (2.23).

2.4.2 Hashin-Shtrikman bounds

The main difficulty in the choice of more accurate test functions $\mathbf{G}$ in (2.74) ($\mathbf{H}$ in (2.75)) is that the curl-free (divergence-free) constraints must be satisfied. We now derive a variational formulation for the effective tensor $\mathbf{D}^*$ that does not have this inconvenience. The first step consist in decomposing any vector-function $\mathbf{P}(\mathbf{x})$ defined on $\mathbb{R}^3$ in the random case and Y in the periodic case, as a constant $\mathbf{P}_0$, a mean-zero curl-free function $\mathbf{P}_1(\mathbf{x})$, and a mean-zero divergence-free function $\mathbf{P}_2(\mathbf{x})$. This is the Helmholtz decomposition. That this is possible is more easily seen in the Fourier domain, as

$$\begin{aligned} \nabla \times \mathbf{G}(\mathbf{x}) = \mathbf{0} \quad &\text{is equivalent to} \quad \mathbf{k} \times \hat{\mathbf{G}}(\mathbf{k}) = \mathbf{0} \\ \nabla \cdot \mathbf{H}(\mathbf{x}) = 0 \quad &\text{is equivalent to} \quad \mathbf{k} \cdot \hat{\mathbf{H}}(\mathbf{k}) = 0. \end{aligned}$$

Let us define the projection operators

$$\begin{aligned} \mathbf{P}_0 = \mathbf{\Gamma}_0\mathbf{P} &= \langle \mathbf{P}\rangle = \hat{\mathbf{P}}(0), \\ \mathbf{P}_1(\mathbf{x}) = \mathbf{\Gamma}_1\mathbf{P} &= \int e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{\Gamma}_1(\mathbf{k})(\hat{\mathbf{P}}(\mathbf{k}) - \hat{\mathbf{P}}(0)), \\ \mathbf{P}_2(\mathbf{x}) = \mathbf{\Gamma}_2\mathbf{P} &= \int e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{\Gamma}_2(\mathbf{k})(\hat{\mathbf{P}}(\mathbf{k}) - \hat{\mathbf{P}}(0)), \end{aligned} \tag{2.77}$$

where

$$\mathbf{\Gamma}_1(\mathbf{k}) = \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}, \qquad \mathbf{\Gamma}_2(\mathbf{k}) = \mathbf{I_3} - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}. \tag{2.78}$$

The integrations in (2.77) are replaced by a discrete summation in the periodic case. It is easy to check that $\Gamma_i^2(\mathbf{k}) = \Gamma_i(\mathbf{k})$ for $i = 1, 2$ and that $\mathbf{k} \times \Gamma_1(\mathbf{k}) = \mathbf{0}$ and $\mathbf{k} \cdot \Gamma_2(\mathbf{k}) = 0$. Therefore, the operators Γ_0, Γ_1, and Γ_2 are projections on the set of constant, mean-zero curl-free, and mean-zero divergence-free functions, respectively. It is then easily checked from its Fourier symbol that the operator Γ_1 is given by

$$\mathbf{\Gamma}_1\mathbf{G} = -\nabla(-\Delta)^{-1}\nabla\cdot(\mathbf{G} - \langle\mathbf{G}\rangle), \tag{2.79}$$

and $\mathbf{\Gamma}_2\mathbf{G} = (\mathbf{I_3} - \mathbf{\Gamma}_1)(\mathbf{G} - \langle\mathbf{G}\rangle)$. Also, these operators are orthogonal to each other, and we have

$$\mathbf{\Gamma}_i\mathbf{\Gamma}_j = \delta_{ij}\mathbf{\Gamma}_i, \qquad \mathbf{\Gamma}_0 + \mathbf{\Gamma}_1 + \mathbf{\Gamma}_2 = \mathbf{I}. \tag{2.80}$$

Let us come back to the derivation of a variational formulation for $\mathbf{D}^*$. Let D_0 be a *constant* reference inverse density coefficient and $\mathbf{G_e}$ the solution of (2.70). We define the *polarization* vector $\mathbf{P_e}$ and the operator $\mathbf{\Gamma}$ as

$$\mathbf{P_e} = (D - D_0)\mathbf{G_e} = \mathbf{H_e} - D_0\mathbf{G_e} \qquad \text{and} \qquad \mathbf{\Gamma} = \frac{\mathbf{\Gamma}_1}{D_0}. \tag{2.81}$$

We obtain that

$$\big((D - D_0)^{-1}\mathbf{I_3} + \mathbf{\Gamma}\big)\mathbf{P_e} = \mathbf{G_e} + \mathbf{\Gamma}\mathbf{P_e} = \mathbf{G_e} + (\langle\mathbf{G_e}\rangle - \mathbf{G_e}) = \mathbf{e}, \tag{2.82}$$

since $\mathbf{\Gamma H_e} = 0$ because $\nabla \cdot \mathbf{H_e} = 0$. Now, $\mathbf{\Gamma}$ is a self-adjoint and non-negative operator and *assuming* that $D(\mathbf{x}) > D_0$ uniformly, then $(D - D_0)^{-1}\mathbf{I_3} + \mathbf{\Gamma}$ is a positive definite self-adjoint operator. This implies that

$$\big\langle (\mathbf{P_e} - \mathbf{P}) \cdot \big((D - D_0)^{-1}\mathbf{I_3} + \mathbf{\Gamma} \big)(\mathbf{P_e} - \mathbf{P}) \big\rangle \geq 0,$$

for *every* test function $\mathbf{P}$. Notice that no curl-free or divergence-free constraint is imposed on $\mathbf{P}$. Because $\mathbf{\Gamma}$ is self-adjoint and $\langle \cdot \rangle$ is equivalent to integration over $\mathbb{R}^3$ by ergodicity, this inequality can be recast using (2.82) as

$$\langle \mathbf{e} \cdot \mathbf{P} \rangle - \mathbf{e} \cdot (\mathbf{D}^* - D\mathbf{I_3})\mathbf{e} \leq \big\langle \mathbf{P} \cdot \big((D - D_0)^{-1}\mathbf{I_3} + \mathbf{\Gamma} \big)\mathbf{P} \big\rangle,$$

because from (2.81) and (2.71), we have that $\langle \mathbf{P_e} \rangle = (\mathbf{D}^* - D_0\mathbf{I_3})\mathbf{e}$. Choosing now $\mathbf{e} = (\mathbf{D}^* - D_0\mathbf{I_3})^{-1}\langle \mathbf{P} \rangle$, we obtain that

$$\langle \mathbf{P} \rangle \cdot (\mathbf{D}^* - D_0\mathbf{I_3})^{-1}\langle \mathbf{P} \rangle \leq \big\langle \mathbf{P} \cdot \big((D - D_0)^{-1}\mathbf{I_3} + \mathbf{\Gamma} \big)\mathbf{P} \big\rangle,$$

for all test function $\mathbf{P}$. Since the minimum is attained for $\mathbf{P} = \mathbf{P_e}$, we have derived the first Hashin-Shtrikman variational principle

$$\langle \mathbf{P} \rangle \cdot (\mathbf{D}^* - D_0\mathbf{I_3})^{-1}\langle \mathbf{P} \rangle = \min_{\mathbf{P}} \big\langle \mathbf{P} \cdot \big((D - D_0)^{-1}\mathbf{I_3} + \mathbf{\Gamma} \big)\mathbf{P} \big\rangle, \tag{2.83}$$

provided $D(\mathbf{x}) \geq D_0$.

The same calculations can be performed for the dual variables, that is to say when the roles of $\mathbf{G}$ and $\mathbf{H}$ are swapped and $\mathbf{\Gamma}_1$ is replaced by $\mathbf{\Gamma}_2$. The dual variational principle we obtain is

$$\langle \mathbf{P} \rangle \cdot ((\mathbf{D}^*)^{-1} - D_0^{-1}\mathbf{I_3})^{-1}\langle \mathbf{P} \rangle = \min_{\mathbf{P}} \big\langle \mathbf{P} \cdot \big((D^{-1} - D_0^{-1})^{-1}\mathbf{I_3} + \tilde{\mathbf{\Gamma}} \big)\mathbf{P} \big\rangle, \tag{2.84}$$

provided $D^{-1}(\mathbf{x}) > D_0^{-1}$, where $\tilde{\mathbf{\Gamma}} = D_0\mathbf{\Gamma}_2$.

The primal principle (2.83) offers a lower bound for the effective tensor $\mathbf{D}^*$ and the dual principle (2.84) an upper bound. We have now switched the difficulty to find curl-free fields to the calculation of the right-hand side of (2.83), for instance, since the operator $\mathbf{\Gamma}$ is not simple. An interesting situation is when the random inverse density D is mean-zero and isotropic, that is to say when

$$\langle D(\mathbf{x})D(\mathbf{x} + \mathbf{y}) \rangle = R(|\mathbf{y}|).$$

Let indeed $\mathbf{Q}$ be a mean-zero isotropic random field, i.e. such that $\langle Q_i \rangle = 0$ and $\langle Q_i(\mathbf{x})Q_j(\mathbf{x} + \mathbf{y}) \rangle = R_{ij}(|\mathbf{y}|)$. Then we have

$$\langle \mathbf{\Gamma}_1\mathbf{Q} \cdot \mathbf{Q} \rangle = \frac{1}{3}\langle \mathbf{Q} \cdot \mathbf{Q} \rangle, \tag{2.85}$$

as can be seen in the Fourier domain. Indeed, since $\hat{Q}_i(0) = 0$,

$$\begin{aligned}
\langle \mathbf{\Gamma}_1\mathbf{Q} \cdot \mathbf{Q} \rangle(\mathbf{x}) &= \int\int e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}} \frac{k_i k_j}{|\mathbf{k}|^2} \langle \hat{Q}_i(\mathbf{k})\hat{Q}_j(\mathbf{q}) \rangle d\mathbf{k}d\mathbf{q} \\
&= \int\int e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}} \frac{k_i k_j}{|\mathbf{k}|^2} R_{ij}(|\mathbf{k}|)\delta(\mathbf{k} + \mathbf{q})d\mathbf{k}d\mathbf{q} \\
&= \int_0^\infty \int_{S^2} k_i k_j R_{ij}(|\mathbf{k}|)d|\mathbf{k}|d\Omega(\mathbf{k}) = \frac{1}{3}\int_0^\infty \mathrm{Tr}(R(|\mathbf{k}|))d|\mathbf{k}| \\
&= \frac{1}{3}\langle \mathbf{Q} \cdot \mathbf{Q} \rangle(\mathbf{x}),
\end{aligned}$$

which is actually independent of $\mathbf{x}$ by stationarity. We can therefore compute the right-hand side of (2.83) when $\mathbf{P}$ is isotropic. Since D is isotropic, we have by symmetry $\mathbf{D}^* = D^*\mathbf{I_3}$,

and it is sufficient to consider $\mathbf{P}$ in (2.83) of the form $\mathbf{P} = P\mathbf{e}_1$, say. Since $\mathbf{\Gamma}\langle\mathbf{P}\rangle = 0$, we then recast (2.83) as

$$\frac{1}{D^* - D_0} \leq \min_{P \text{ isotropic}} \big\langle \big(\frac{1}{D - D_0} \frac{P^2}{\langle P\rangle^2} + \frac{1}{3D_0} \frac{(P - \langle P\rangle)^2}{\langle P\rangle^2} \big) \big\rangle. \tag{2.86}$$

Defining $\delta = (3D_0)^{-1}(D - D_0)$, we obtain that

$$\frac{1}{D^* - D_0} \leq \frac{1}{3D_0} \Big(\min_{P \text{ isotropic}} \big\langle \frac{1+\delta}{\delta} \big(\frac{P}{\langle P\rangle} - \frac{\delta}{1+\delta} \big)^2 \big\rangle + \big\langle \frac{1}{1+\delta} \big\rangle \Big). \tag{2.87}$$

We now use the following optimization result

$$\min_{\langle q\rangle = \beta} \langle a q^2 \rangle = \langle a^{-1}\rangle^{-1} \beta^2, \tag{2.88}$$

for a positive isotropic field a, and the minimum is realized for $q = \beta \langle a^{-1}\rangle^{-1} a^{-1}$. Since D is isotropic, so is δ. The minimum in (2.87) with the isotropy constraint relaxed is still attained by an isotropic field, and we have

$$\frac{1}{D^* - D_0} \leq \frac{1}{3D_0} \big\langle \frac{1}{1+\delta} \big\rangle \big(\big\langle \frac{1+\delta}{\delta} \big\rangle + \frac{1}{3D_0} \big),$$

which we recast as

$$D^* \geq D_0 \big(1 + 3 \big\langle \frac{\delta}{1+\delta} \big\rangle \big\langle \frac{1}{1+\delta} \big\rangle^{-1} \big), \qquad \delta = \frac{D - D_0}{3D_0} > 0. \tag{2.89}$$

A similar calculation shows that the dual variational principle (2.84) simplifies in the isotropic case to the same expression with $\delta < 0$:

$$D^* \leq D_0 \big(1 + 3 \big\langle \frac{\delta}{1+\delta} \big\rangle \big\langle \frac{1}{1+\delta} \big\rangle^{-1} \big), \qquad \delta = \frac{D - D_0}{3D_0} < 0. \tag{2.90}$$

2.5 Homogenization with boundaries and interfaces

2.5.1 The time dependent case

This section is concerned with the homogenization of rough boundaries and interfaces. We consider a two dimensional setting where two homogeneous media are separated by a $\varepsilon-$periodic interface $z = h(x/\varepsilon)$. We assume that h is a smooth $1-$periodic function with range $[-H, 0]$. The wave equation (2.3) for the pressure field p_ε takes the form

$$\kappa_\varepsilon \frac{\partial^2 p_\varepsilon(t, \mathbf{x})}{\partial t^2} - D_\varepsilon(\mathbf{x}) \Delta p_\varepsilon(t, \mathbf{x}) = S(\mathbf{x}), \tag{2.91}$$

with vanishing initial conditions and for $t \in \mathbb{R}^+$ and $\mathbf{x} = (x, z) \in \mathbb{R}^2$ such that $z \neq h(x/\varepsilon)$, where

$$\kappa_\varepsilon(x, z) = \begin{cases} \kappa^+, & z > h\big(\frac{x}{\varepsilon}\big), \\ \kappa^-, & z < h\big(\frac{x}{\varepsilon}\big), \end{cases} \qquad D_\varepsilon(x, z) = \begin{cases} D^+, & z > h\big(\frac{x}{\varepsilon}\big), \\ D^-, & z < h\big(\frac{x}{\varepsilon}\big). \end{cases} \tag{2.92}$$

Furthermore, the pressure and the normal component of the velocity field are continuous across the interface, so that their jumps vanish

$$[p] = 0, \quad \big[D \frac{\partial p}{\partial n}\big] = 0, \quad z = h\big(\frac{x}{\varepsilon}\big). \tag{2.93}$$

Let us now assume that

$$p_\varepsilon(t,x,z) = p_0(t,x,\frac{x}{\varepsilon},z) + \varepsilon p_1(t,x,\frac{x}{\varepsilon},z) + \varepsilon^2 p_2(t,x,\frac{x}{\varepsilon},z) + O(\varepsilon^3), \tag{2.94}$$

where the functions p_i are $1-$periodic with respect to the third variable. Notice that the second jump condition for $p = p(t,x,y,z)$ now reads

$$\big[D\big(\frac{h'}{\varepsilon^2}\frac{\partial p}{\partial y} + \frac{h'}{\varepsilon}\frac{\partial p}{\partial x} - \frac{\partial p}{\partial z}\big)\big] = 0, \quad z = h(y).$$

Upon plugging (2.94) into (2.91) and (2.93) and equating like powers of ε, we obtain first that

$$\begin{aligned} &D\frac{\partial^2 p_0}{\partial y^2} = 0, \quad z \neq h(y), \\ &[p_0] = 0, \quad \big[D\frac{\partial p_0}{\partial y}\big] = 0, \quad z = h(y). \end{aligned} \tag{2.95}$$

Because p_0 is periodic in y, we deduce that $p_0 = p_0(x,z)$ independent of y. The second equation is

$$\begin{aligned} &D\frac{\partial^2 p_1}{\partial y^2} = 0, \quad z \neq h(y), \\ &[p_1] = 0, \quad \big[D\big(\frac{\partial p_1}{\partial y} + \frac{\partial p_0}{\partial x}\big)\big] = 0, \quad z = h(y), \end{aligned} \tag{2.96}$$

because p_0 is independent of y. We anticipate here that $\frac{\partial p_0}{\partial x}$ is a smooth function. Since it does not depend on y, it cannot be discontinuous at $z = h(y)$. We see therefore that

$$p_1(x,y,z) = \theta_1(y,z)\frac{\partial p_0}{\partial x}(x,z), \tag{2.97}$$

where θ_1 is the mean-zero $1-$periodic solution of

$$\begin{aligned} &D\frac{\partial^2 \theta_1}{\partial y^2} = 0, \quad z \neq h(y), \\ &[\theta_1] = 0, \quad \big[D\big(\frac{\partial \theta_1}{\partial y} + 1\big)\big] = 0, \quad z = h(y). \end{aligned} \tag{2.98}$$

We shall come back later to the calculation of θ_1. The next order equation yields

$$\begin{aligned} &\kappa\frac{\partial^2 p_0}{\partial t^2} + \frac{\partial}{\partial y}D\big(\frac{\partial p_2}{\partial y} + \frac{\partial p_1}{\partial x}\big) + \frac{\partial}{\partial x}D\big(\frac{\partial p_1}{\partial y} + \frac{\partial p_0}{\partial x}\big) + D\frac{\partial^2 p_0}{\partial z^2} = S(\mathbf{x}), \quad z \neq h(y) \\ &[p_2] = 0, \quad \big[D\big(h'\frac{\partial p_2}{\partial y} + h'\frac{\partial p_1}{\partial x} - \frac{\partial p_0}{\partial z}\big)\big] = 0, \quad z = h(y). \end{aligned} \tag{2.99}$$

This equation admits a solution provided some compatibility condition is ensured. This condition is obtained by averaging the above equation over $(0,1)$ in y. We deduce from (2.98) that the derivative in y of $D(\frac{\partial \theta_1}{\partial y}+1)$ for $z \neq h(y)$ and its jumps at $z = h(y)$ vanish. Therefore,

$$D_{\text{eff}}(z) = D(y,z)\big(\frac{\partial \theta_1}{\partial y}(y,z) + 1\big) \tag{2.100}$$

independent of y. Furthermore, we have

$$\int_0^1 \frac{\partial}{\partial x}D\big(\frac{\partial p_1}{\partial y} + \frac{\partial p_0}{\partial x}\big)dy = D_{\text{eff}}(z)\frac{\partial^2 p_0}{\partial x^2}.$$

For every $z \in \mathbb{R}$, $h^{-1}(z)$ is a finite set of points $y_i(z)$, $1 \le i \le I(z)$, which is empty for $z < -H$ and $z > 0$. The function $D(\frac{\partial p_2}{\partial y} + \frac{\partial p_1}{\partial x})$ is $1-$periodic in y by assumption. Therefore, the integral of its derivative (in the sense of distributions) vanishes and we have thanks to the jump conditions in (2.99) that

$$\int_0^1 \frac{\partial}{\partial y} D\big(\frac{\partial p_2}{\partial y} + \frac{\partial p_1}{\partial x}\big) + \big(\sum_{i=1}^{I(z)} \frac{[D](y_i)}{h'(y_i)}\big) \frac{\partial p_0}{\partial z} = 0.$$

Introducing

$$\langle D\rangle(z) = \int_0^1 D(y,z)dy, \quad \text{and} \quad \langle \kappa\rangle(z) = \int_0^1 \kappa(y,z)dy, \tag{2.101}$$

we verify that

$$\big(\sum_{i=1}^{I(z)} \frac{[D](y_i)}{h'(y_i)}\big) = \frac{\partial \langle D\rangle}{\partial z}.$$

Upon integrating (2.99) in y over $(0,1)$, we obtain

$$\langle \kappa\rangle \frac{\partial^2 p_0}{\partial t^2} - D_{\text{eff}} \frac{\partial^2 p_0}{\partial x^2} - \frac{\partial}{\partial z}\big(\langle D\rangle \frac{\partial p_0}{\partial z}\big) = S(\mathbf{x}). \tag{2.102}$$

This equation holds in the whole physical domain $(x,z) \in \mathbb{R}^2$. For $z > 0$, we observe that $\frac{\partial \theta_1}{\partial y} = 0$, hence $D_{\text{eff}}(z) = D^+$. Moreover, $\langle D\rangle = D^+$ and $\langle \kappa\rangle = \kappa^+$. Similarly, $D_{\text{eff}}(z) = D^-$, $\langle D\rangle = D^-$, and $\langle \kappa\rangle = \kappa^-$ for $z < -H$. Therefore (2.102) is the same equation as (2.91) outside the layer $-H \le z \le 0$. Inside this layer, the heterogeneities of the fast oscillating boundary have been homogenized. Notice that the homogenized density tensor $\boldsymbol{\rho} = \mathbf{D}^{-1}$ is again asymmetrical, since in general $D_{\text{eff}} \neq \langle D\rangle$.

In order to obtain an explicit expression for D_{eff}, we now assume that $h(0) = 0$ and that h is strictly decreasing from $y = 0$ to $y = y_1(-H)$ and then strictly increasing from $y = y_1(-H)$ to $y = 1$ where $h(1) = h(0)$ by periodicity. For $-H < z < 0$, we denote by $y_1(z)$ and $y_2(z)$ the functions such that $h(y_1(z)) = z$, $h(y_2(z)) = z$ and $y_1(z) < y_2(z)$. Therefore, for $-H < z < 0$, $D(y,z) = D^+$ for $y \in (y_1(z), y_2(z))$ and $D(y,z) = D^-$ for $y \in (0, y_1(z)) \cup (y_2(z), 1)$.

Let us now calculate θ_1 in (2.98). Clearly, for z fixed, θ_1 is linear on $(0, y_1)$, (y_1, y_2) and $(y_2, 1)$ of slope (a, b, c), respectively. By periodicity, we obtain that $a = c$ and $b = -(y_2 - y_1)^{-1}(1 - y_2 + y_1)a$. Now the jump conditions both yield that $D^-(a+1) = D^+(b+1)$, from which we deduce that

$$a = \frac{(D^+ - D^-)(y_2 - y_1)}{(1-(y_2-y_1))D^+ + (y_2-y_1)D^-}, \quad b = \frac{(D^+ - D^-)(y_2 - y_1 - 1)}{(1-(y_2-y_1))D^+ + (y_2-y_1)D^-}.$$

Therefore,

$$D_{\text{eff}} = D^-(a+1) = D^+(b+1) = \frac{D^- D^+}{(1-(y_2-y_1))D^+ + (y_2-y_1)D^-}. \tag{2.103}$$

The same method can be used to homogenize rough boundaries. Consider first the propagation of the pressure waves with Neumann boundary conditions

$$\begin{aligned} &\kappa^+ \frac{\partial^2 p_\varepsilon(t,\mathbf{x})}{\partial t^2} - D^+ \Delta p_\varepsilon(t,\mathbf{x}) = S(\mathbf{x}), \quad z \ge h\big(\frac{x}{\varepsilon}\big), \\ &\frac{\partial p_\varepsilon}{\partial n} = 0, \quad z = h\big(\frac{x}{\varepsilon}\big). \end{aligned} \tag{2.104}$$

This equation is equivalent to (2.91) with $D^- = 0$ as can be easily checked, with the difference that the source term $S(\mathbf{x})$ only contributes when $z > h(x/\varepsilon)$. We define then $\langle S\rangle(x,z) = \int_0^1 S(x,y,z)dy$, where $S(x,y,z) = S(x,z)$ when $z > h(y)$ and $S(x,y,z) = 0$ otherwise. Now because $D^- = 0$, we deduce from (2.100) that $D_{\text{eff}}(z) = 0$, and (2.102) becomes

$$\langle\kappa\rangle\frac{\partial^2 p_0}{\partial t^2} - \frac{\partial}{\partial z}\big(\langle D\rangle\frac{\partial p_0}{\partial z}\big) = \langle S\rangle(\mathbf{x}), \qquad (x,z) \in \mathbb{R}\times(-H,\infty). \tag{2.105}$$

Boundary conditions at $z = -H$ are not necessary as $\langle D\rangle(-H) = \langle\kappa\rangle(-H) = \langle S\rangle(-H) = 0$. When h is strictly decreasing and then strictly increasing on one period, (2.105) simplifies to

$$\kappa^+\frac{\partial^2 p_0}{\partial t^2} - \frac{D^+}{y_2 - y_1}\frac{\partial}{\partial z}\big((y_2 - y_1)\frac{\partial p_0}{\partial z}\big) = S(\mathbf{x}), \quad (x,z) \in \mathbb{R}\times(-H,\infty). \tag{2.106}$$

Dirichlet conditions ($p_\varepsilon = 0$ on $z = h(x/\varepsilon)$) can also be considered. We then easily deduce from (2.95) that $p_0 = 0$ on $z = h(y)$. Since p_0 is independent of y, this simply implies that $p_0 = 0$ on $z = 0$. It also solves the already homogeneous equation (2.91) for $z > 0$. The roughness of the boundary is then not seen in the case of Dirichlet boundary conditions.

2.5.2 Plane wave reflection and transmission

Instead of the evolution problem (2.91), we can also consider the reflection and transmission of planes waves from a rough interface

$$\begin{aligned}
&\kappa_\varepsilon\omega^2 p_\varepsilon(\mathbf{x}) + D_\varepsilon(\mathbf{x})\Delta p_\varepsilon(\mathbf{x}) = 0, \quad z \neq h\big(\frac{x}{\varepsilon}\big),\\
&p_\varepsilon = e^{i\mathbf{k}\cdot\mathbf{x}} + p_\varepsilon^R, \quad z > h\big(\frac{x}{\varepsilon}\big),\\
&p_\varepsilon = p_\varepsilon^T, \quad z < h\big(\frac{x}{\varepsilon}\big),\\
&[p_\varepsilon] = 0, \quad \big[D\frac{\partial p_\varepsilon}{\partial n}\big] = 0, \quad z = h\big(\frac{x}{\varepsilon}\big).
\end{aligned} \tag{2.107}$$

where k_ε and D_ε are defined in (2.92) and

$$\kappa^+\omega^2 = D^+k^2, \qquad k = |\mathbf{k}|, \quad \mathbf{k} = (k_x, k_z).$$

Let us introduce $p_\varepsilon = e^{ik_x x}\tilde{p}_\varepsilon = e^{ik_x x}(e^{ik_z z} + \tilde{p}_\varepsilon^R)$ for $z > h(x/\varepsilon)$ and $p_\varepsilon = e^{ik_x x}\tilde{p}_\varepsilon^T$ for $z < h(x/\varepsilon)$. Here, $k_z < 0$. Using the ansatz (2.94), we obtain that $\tilde{p}_\varepsilon^R$ and $\tilde{p}_\varepsilon^T$ only depend on z. Since they solve a homogeneous equation for $z > 0$ and $z < -H$, respectively, we have that $\tilde{p}_\varepsilon^R(z) = Re^{-ik_z z}$ for $z > 0$ and $\tilde{p}_\varepsilon^T = Te^{i\tilde{k}_z z}$ for $z < -H$, where $\tilde{k}_z < 0$ is defined by

$$\kappa^-\omega^2 = D^-(k_x^2 + (\tilde{k}_z)^2).$$

For $-H < z < 0$, we define $p_\varepsilon = e^{ik_x x}w_\varepsilon$ for $-H < z < 0$. The leading order term w_0 in w_ε satisfies then the following equation, thanks to (2.102)

$$(\omega^2\langle\kappa\rangle - k_x^2 D_{\text{eff}})w_0 + \frac{\partial}{\partial z}\big(\langle D\rangle\frac{\partial w_0}{\partial z}\big) = 0, \quad -H < z < 0. \tag{2.108}$$

It remains to find boundary conditions for w_0. The continuity of u_0 and $\partial_x u_0$ at $z = 0$ and $z = -H$ yield that

$$\begin{aligned}
&1 + R = w(0), \quad ik_z(1 - R) = w'(0),\\
&T = w(-H), \quad\ \ i\tilde{k}_z T = w'(-H).
\end{aligned} \tag{2.109}$$

Upon eliminating R and T, we find that

$$w'(0) + ik_z w(0) = 2ik_z \quad \text{and} \quad w'(-H) - i\tilde{k}_z w(-H) = 0. \tag{2.110}$$

Once (2.108) and (2.110) are solved for w_0, the response and transmission coefficients are obtained from (2.109).

In the case of a rough boundary with Neumann boundary conditions, where $D^- = 0$, the above analysis holds with $T = 0$. Introducing the impedance

$$\zeta = \frac{\partial_z p_0}{p_0}(z=0) = \frac{\partial_z w_0}{w_0}(z=0), \tag{2.111}$$

which is a real coefficient, we obtain that

$$R = \frac{ik_z - \zeta}{ik_z + \zeta}. \tag{2.112}$$

Clearly $|R| = 1$ as expected from conservation of energy. In the limit $\varepsilon \to 0$ in the case of Neumann boundary conditions, the pressure field p_ε converges to a function p_0 that solves

$$\begin{aligned} &k^2 p_0 + \frac{\partial^2 p_0}{\partial z^2} = 0, \quad z > 0 \\ &p_0 = e^{i\mathbf{k}\cdot\mathbf{x}} + p_0^R \\ &\frac{\partial p_0}{\partial n} + \zeta p_0 = 0, \quad z = 0 \qquad \big(\frac{\partial}{\partial n} = -\frac{\partial}{\partial z} \text{ on this surface }\big). \end{aligned} \tag{2.113}$$

Consider for example the case of a comb-like surface, where the periodic surface in the (y, z) plane is composed of the line $\mathbb{R} \times \{-H\}$ and the segments $\{n\} \times (-H, 0)$ for $n \in Z$. This surface cannot be represented as a graph of a function, but can be approximated by piecewise linear functions $h_\eta(y)$ where $h_\eta(0) = h_\eta(1) = 0$ and $h_\eta(y) = -H$ on $(\eta, 1-\eta)$, where η is a small parameter sent to 0. In this limit, the equation for w_0 is

$$\begin{aligned} &k^2 w_0 + \frac{\partial^2 w_0}{\partial z^2} = 0 \\ &w_0'(0) + ik_z w_0(0) = 2ik_z \\ &w_0'(-H) = 0. \end{aligned} \tag{2.114}$$

Upon solving for w_0, we obtain that

$$\zeta = -k \tan kH. \tag{2.115}$$

Notice that this impedance can be positive or negative depending on the size H of the combs.

Chapter 3

Geometric Optics

3.1 Introduction

Recall that in the variable $\mathbf{y} = (t, \mathbf{x})$ a differential operator $L(\mathbf{y}, \mathbf{D_y})$ is of order $m \geq 0$ if its highest-order differentiation is of order m. Then $L_m(\mathbf{y}, \mathbf{D_y})$ is the homogeneous differential operator of order m such that $L - L_m$ is at most of order $m-1$. The *characteristic variety* of L, denoted by Char$L$, is then the subset of points $(\mathbf{y}, \boldsymbol{\eta}) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \backslash \{\mathbf{0}\}$ such that

$$\det L_m(\mathbf{y}, \boldsymbol{\eta}) = 0. \tag{3.1}$$

Geometric Optics is concerned with highly-oscillatory solutions of differential equations. High oscillations mean high frequencies. Now differential operators of order m multiply frequencies by a coefficient of order $|\boldsymbol{\eta}|^m$, which is much larger than $|\boldsymbol{\eta}|^{m-1}$ when $|\boldsymbol{\eta}| \gg 1$. This implies that the leading term $L_m(\mathbf{y}, \mathbf{D_y})$ plays a crucial role in geometric optics.

Consider an operator such that $L = L_m$ and look for a plane wave solution of the form

$$L_m(\mathbf{y}, \mathbf{D_y})[e^{i\mathbf{y}\cdot\boldsymbol{\eta}}\mathbf{a}] = i^m L_m(\mathbf{y}, \boldsymbol{\eta}) e^{i\mathbf{y}\cdot\boldsymbol{\eta}}\mathbf{a} = 0. \tag{3.2}$$

This will happen locally if and only if $(\mathbf{y}, \boldsymbol{\eta}) \in$ Char$L$. If $L_m(\mathbf{D_y})$ has constant coefficients, then $e^{i\mathbf{y}\cdot\boldsymbol{\eta}}\mathbf{a}$ is a global solution of $L_m\mathbf{u} = 0$ if and only if $\det L_m(\boldsymbol{\eta}) = 0$, in which case $e^{i\mathbf{y}\cdot\boldsymbol{\eta}}\mathbf{a}$ is a global solution provided that $\mathbf{a}$ is in the kernel of $L_m(\boldsymbol{\eta})$.

Let us insist on the high frequency regime by introducing $\hat{\boldsymbol{\eta}} = \boldsymbol{\eta}/|\boldsymbol{\eta}|$ and $\varepsilon = |\boldsymbol{\eta}|^{-1} \ll 1$. Then we verify that

$$L_m(\mathbf{D_y}) e^{\frac{i\hat{\boldsymbol{\eta}}\cdot\mathbf{y}}{\varepsilon}}\mathbf{a} = \frac{i^m}{\varepsilon^m} L_m(\hat{\boldsymbol{\eta}}) e^{\frac{i\hat{\boldsymbol{\eta}}\cdot\mathbf{y}}{\varepsilon}}\mathbf{a} = 0,$$

provided that $(\mathbf{y}, \hat{\boldsymbol{\eta}})$ is in the characteristic variety of L_m and $\mathbf{a}$ is an associated eigenvector in the kernel of $L_m(\hat{\boldsymbol{\eta}})$.

In the case of non-constant coefficients in the linear differential operator L, we no longer expect plane waves of the form $e^{i\boldsymbol{\eta}\cdot\mathbf{y}}$ to be solutions. These need to be generalized to the following form

$$\mathbf{u}_0(\mathbf{y}) = e^{i\frac{\phi(\mathbf{y})}{\varepsilon}}\mathbf{a}_0(\mathbf{y}). \tag{3.3}$$

Note that the above plane waves were such that $\phi(\mathbf{y}) = \boldsymbol{\eta}\cdot\mathbf{y}$ and $\nabla_{\mathbf{y}}\phi(\mathbf{y}) = \boldsymbol{\eta}$. Plugging this ansatz into the differential operator, we verify that

$$L(\mathbf{y}, \mathbf{D_y})\mathbf{u}_0(\mathbf{y}) = \frac{1}{\varepsilon^m} L_m(\mathbf{y}, i\nabla_{\mathbf{y}}\phi(\mathbf{y}))\mathbf{u}_0(\mathbf{y}) + O(\varepsilon^{1-m}), \tag{3.4}$$

assuming that ϕ and $\mathbf{a}_0$ are smooth functions.

Exercise 3.1.1 Verify the above statement.

We thus see that different behaviors emerge depending on whether $(\mathbf{y}, \nabla\phi(\mathbf{y}))$ belongs to the characteristic variety of L or not. When the characteristic variety is empty, i.e., $\det L_m(\mathbf{y}, \boldsymbol{\eta}) \equiv |\boldsymbol{\eta}|^m L_m(\mathbf{y}, \hat{\boldsymbol{\eta}}) \neq 0$, then we say that the operator L is *elliptic*. Elliptic operators do not admit oscillatory solutions unless there is an oscillatory source. Indeed we deduce from (3.4) that $\mathbf{a}_0$ has to vanish. We refer to [26, Chap.4] for additional details on the geometric optics theory for elliptic equations. However, for hyperbolic equations, the characteristic variety is not empty, and (3.4) tells us that locally, highly oscillatory solutions of $L\mathbf{u} = 0$ require us to construct phases $\phi(\mathbf{y})$ such that $(\mathbf{y}, \nabla\phi(\mathbf{y})) \in \operatorname{Char} L$. This latter statement is in fact equivalent to $\phi(\mathbf{y})$ being a solution to the famous *eikonal* equation.

3.2 Second-order scalar equation

Consider the second-order scalar equation

$$\frac{\partial^2 p}{\partial t^2} - c^2(\mathbf{x})\Delta p = 0, \qquad p(0,\mathbf{x}) = p_0(\mathbf{x}), \quad \frac{\partial p}{\partial t}(0,\mathbf{x}) = 0, \tag{3.5}$$

which corresponds to $L(\mathbf{y}, \mathbf{D}) = L_2(\mathbf{y}, \mathbf{D})$ with symbol $L(\mathbf{y}, \boldsymbol{\eta}) = -\tau^2 + c^2(\mathbf{x})|\boldsymbol{\xi}|^2$. We recall that $\mathbf{y} = (t, \mathbf{x})$ and $\boldsymbol{\eta} = (\tau, \boldsymbol{\xi})$. We thus find that

$$\operatorname{Char} L = \{(\mathbf{y}, \boldsymbol{\eta}), \text{ such that } -\tau^2 + c^2(\mathbf{x})|\boldsymbol{\xi}|^2 = 0\}. \tag{3.6}$$

The characteristic variety may be decomposed into two leaves parameterized by $\tau = \pm c(\mathbf{x})|\boldsymbol{\xi}|$. This shows that information propagates with speed $\pm c(\mathbf{x})$ since τ is real-valued when $\boldsymbol{\xi}$ is. We thus observe that $(\mathbf{y}, \nabla\phi(\mathbf{y})) \in \operatorname{Char} L$ is equivalent to the classical *eikonal* equation

$$L(\mathbf{y}, i\nabla\phi(\mathbf{y})) = \Big(\frac{\partial\phi}{\partial t}\Big)^2 - c^2(\mathbf{x})|\nabla\phi|^2 = 0, \tag{3.7}$$

which again has the two possible solutions

$$\frac{\partial\phi}{\partial t} = \pm c(\mathbf{x})|\nabla\phi|. \tag{3.8}$$

Before we further analyze the eikonal equation, we recast the high-frequency regime as a regime with highly oscillatory functions.

3.2.1 High Frequency Regime

Consider the framework where the typical distance of propagation L of the waves is much larger than the typical wavelength λ in the system. We introduce the small adimensionalized parameter

$$\varepsilon = \frac{\lambda}{L} \ll 1. \tag{3.9}$$

We thus rescale space $\mathbf{x} \to \varepsilon^{-1}\mathbf{x}$ and since $l = c \times t$ rescale time accordingly $t \to \varepsilon^{-1}t$ to obtain the equation

$$\varepsilon^2 \frac{\partial^2 p_\varepsilon}{\partial t^2} = c_\varepsilon^2(\mathbf{x})\varepsilon^2\Delta p_\varepsilon, \qquad p_\varepsilon(0,\mathbf{x}) = p_{0\varepsilon}(\varepsilon^{-1}\mathbf{x}), \quad \frac{\partial p_\varepsilon}{\partial t}(0,\mathbf{x}) = h_{0\varepsilon}(\varepsilon^{-1}\mathbf{x}). \tag{3.10}$$

Note that the terms ε^2 cancel in the above partial differential equation so that the high frequency regime is really encoded in the highly oscillatory initial conditions. Energy conservation is then recast as

$$\mathcal{E}(t) = \frac{1}{2\rho_0}\int_{\mathbb{R}^d}\Big(|\varepsilon\nabla p_\varepsilon|^2(t,\mathbf{x}) + c^{-2}(\mathbf{x})\Big|\varepsilon\frac{\partial p_\varepsilon}{\partial t}\Big|^2(t,\mathbf{x})\Big)d\mathbf{x} = \mathcal{E}(0). \tag{3.11}$$

One of our objective in such a regime is to characterize the spatial distribution of the energy density, at least in an approximate sense as $\varepsilon \to 0$. The high frequency regime of the wave field is apparent in its initial condition p_0, which depends on ε. We thus want a highly oscillatory initial field, and want moreover that it be of finite energy. In a homogeneous medium, the simplest highly oscillatory fields are the plane waves $\exp(i\mathbf{y}\cdot\boldsymbol{\eta}/\varepsilon)$, which may be evaluated at $t = 0$ to yield $\exp(i\mathbf{x}\cdot\boldsymbol{\xi}/\varepsilon)$. Since such a function is not of bounded energy, we multiply it by a, say, smooth and compactly supported function $a_0(\mathbf{x})$. An initial condition of the form $e^{i\mathbf{x}\cdot\boldsymbol{\xi}/\varepsilon}a_0(\mathbf{x})$ both has high frequency oscillations (of order ε^{-1}) and has bounded energy.

In media with non-constant coefficients, plane wave solutions rarely exist. Even in homogeneous media, one may be interested in highly oscillatory initial conditions that may be constant along surfaces that are not necessarily hyperplanes (the plane waves $e^{i\mathbf{x}\cdot\boldsymbol{\xi}/\varepsilon}$ are constant on hyperplanes $\mathbf{x}\cdot\boldsymbol{\xi}$ constant). This encourages us to consider initial conditions of the form

$$p_0(\mathbf{x}) = \exp\big(\frac{i\phi(\mathbf{x})}{\varepsilon}\big)a_0(\mathbf{x}). \tag{3.12}$$

When $\phi(\mathbf{x}) = \mathbf{x}\cdot\boldsymbol{\xi}$, we retrieve the preceding plane waves. Once we have initial conditions of the above form, the main question that arises is whether the solution of the wave equation (3.5) admits a useful asymptotic expansion. The answer consists of looking for solutions of the same form as that of the initial conditions, namely, a highly oscillatory exponential phase multiplied by a slowly varying amplitude. The theory of geometric optics provides a general framework to solve for the phase and the amplitude.

3.2.2 Geometric Optics Expansion

Following the preceding discussion, we define the following ansatz

$$p_\varepsilon(\mathbf{y}) = e^{i\phi(\mathbf{y})/\varepsilon}a_\varepsilon(\mathbf{y}), \qquad a_\varepsilon(\mathbf{y}) = a_0(\mathbf{y}) + \varepsilon a_1(\mathbf{y}) + \dots. \tag{3.13}$$

and wish to compare the above ansatz to the high frequency solution $p_\varepsilon(t,\mathbf{x})$. Recall that $c(\mathbf{x})$ is independent of ε in this chapter. Some algebra shows that:

$$\begin{aligned} L\big(e^{i\phi(\mathbf{y})/\varepsilon}a(\mathbf{y})\big) &= e^{i\phi(\mathbf{y})/\varepsilon}\times \\ &\times\big(\big(\frac{i}{\varepsilon}\big)^2 L(\mathbf{y},\nabla_{\mathbf{y}}\phi)a + \frac{2i}{\varepsilon}V_\phi a + \frac{i}{\varepsilon}(L(\mathbf{y},\mathbf{D}_{\mathbf{y}})\phi)a + (L(\mathbf{y},\mathbf{D}_{\mathbf{y}})a)\big) \end{aligned} \tag{3.14}$$

where we have defined the vector field

$$V_\phi(\mathbf{y}) = \frac{\partial\phi}{\partial t}(\mathbf{y})\frac{\partial}{\partial t} - c^2(\mathbf{x})\frac{\partial\phi}{\partial x_j}(\mathbf{y})\frac{\partial}{\partial x_j}. \tag{3.15}$$

Note that as is usual in differential geometry, vector fields are identified with first-order differentials, i.e., the basis elements $\mathbf{e}_i$ of $\mathbb{R}^d$ are identified with $\frac{\partial}{\partial x_i}$; $1 \le i \le d$.

Exercise 3.2.1 Verify the above derivation.

In order for $e^{i\phi(\mathbf{y})/\varepsilon}a_\varepsilon(\mathbf{y})$ to solve the wave equation, at least approximately, we see that the leading term in (3.14) must vanish, which implies that the phase ϕ must satisfy the **eikonal** equation

$$L(\mathbf{y}, i\nabla\phi(\mathbf{y})) = \big(\frac{\partial\phi}{\partial t}\big)^2 - c^2(\mathbf{x})|\nabla\phi|^2 = 0. \tag{3.16}$$

In order to justify the above asymptotic expansion, we need to ensure that the phase function $\phi(\mathbf{y})$ is uniquely determined. This is so for instance when $|\nabla\phi(0,\mathbf{x})|$ never vanishes on the support of $a_0(\mathbf{x})$, in which case either choice in (3.8) can be made. We will come back to the solution of the eikonal equation in the next section. For the moment we assume that the eikonal equation is uniquely solvable on an interval $(0,T)$ once a choice of sign has been made in (3.8).

The vector field in (3.15) is now uniquely defined and it remains to find an equation for $a_\varepsilon(\mathbf{y})$, at least approximately. We plug the expansion for a_ε into (3.14), with a replaced by a_ε and equate like powers of ε. The term of order ε^{-1} provides that

$$2V_\phi a_0 + (L\phi)a_0 = 0. \tag{3.17}$$

This is a **transport** equation for a_0. We verify that $V_\phi\phi = 0$, which implies that ϕ is constant along the integral curves of the vector field V_ϕ, which are called **rays**. We see that a_0 is also transported along the rays, except for the presence of the "absorption" coefficient (which does not need to have a constant sign) $L\phi$. Because we assume that $|\nabla\phi| \neq 0$ so that $\partial_t\phi \neq 0$, the integral curves of V_ϕ indeed are transverse to the hyperplane $t = 0$ so that the equation for $a_0(\mathbf{y})$ with $a_0(0,\mathbf{x})$ known admits a unique smooth solution.

The higher-order terms in the expansion yield in turn that

$$2V_\phi a_n + (L\phi)a_n + La_{n-1} = 0, \qquad n \geq 1. \tag{3.18}$$

For the same reasons as before, the above equation for a_n admits a unique solution.

Formally, if the above construction is carried out for all $n \geq 1$, we find that

$$Lp^\varepsilon = L\big(e^{i\phi(\mathbf{y})/\varepsilon}a_\varepsilon(\mathbf{y})\big) = O(\varepsilon^\infty) \ \sim 0.$$

We refer to [26] for additional details on the notation, which we will not use in the sequel.

Rather we now want to understand the accuracy of the above procedure provided that the above expansion is truncated at order $n \geq 0$, say. This is done by an energy estimate. Consider

$$p^\varepsilon(\mathbf{y}) = e^{i\phi(\mathbf{y})/\varepsilon}\sum_{j=0}^{n} a_j(\mathbf{y})\varepsilon^j.$$

Then we verify that

$$\varepsilon^2 Lp^\varepsilon = \varepsilon^{2+n}La_n e^{i\phi(\mathbf{y})/\varepsilon} = \varepsilon^2 f_\varepsilon(\mathbf{y}), \quad \text{with} \quad \|f_\varepsilon(t,\cdot)\|_{H^s(\mathbb{R}^d)} = O(\varepsilon^{n-s}), \quad 0 \leq s \leq n,\ 0 \leq t \leq T.$$

Exercise 3.2.2 Verify the last statement.

The error $\delta_\varepsilon = p_\varepsilon - p^\varepsilon$ thus satisfies the equation

$$\varepsilon^2 L\delta_\varepsilon = \varepsilon^2 f_\varepsilon, \qquad \delta_\varepsilon(0,\mathbf{x}) = \frac{\partial\delta_\varepsilon}{\partial t}(0,\mathbf{x}) = 0.$$

This shows the existence of a constant such that, for instance,

$$\|p^\varepsilon(t,\cdot) - p_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^n, \quad \text{uniformly on } 0 \leq t \leq T. \tag{3.19}$$

Proof. This is based on the energy method. Consider the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2(\mathbf{x})\Delta u = f,$$

with vanishing initial conditions at $t = 0$, and the total energy

$$\mathcal{E}(t) = \frac{1}{2\rho_0}\int_{\mathbb{R}^d}\Big(|\varepsilon\nabla u|^2(t,\mathbf{x}) + c^{-2}(\mathbf{x})\Big|\varepsilon\frac{\partial u}{\partial t}\Big|^2(t,\mathbf{x})\Big)d\mathbf{x}.$$

We find that

$$\dot{\mathcal{E}}(t) = \frac{1}{\rho_0}\int_{\mathbb{R}^d}\Big(\varepsilon\nabla u\cdot\varepsilon\nabla\frac{\partial u}{\partial t} + c^{-2}(\mathbf{x})\varepsilon\frac{\partial u}{\partial t}\varepsilon\frac{\partial^2 u}{\partial t^2}\Big)d\mathbf{x} = \frac{1}{\rho_0}\int_{\mathbb{R}^d}\varepsilon f(t)\varepsilon\frac{\partial u}{\partial t}d\mathbf{x},$$

after integrations by parts. As in the proof of Theorem 2.1.1, we find that

$$\|\varepsilon\frac{\partial u}{\partial t}\|(t) \leq C_T\varepsilon\|f\|_{\mathcal{C}(0,T;L^2(\mathbb{R}^d))},$$

Here $\|\cdot\|$ is the $L^2(\mathbb{R}^d)$ norm.

Exercise 3.2.3 Verify the the last statement in detail.

After integration in time, this shows that

$$\|u(t)\| \leq C_T\|f\|_{\mathcal{C}(0,T;L^2(\mathbb{R}^d))}, \quad 0 \leq t \leq T.$$

This concludes the proof of the bound on δ_ε. Of course, the proof holds independent of ε and we could have chosen $\varepsilon = 1$ since all derivatives in the wave equation are second-order, and thus the powers ε^2 cancel out. We have kept the ε-dependence to insist on the fact that this is the high-frequency energy estimate $\mathcal{E}(t)$ that is needed here. □

3.3 The Eikonal equation

We refer to the Appendix in Chapter 5 of [26] and to Chapter 3 in [15] for the details of the theory.

The eikonal equation takes the form

$$L(\mathbf{y}, \nabla\phi) = 0. \tag{3.20}$$

A better notation for $\nabla\phi(\mathbf{y})$ would be the one form $d\phi(\mathbf{y})$ since the theory is more natural geometrically when $(\mathbf{y}, d\phi(\mathbf{y}))$ is seen as a covariant vector rather than a contravariant vector. In Euclidean geometry, both types of vectors have the same coordinate expressions and we will use the notation in (3.20).

The objective is to solve the above equation by the method of characteristics. Let us assume that $\phi(\mathbf{y})$ is known on a hypersurface M of $\mathbb{R}^{d+1}$, for instance on the hyperplane $t = 0$, so that $\phi_{|M} = g$. Then the derivatives of ϕ in the directions tangent to M are also known, since

$$\mathbf{e}\cdot\nabla\phi(\mathbf{z}) = \mathbf{e}\cdot\nabla g(\mathbf{z}), \qquad \mathbf{e} \in T_{\mathbf{z}}M.$$

Here $T_{\mathbf{z}}M$ is the tangent space to M at $\mathbf{z}$. It thus remains one unknown directional derivative of $\phi(\mathbf{z})$, namely that in the direction orthogonal to M. We use the equation (3.20) to determine such a derivative. We need however to make sure that (3.20) indeed determines that derivative. This is the case provided that $\nabla_{\boldsymbol{\eta}}L(\mathbf{y}, \nabla\phi)$ is not tangent to M at $\mathbf{y} \in M$. Otherwise,

$L(\mathbf{y}, \nabla\phi) = 0$ only gives information about the derivatives of ϕ that are tangent to M, and we already know those.

For instance for the wave equation with M the hyperplane $t = 0$, the eikonal equation for $L(\mathbf{y}, \boldsymbol{\eta}) = \tau^2 - c^2(\mathbf{x})|\boldsymbol{\xi}|^2$ implies that

$$\frac{\partial}{\partial\tau}(\tau^2 - c^2(\mathbf{x})|\boldsymbol{\xi}|) = 2\tau \neq 0,$$

and that

$$\frac{\partial\phi}{\partial t} = \pm|\nabla\phi| = \pm|\nabla g|.$$

The constraint $\nabla_{\boldsymbol{\eta}} L(\mathbf{y}, \nabla\phi)$ is not tangent to M thus may provide several solutions. We pick one of them. We have thus defined $\nabla\phi(\mathbf{z})$ in all directions for all $\mathbf{z} \in M$. It remains to find $\phi(\mathbf{z})$ away from M. This can be achieved by the method of characteristics.

Let us first differentiate the equation (3.20):

$$\frac{\partial L}{\partial y_j}(\mathbf{y}, \nabla\phi(\mathbf{y})) + \nabla_{\boldsymbol{\eta}} L(\mathbf{y}, \nabla\phi(\mathbf{y})) \cdot \nabla\frac{\partial\phi(\mathbf{y})}{\partial y_j} = 0,$$

by the chain rule. Let us now construct curves in the phase space $(\mathbf{y}(s), \boldsymbol{\eta}(s))$ such that $\nabla\phi(\mathbf{y}(s)) = \boldsymbol{\eta}(s)$. Differentiating the latter equality yields

$$\dot{\eta}_i(s) = \sum_j \frac{\partial^2\phi}{\partial y_i \partial y_j}(\mathbf{y}(s))\dot{y}_j(s).$$

Upon defining

$$\dot{y}_j(s) = \frac{\partial}{\partial\eta_j} L(\mathbf{y}(s), \nabla\phi(\mathbf{y}(s))),$$

we thus obtain that

$$\frac{\partial L}{\partial y_j}(\mathbf{y}(s), \nabla\phi(\mathbf{y}(s))) + \dot{\eta}_j(s) = 0.$$

This is nothing but the system of Hamilton's equations

$$\dot{\mathbf{y}}(s) = \nabla_{\boldsymbol{\eta}} L(\mathbf{y}(s), \boldsymbol{\eta}(s)), \qquad \dot{\boldsymbol{\eta}}(s) = -\nabla_{\mathbf{y}} L(\mathbf{y}(s), \boldsymbol{\eta}(s)). \tag{3.21}$$

Such a system admits a unique solution for given initial conditions $\mathbf{y}(0), \boldsymbol{\eta}(0)$. The curves $(\mathbf{y}(s), \boldsymbol{\eta}(s))$ are then called the **bicharacteristic** curves and their spatial projections $\mathbf{y}(s)$ are called **rays**. Note that we already defined rays as the integral curves of V_ϕ in the preceding section.

Exercise 3.3.1 (not so easy.) Verify that, for the example given in the preceding section, the two definitions of rays coincide by showing that the spatial projection of the bicharacteristics are indeed integral curves for V_ϕ. Hint: use the fact that $\mathbf{k} = \nabla_{\mathbf{x}}\phi$ in the construction of the bicharacteristics.

Let us recapitulate. We know $\phi(\mathbf{y})$ on M and have been able to deduce $\nabla\phi(\mathbf{y})$ on M using (3.20) and the fact that M is not characteristic for $L(\mathbf{y}, \boldsymbol{\eta})$, i.e., that $\nabla_{\boldsymbol{\eta}} L$ is not tangent to M. This provides us with "initial" conditions $\mathbf{y}(0) = \mathbf{y} \in M$ and $\boldsymbol{\eta}(0) = \nabla\phi(\mathbf{y})$. We thus solve for the bicharacteristics $(\mathbf{y}(s), \boldsymbol{\eta}(s))$. This gives us $\nabla\phi(\mathbf{y}(s)) = \boldsymbol{\eta}(s)$.

Application of the inverse function theorem shows that for sufficiently small times, the rays $\mathbf{y}(s; \mathbf{y}, \nabla\phi(\mathbf{y}))$ starting at $\mathbf{y} \in M$ with direction $\nabla\phi(\mathbf{y})$ so constructed cover a neighborhood of M. Thus for any point $\mathbf{z}$ in the vicinity of M, there is a unique $\mathbf{y} \in M$ and a unique s_0 such

that $\mathbf{z} = \mathbf{y}(s_0)$ with $\mathbf{y}(0) = \mathbf{y}$ and $\boldsymbol{\eta}(0) = \nabla\phi(\mathbf{y})$. Let $\gamma(s)$ be such a curve. On the curve, we have $\nabla\phi(\mathbf{y}(s)) = \boldsymbol{\eta}(s)$, which implies that

$$\frac{d}{ds}\phi(\mathbf{y}(s)) = \nabla\phi(\mathbf{y}(s)) \cdot \dot{\mathbf{y}}(s) = \boldsymbol{\eta}(s) \cdot \dot{\mathbf{y}}(s).$$

$$\phi(\mathbf{z}) = \phi(\mathbf{y}) + \int_0^{s_0} \boldsymbol{\eta}(s) \cdot \dot{\mathbf{y}}(s) ds = \phi(\mathbf{y}) + \int_0^{s_0} \boldsymbol{\eta}(s) \cdot \nabla_{\boldsymbol{\eta}} L(\mathbf{y}(s), \boldsymbol{\eta}(s)) ds.$$

Note that the above construction for $\phi(\mathbf{y})$ only works in the vicinity of M, i.e., for sufficiently small times $0 < s < T$. Indeed, for large times, it may be that $\mathbf{y}(s; \mathbf{y}, \nabla\phi(\mathbf{y}))$ and $\mathbf{y}(s'; \mathbf{y}', \nabla\phi(\mathbf{y}'))$ are equal. In such a case, we find that $\phi(\mathbf{z})$ becomes multi-valued and thus ceases to be a smooth solution of the eikonal equation (3.20).

Exercise 3.3.2 Let M be $\mathbb{R}^2$, a surface in $\mathbb{R}^2 \times \mathbb{R}$ corresponding to $t = 0$. Consider the wave equation with $L(\mathbf{y}, \boldsymbol{\eta}) = \tau^2 - c^2|\boldsymbol{\xi}|^2$ with c constant. Let $\phi(0, \mathbf{x}) = g(r)$ be a function in $\mathbb{R}^2$ equal to r for $1 < r < \infty$ and equal to 0 otherwise. Choose the sheet of the eikonal equation given by $\partial_t\phi = -c|\nabla\phi|$.

Solve the eikonal equation by the method of characteristics. Until what time can one solve it? Show that at the final time, the solution of the eikonal equation cannot be smooth.

The above construction for the solution of the eikonal equation is only local in time, but there is a more fundamental problem. When the solution of the eikonal equation stops being defined, it means that solutions of the form $e^{i\phi/\varepsilon}a_\varepsilon$ are no longer sufficiently rich to represent wave propagation.

3.4 First-order hyperbolic systems

The construction in section 3.2.2 can be generalized to arbitrary first-order hyperbolic systems of equations. Consider the equation

$$L\mathbf{u}(\mathbf{y}) = 0, \qquad \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \tag{3.22}$$

where

$$L(\mathbf{y}, \mathbf{D}) = \sum_{m=0}^{d} A_m(\mathbf{y}) \frac{\partial}{\partial y_m} + B(\mathbf{y}) = L_1(\mathbf{y}, \mathbf{D}) + B(\mathbf{y}). \tag{3.23}$$

We can then look for solutions of the form

$$\mathbf{u}(\mathbf{y}) = e^{i\phi(\mathbf{y})/\varepsilon}\mathbf{a}_\varepsilon(\mathbf{y}), \quad \mathbf{a}_\varepsilon(\mathbf{y}) = \mathbf{a}_0(\mathbf{y}) + \varepsilon\mathbf{a}_1(\mathbf{y}) + \dots.$$

We then verify that

$$\begin{aligned} L(\mathbf{y}, \mathbf{D})\Big(e^{i\phi(\mathbf{y})/\varepsilon}\mathbf{a}_\varepsilon(\mathbf{y})\Big) &\sim e^{i\phi(\mathbf{y})/\varepsilon} \times \\ &\times \Big(\frac{1}{\varepsilon} L_1(\mathbf{y}, i\nabla\phi(\mathbf{y}))\mathbf{a}_0 + \sum_{j=0}^{\infty} \varepsilon^j \big(L_1(\mathbf{y}, i\nabla\phi(\mathbf{y}))\mathbf{a}_{j+1}(\mathbf{y}) + L(\mathbf{y}, \nabla)a_j(\mathbf{y})\big)\Big). \end{aligned} \tag{3.24}$$

The leading term shows that $L_1(\mathbf{y}, i\nabla\phi(\mathbf{y}))\mathbf{a}_0$ should vanish to verify (3.22) approximately. This implies that

$$\det L_1(\mathbf{y}, i\nabla\phi(\mathbf{y})) = 0. \tag{3.25}$$

This is the **eikonal** equation for $\phi(\mathbf{y})$. The graph of $\nabla\phi$, i.e, $(\mathbf{y}, \nabla\phi(\mathbf{y}))$ belongs to the characteristic variety of $L(\mathbf{y}, \mathbf{D})$. The solutions of the above equation are obtained as in

section 3.3: we first obtain a possible solution of $\partial_t \phi$ on M based on (3.25) and then once the corresponding sheet of the characteristic variety has been chosen, we solve the eikonal equation by the method of characteristics.

The construction of $\mathbf{a}_j$ is somewhat more complicated and the reader is referred to [26, §5.3] for the details, where it is shown that the construction can be done, at least for sufficiently small times so that the eikonal equation can be solved.

A theorem by P.Lax (1957) then shows that $e^{i\phi(\mathbf{y})/\varepsilon}\mathbf{a}_\varepsilon(\mathbf{y})$ so constructed is indeed a solution of the first order hyperbolic system up to a term of order $O(\varepsilon^\infty)$.

Chapter 4

Random perturbations

This chapter addresses the variations in the wave fields caused by small fluctuations in the underlying medium. By small we mean sufficiently small so that simple asymptotic expansions provide a good approximation for how the field is modified.

4.1 Statistical description of continuous random fields.

This section introduces some notation on the random perturbations that will be useful in the sequel. Let $f(t)$ be a random process depending on a one-dimensional variable t and taking values in $\mathbb{R}^d$. By a random process, we mean that the process is a function of the realization $\omega \in \Omega$ so that at each t in an interval $I \subset \mathbb{R}$, $\omega \mapsto f(t;\omega)$ is a random variable on the probability space $(\Omega, \mathcal{F}, P)$.

In the probability space, $(\Omega, \mathcal{F})$ is a *measurable* space, i.e., a set (state space) Ω that comes with a $\sigma-$algebra $\mathcal{F}$, which should be thought of "all the reasonable" subsets of Ω). Then P is a function $\mathcal{F} \to [0,1]$ (a measure), which to each $A \in \mathcal{F}$ associates a probability $P(A)$. The measure P is a probability measure when $P(\Omega) = 1$. Thus $P(A)$ roughly indicates the probability that the realization ω be in A. When all this is true, we call $(\Omega, \mathcal{F}, P)$ a probability space.

Let $(\mathbb{R}^d, \mathcal{B})$ be a measurable space with $\mathcal{B}$ the *Borel* $\sigma-$algebra on $\mathbb{R}^d$ (think of all products of intervals on $\mathbb{R}^d$ and then all "reasonable" unions and intersections of such products and you get an idea of $\mathcal{B}$ [12, 13, 25]). A random variable f on $(\Omega, \mathcal{F})$ is an $\mathcal{F}$-measurable function from Ω to $\mathbb{R}^d$. It induces a probability measure on $\mathbb{R}^d$, μ_f, defined by

$$\mu_f(B) = P(f^{-1}(B)). \tag{4.1}$$

Then if $\int_\Omega |f(\omega)| dP(\omega) < \infty$, then we define the *expectation* of f as

$$\mathbb{E}\{f\} = \int_\Omega f(\omega) dP(\omega) = \int_{\mathbb{R}^d} f d\mu_f(f). \tag{4.2}$$

A stochastic process is thus a parameterized family of random variables $\{f_t\}_{t\in I}$ defined on $(\Omega, \mathcal{F}, P)$ and taking values in $\mathbb{R}^d$.

We can also view $t \mapsto f(t;\omega)$ as a path of f in Ω. We can thus identify ω with that path, and may thus regard Ω as a subset of $\tilde{\Omega} = (\mathbb{R}^n)^I$ of all functions from I to $\mathbb{R}^n$ (an enormous space!). We refer to [12, 13, 25] for more details. The notation that follows comes almost verbatim from the book by Tatarski "Wave propagation in turbulent media".

Correlation function and power spectrum. Let $f(t)$ be a random process. The correlation function of $f(t)$ is defined by

$$B_f(t_1, t_2) = \big\langle [f(t_1) - \langle f(t_1)\rangle][f(t_2) - \langle f(t_2)\rangle]\big\rangle. \tag{4.3}$$

Here $\langle\cdot\rangle$ refers to the ensemble average over all possible realizations (with respect to the measure P introduced above). Thus the correlation function is a second-order moment of the process $f(t)$. We say that $f(t)$ is **stationary** if $\langle f(t)\rangle$ is independent of t and $B_f(t_1, t_2) = B_f(t_1 - t_2)$. Often we assume $\langle f(t)\rangle = 0$. Then we have the stochastic Fourier-Stieltjes integral with random complex amplitude

$$f(t) = \int_{\mathbb{R}} e^{i\omega t} d\varphi(\omega). \tag{4.4}$$

Since B_f depends on $t_1 - t_2$ we verify that

$$\langle d\varphi(\omega_1) d\varphi^*(\omega_2)\rangle = \delta(\omega_1 - \omega_2) W(\omega_1) d\omega_1 d\omega_2, \tag{4.5}$$

where $W(\omega) \geq 0$ is the **spectal density** of $f(t)$. We check that

$$B_f(t) = \int_{\mathbb{R}} e^{i\omega t} W(\omega) d\omega. \tag{4.6}$$

We see that stationary processes $f(t)$ generate non-negative spectral densities. The Wiener-Kinchin theorem states the converse: If $W(\omega)$ is non-negative, then there is a stationary random process $f(t)$ with spectral density $W(\omega)$. Note that

$$\langle |f(t)|^2\rangle = \int_{\mathbb{R}} W(\omega) d\omega,$$

whence the name spectral density of the power, or power spectrum.

Stationary increments. More general random functions than stationary may be described by stationary increments. They are then described by their structure function

$$D_f(t_1, t_2) = \big\langle [f(t_1) - f(t_2)]^2\big\rangle. \tag{4.7}$$

Stationary increments mean that D_f depends on $\tau = t_2 - t_1$. Stationary functions are special cases of functions with stationary increments, for which we have

$$D_f(\tau) = 2[B_f(0) - B_f(\tau)].$$

When $B_f(\infty) = 0$ we have

$$2B_f(t) = D_f(\infty) - D_f(t),$$

so D_f characterizes the stationary process.

For all processes with stationary increments, we have that

$$D_f(t) = 2\int_{\mathbb{R}} (1 - \cos\omega t) W(\omega) d\omega,$$

and that a random function with stationary increments can be represented as

$$f(t) = f(0) + \int_{\mathbb{R}} (1 - e^{i\omega t}) d\varphi(\omega), \tag{4.8}$$

for some random amplitudes such that

$$\langle d\varphi(\omega_1) d\varphi^*(\omega_2)\rangle = \delta(\omega_1 - \omega_2) W(\omega_1) d\omega_1 d\omega_2. \tag{4.9}$$

For this we need to make sure that $W(\omega)$ is integrable.

Homogeneous random fields. In higher dimensions random processes are replaced by random *fields* (it's just terminology). We define the correlation function $B_f(\mathbf{x}_1, \mathbf{x}_2)$ as

$$B_f(\mathbf{x}_1, \mathbf{x}_2) = \big\langle [f(\mathbf{x}_1) - \langle f(\mathbf{x}_1)\rangle][f(\mathbf{x}_2) - \langle f(\mathbf{x}_2)\rangle]\big\rangle. \tag{4.10}$$

Stationary random functions are replaced by *homogeneous* random fields, i.e. fields for which the correlation function $B_f(\mathbf{x}_1, \mathbf{x}_2)$ depends only on $\mathbf{x}_1 - \mathbf{x}_2$. When B_f depends only on $|\mathbf{x}_1 - \mathbf{x}_2|$ the field is isotropic.

We have the stochastic Fourier-Stieltjes integral

$$f(\mathbf{x}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{x}} d\varphi(\mathbf{k}), \tag{4.11}$$

where for some $\Phi \geq 0$,

$$\langle d\varphi(\mathbf{k}_1) d\varphi(\mathbf{k}_2)\rangle = \delta(\mathbf{k}_1 - \mathbf{k}_2)\Phi(\mathbf{k}_1) d\mathbf{k}_1 d\mathbf{k}_2. \tag{4.12}$$

We find that

$$\Phi(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \cos(\mathbf{k}\cdot\mathbf{x}) B_f(\mathbf{x}) d\mathbf{x}.$$

Locally homogeneous random fields. We define the structure function

$$D_f(\mathbf{x}_1, \mathbf{x}_2) = \big\langle [f(\mathbf{x}_1) - f(\mathbf{x}_2)]^2\big\rangle, \tag{4.13}$$

which depends on $\mathbf{x}_1 - \mathbf{x}_2$ for locally homogeneous fields. A locally homogeneous field may then be represented as

$$f(\mathbf{x}) = f(\mathbf{0}) + \int_{\mathbb{R}^d} (1 - e^{i\mathbf{k}\cdot\mathbf{x}}) d\varphi(\mathbf{k}), \tag{4.14}$$

where $f(\mathbf{0})$ is a random variable. We find that

$$D_f(\mathbf{x}) = 2\int_{\mathbb{R}^d} (1 - \cos\mathbf{k}\cdot\mathbf{x})\Phi(\mathbf{k}) d\mathbf{k},$$

where $\Phi(\mathbf{k})$ is defined as for homogeneous fields.

4.2 Regular Perturbation method

Let us consider the wave equation

$$\begin{aligned} &\frac{1}{c^2(\mathbf{x})}\frac{\partial^2 p}{\partial t^2} - \Delta p = 0, \qquad t > 0,\, \mathbf{x} \in \mathbb{R}^d \\ &p(0, \mathbf{x}) = 0, \qquad \frac{\partial p}{\partial t}(0, \mathbf{x}) = h(\mathbf{x}). \end{aligned} \tag{4.15}$$

The above equation may be recast as

$$\frac{\partial^2 p}{\partial t^2} - c^2(\mathbf{x})\Delta p = h(\mathbf{x})\delta(t), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{4.16}$$

with the condition that $p(t, \mathbf{x}) \equiv 0$ for $t < 0$. This may be verified by recalling that the derivative of the Heaviside function is the delta function in the sense of distributions. Upon

taking the Fourier transform in time $\mathcal{F}_{t\to\tau}$ and using the dispersion relation $\tau = c_0 k$, for some constant background c_0, we find that

$$\Delta\hat{p} + k^2 n^2(\mathbf{x})\hat{p} = -h(\mathbf{x}), \tag{4.17}$$

where the index of refraction is given by

$$n(\mathbf{x}) = \frac{c_0}{c(\mathbf{x})}, \qquad n^2(\mathbf{x}) = \frac{c_0^2}{c^2(\mathbf{x})} = 1 + \varepsilon\mu(\mathbf{x}), \tag{4.18}$$

where $\varepsilon \ll 1$ is a small parameter measuring the amplitude of the fluctuations in the underlying medium and $\mu(\mathbf{x})$ is a random field on $\mathbb{R}^d$. We thus assume that the sound speed is equal to a constant background plus a small perturbation. Note that we are in the low frequency regime here, where $\lambda \sim L$ and the correlation length $l \sim L$ is also of the same order as the medium. The only small parameter is thus the size of the random fluctuations.

Let us consider the case of an incoming plane wave $e^{i\mathbf{k}\cdot\mathbf{x}}$ onto a compact domain. This is modeled by an index of refraction given by (4.18). Note that for $k = |\mathbf{k}|$, the plane wave is a solution of the homogeneous equation $\Delta p + k^2 p = 0$. We thus look for a solution of

$$\Delta p + k^2 n^2(\mathbf{x}) p = 0, \tag{4.19}$$

that solves the following *Lipman- Schwinger* equation

$$p(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \int_{\mathbb{R}^d} G(\mathbf{x},\mathbf{y}) k^2 \varepsilon\mu(\mathbf{y}) p(\mathbf{y}) d\mathbf{y}, \tag{4.20}$$

where $G(\mathbf{x},\mathbf{y})$ is the Green's function associated to the homogeneous Helmholtz equation

$$\Delta G(\mathbf{x},\mathbf{y}) + k^2 G(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \tag{4.21}$$

with appropriate radiation conditions at infinity (i.e., so that there is no radiation coming from infinity). In three dimensions we have the usual form

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{x}-\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}. \tag{4.22}$$

Let us recast symbolically the Lipman-Schwinger equation as

$$p = p_0 + \varepsilon G V p.$$

Then we observe that, for μ a mean zero stationary random process, we have

$$\langle p\rangle = (I + \varepsilon^2\langle GVGV\rangle)p_0 + O(\varepsilon^4),$$

which, neglecting $O(\varepsilon^4)$ terms, may be recast as

$$(\Delta + k^2)\langle p\rangle + \langle VGV\rangle p = 0. \tag{4.23}$$

We have thus replaced a heterogeneous equation by a homogenized equation for the ensemble average of p. Note that for $\mu(\mathbf{x})$ a stationary random process with correlation $R(\mathbf{x})$, we obtain that

$$(\Delta + k^2)\langle p\rangle + \frac{\varepsilon^2 k^4}{4\pi}\int_{\mathbb{R}^3} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} R(\mathbf{x}-\mathbf{y})\langle p\rangle(\mathbf{y}) d\mathbf{y} = 0. \tag{4.24}$$

The above equation was obtained by J.B. Keller in 1964.

This is a convolution equation, which may thus admits plane waves as exact solutions. Looking for $\langle p \rangle = e^{i\boldsymbol{\xi}\cdot\mathbf{x}}$, we find the *dispersion relation*

$$\xi^2 = k^2 + \frac{\varepsilon^2 k^4}{4\pi} \int_{\mathbb{R}^3} \frac{e^{ik|\mathbf{y}|}}{|\mathbf{y}|} R(\mathbf{y}) e^{-i\boldsymbol{\xi}\cdot\mathbf{y}} d\mathbf{y}. \tag{4.25}$$

Here $\xi^2 = -(i\boldsymbol{\xi}) \cdot (i\boldsymbol{\xi})$. However we observe that ξ is not real-valued. Let us consider $\boldsymbol{\xi} = \rho\mathbf{k}$ for some $\rho > 0$. We can then to first order replace $e^{-i\boldsymbol{\xi}\cdot\mathbf{y}}$ by $e^{-i\mathbf{k}\cdot\mathbf{y}}$ in the above expression to get

$$\xi^2 = k^2 + \frac{\varepsilon^2 k^4}{4\pi} \int_{\mathbb{R}^3} \frac{e^{ik|\mathbf{y}|}}{|\mathbf{y}|} R(\mathbf{y}) e^{-i\mathbf{k}\cdot\mathbf{y}} d\mathbf{y}.$$

For $R(\mathbf{y}) = R(|\mathbf{y}|)$ (isotropic random medium) we find that

$$\frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{e^{i(k|\mathbf{y}|-\mathbf{k}\cdot\mathbf{y})}}{|\mathbf{y}|} R(\mathbf{y}) d\mathbf{y} = \int_0^\infty R(|\mathbf{y}|)\big(\sin 2k|\mathbf{y}| + i(1-\cos 2k|\mathbf{y}|)\big) d|\mathbf{y}|.$$

Exercise 4.2.1 Check the above formula.

So for instance for $R(|\mathbf{y}|) = e^{-\alpha|\mathbf{y}|}$, we verify that both the real and imaginary parts in the above expression are non-negative. This allows us to conclude that $\Re\xi > k$ and that $\Im\xi > 0$, at least for sufficiently small ε. This means that the mean wave propagates more slowly than in a homogeneous medium since $\tau = c_0 k = c_{in}\Re\xi$ and that it is *attenuated* by the inhomogeneities.

The attenuation is **not** intrinsic attenuation, since energy is conserved by the wave equation and the Helmholtz equation is the same phenomenon seen in the Fourier domain. What happens is that the *coherent* part of the signal decays exponentially. As the waves propagate through the random medium, energy is scattered into other directions. Because this scattering is *incoherent*, it is "lost" when we consider $\langle p \rangle$. Higher moments of p need to be considered as well. This will be done by means of the Wigner transform of the wave fields in later chapters.

4.3 Random Geometric Optics

For the same equation

$$\Delta p + k^2 n^2(\mathbf{x}) p = 0, \tag{4.26}$$

let us consider geometric optics solutions

$$p(\mathbf{x}) = e^{ikS(\mathbf{x})} a(\mathbf{x}), \tag{4.27}$$

in the *high frequency* regime, i.e., when $kL \gg 1$, where L is the typical distance of propagation we are interested in.

Note that this time, we are interested in frequencies of order k so that the "low-order term", at least in terms of derivatives, is as important as the "high-order term" Δp. As a "characteristic variety" for the Helmholtz equation, we thus really want to consider

$$L(\mathbf{x}, \boldsymbol{\xi}) = -|\boldsymbol{\xi}|^2 + k^2 n^2(\mathbf{x}) = 0. \tag{4.28}$$

Let us plug the anzatz (4.27) into (4.26) and equate like powers of k. The eikonal equation, which is the leading term in $O(k^2)$ now becomes

$$L(\mathbf{x}, k\nabla S(\mathbf{x})) = k^2(-|\nabla S|^2(\mathbf{x}) + n^2(\mathbf{x})) = 0. \tag{4.29}$$

Note that the formalism is the same as in the preceding chapters: the eikonal equation captures the largest homogeneous terms in the equation. Here "largest" means in terms of k, and not $\boldsymbol{\xi}$, the dual variable to $\mathbf{x}$.

The above variety has two regular branches and we consider

$$\tilde{L}(\mathbf{x}, \boldsymbol{\xi}) = |\boldsymbol{\xi}| - n(\mathbf{x}) = 0, \tag{4.30}$$

where we have replaced $\boldsymbol{\xi}$ by the reduced $\boldsymbol{\xi}/k$. The eikonal equation thus becomes

$$\tilde{L}(\mathbf{x}, \nabla S(\mathbf{x})) = |\nabla S|(\mathbf{x}) - n(\mathbf{x}) = 0. \tag{4.31}$$

The above eikonal equation can then be solved by the method of characteristics. As before we find rays $(\mathbf{x}(s), \boldsymbol{\xi}(s))$ such that $\nabla S(\mathbf{x}(s)) = \boldsymbol{\xi}(s)$. This is achieved by the Hamiltonian system

$$\begin{aligned} \dot{\mathbf{X}}(s) &= \hat{\boldsymbol{\Xi}}(s), & \mathbf{X}(0) &= \mathbf{x}, \\ \dot{\boldsymbol{\Xi}}(s) &= \nabla n(\mathbf{X}(s)) & \boldsymbol{\Xi}(0) &= \boldsymbol{\xi}. \end{aligned} \tag{4.32}$$

We verify that

$$\frac{d}{ds} S(\mathbf{X}(s)) = n(\mathbf{X}(s)) = |\boldsymbol{\Xi}(s)|, \tag{4.33}$$

so that

$$S(\mathbf{X}(s)) = S(\mathbf{x}) + \int_0^s n(\mathbf{X}(u)) du. \tag{4.34}$$

This fully characterizes the phase $S(\mathbf{x})$.

Let us now consider a simplified version where $\mathbf{x} = (x_1, x_2)$ in two space dimensions and where $S(0, x_2) = 0$. We also assume that $n(\mathbf{x}) = 1 + \sigma\mu(\mathbf{x})$ with $\varepsilon \ll 1$. We thus have the phase at $x_1 = 0$ and are interested in the phase at $x_1 > 0$.

Characteristics start at $(0, x_2)$ and with direction $\mathbf{e}_1 + O(\sigma) \approx \mathbf{e}_1$. We want to find the right scaling in time t so that fluctuations of order $O(1)$ can be observed in the phase with respect to propagation in a homogeneous medium.

For $\sigma \ll 1$, we observe that $\hat{\boldsymbol{\Xi}}(s) = (1, 0)^t + O(\sigma s)$ for $s \ll \sigma^{-1}$. In that regime we thus have $\mathbf{X}(s) = \mathbf{X}(0) + s\hat{\boldsymbol{\Xi}}(0) + O(\sigma s^2)$ so that for $\sigma s^2 \ll 1$, we have approximatively that

$$\frac{d(S - s)}{ds} = \sigma\mu(\mathbf{X}(0) + s\hat{\boldsymbol{\Xi}}(0)).$$

For large times $s = t/\varepsilon$, we can approximate the phase as

$$S(\frac{t}{\varepsilon}) = \frac{t}{\varepsilon} + \varepsilon^\alpha S_\varepsilon(t),$$

where

$$\frac{dS_\varepsilon}{dt} = \frac{\sigma}{\varepsilon^{1+\alpha}} \mu(\frac{t}{\varepsilon}, x_2). \tag{4.35}$$

Let us now choose ε such that $\sigma = \varepsilon^{1/2+\alpha}$ so that the above equation is recast as

$$\frac{dS_\varepsilon(t; x_2)}{dt} = \frac{1}{\sqrt{\varepsilon}} \mu(\frac{t}{\varepsilon}, x_2), \qquad S_\varepsilon(0; x_2) = 0. \tag{4.36}$$

Since we want $\sigma t^2/\varepsilon^2 \ll 1$, this implies that $\alpha > 3/2$. The value of $3/2$ is by no means optimal because the estimate $\sigma t^2/\varepsilon^2 \ll 1$ is very conservative.

In the limit $\varepsilon \to 0$, the above phase function converges to a stochastic process $S(t; x_2)$ given by

$$S(t; x_2) = \sigma_0(t; x_2) W_t, \tag{4.37}$$

where W_t is *Brownian motion* and where the variance σ_0^2 is given by

$$\sigma_0^2(t; x_2) = 2 \int_0^\infty \mathbb{E}\{\mu(0, x_2)\mu(\tau, x_2)\} d\tau. \tag{4.38}$$

If μ is homogeneous in both directions x_1 and x_2, then the above variance is independent of x_2. That this is so for a large class of processes μ comes from a much more general result that can be found in Appendix A.

The above result is only valid when the method of characteristics provides a unique solution to $S(\mathbf{x})$ for $x_1 > 0$. It is shown in [31] that caustics form with probability one when $\varepsilon \sim \sigma^{2/3}$, which corresponds to $\alpha = 1$. In this scaling, it can be shown that $\mathbf{X}(s/\varepsilon)$ deviates by an amount of order $O(1)$ from its value in a homogeneous medium. This is therefore the regime where characteristics cross (with probability one somewhere), whereby invalidating the validity of the geometric optics ansatz.

Let us consider the regime $\varepsilon = \sigma^{2/3}$ more carefully. We first recast the Hamiltonian system (4.32) as

$$\begin{aligned} \dot{\mathbf{X}}(s) &= \hat{\boldsymbol{\Xi}}(s), & \mathbf{X}(0) &= \mathbf{x}, \\ \dot{\hat{\boldsymbol{\Xi}}}(s) &= \big(I - \hat{\boldsymbol{\Xi}}(s) \otimes \hat{\boldsymbol{\Xi}}(s)\big) \frac{\nabla n(\mathbf{X}(s))}{n(\mathbf{X}(s))}, & \hat{\boldsymbol{\Xi}}(0) &= \boldsymbol{\xi}. \end{aligned} \tag{4.39}$$

Here we have used the fact that $|\boldsymbol{\Xi}(s)| - n(\mathbf{X}(s)) = 0$ to find an equation for $(\mathbf{X}, \hat{\boldsymbol{\Xi}})$.

Exercise 4.3.1 Derive (4.39) from (4.32).

We now restrict ourselves to the two-dimensional setting and define $\boldsymbol{\xi}^\perp$ as the rotation by $\pi/2$ of $\boldsymbol{\xi} \in S^1$. The above system may then be recast as

$$\begin{aligned} \dot{\mathbf{X}}(s) &= \hat{\boldsymbol{\Xi}}(s), & \mathbf{X}(0) &= \mathbf{x}, \\ \dot{\hat{\boldsymbol{\Xi}}}(s) &= \hat{\boldsymbol{\Xi}}^\perp(s) \cdot \frac{\nabla n(\mathbf{X}(s))}{n(\mathbf{X}(s))} \hat{\boldsymbol{\Xi}}^\perp(s), & \hat{\boldsymbol{\Xi}}(0) &= \boldsymbol{\xi}. \end{aligned} \tag{4.40}$$

Let us now assume that $n = 1 + \sigma\mu$ with $\mu(\mathbf{x})$ a mean zero stationary random field and $\sigma = \varepsilon^{3/2}$. For times $s = t/\varepsilon$, which are thus such that $s\sigma = t\sqrt{\varepsilon} \ll 1$, we have $\hat{\boldsymbol{\Xi}}(s) = \hat{\boldsymbol{\Xi}}(0) + O(\sqrt{\varepsilon})$. Upon neglecting terms that will not contribute to order $O(1)$ at times of order $s = t/\varepsilon$, we deduce that

$$\begin{aligned} \dot{\mathbf{X}}(s) &= \hat{\boldsymbol{\Xi}}(s), & \mathbf{X}(0) &= \mathbf{x}, \\ \dot{\hat{\boldsymbol{\Xi}}}(s) &= \sqrt{\varepsilon} \frac{\partial \mu}{\partial x_2}(\mathbf{X}(s)) \boldsymbol{\xi}^\perp, & \hat{\boldsymbol{\Xi}}(0) &= \boldsymbol{\xi}. \end{aligned}$$

This implies that $\mathbf{X}(s) = s\boldsymbol{\xi} + \mathbf{X}_\varepsilon(t)$, where $\mathbf{X}_\varepsilon(t)$ is a priori not as large as $s = t/\varepsilon$. We thus recast the above system (now in the t-variable) as

$$\begin{aligned} \dot{\mathbf{X}}_\varepsilon(t) &= \boldsymbol{\Xi}_\varepsilon(t), & \mathbf{X}_\varepsilon(0) &= \mathbf{x}, \\ \dot{\boldsymbol{\Xi}}_\varepsilon(t) &= \frac{1}{\sqrt{\varepsilon}} \frac{\partial \mu}{\partial x_2} \Big(\frac{t}{\varepsilon} \boldsymbol{\xi} + \mathbf{X}_\varepsilon(t) \Big) \boldsymbol{\xi}^\perp, & \boldsymbol{\Xi}_\varepsilon(0) &= \boldsymbol{\xi}. \end{aligned} \tag{4.41}$$

Exercise 4.3.2 Verify the above equation. Note that we have defined implicitly $\boldsymbol{\Xi}_\varepsilon(t) = \varepsilon^{-1}(\hat{\boldsymbol{\Xi}}(s) - \hat{\boldsymbol{\Xi}}(0))$.

Note that only the second component of $\mathbf{\Xi}_\varepsilon(t)$ evolves in time, which implies the same result for $\mathbf{X}_\varepsilon(t)$. We thus further simplify (with no additional term neglected) that

$$\begin{aligned} \dot{X}_\varepsilon(t) &= \Xi_\varepsilon(t), & X_\varepsilon(0) &= x_2, \\ \dot{\Xi}_\varepsilon(t) &= \frac{1}{\sqrt{\varepsilon}} \frac{\partial \mu}{\partial x_2}\Big(\frac{t}{\varepsilon}, X_\varepsilon(t)\Big), & \Xi_\varepsilon(0) &= 0. \end{aligned} \tag{4.42}$$

This is the correct scaling to show order $O(1)$ modifications caused by the randomness. The results in the appendix show that $\Xi_\varepsilon(t)$ becomes an $O(1)$ process and so $X_\varepsilon(t)$ as well. Since $\mathbf{X}(s) = s\boldsymbol{\xi} + \mathbf{X}_\varepsilon(t)$, we observe that trajectories starting at $(0, x_2)$ have deviated by an order $O(1)$ at the large time $s = t/\varepsilon$ (large compared to the correlation length of the medium, which we have scaled as $O(1)$ here) from where they would be should the medium be homogeneous, i.e., (s, x_2). It is then shown in [31] that in this regime, rays cross with probability one, whence generating caustics at those points where an infinite number of rays come tangentially (whereby concentrating signals and creating very bright areas, as at the bottom of a pool illuminated from the top).

We are "almost" in the regime considered in the appendix. The reason is here that the random variable depends continuously on the fast scale t/ε but also on the unknown X_ε.

Let us assume that

$$\frac{\partial \mu}{\partial x_2}(t, x) = \sum_{j \in J} e^{-ik_j x} \alpha_j(t),$$

where $\alpha_j(t)$ is a mean-zero random process and the $\{\alpha_j(t)\}_{j \in J}$ are jointly Markov with infinitesimal generator Q. We can now apply the theory with $\mathbf{x} = (x, \xi)$ and $\mathbf{y} = \{\alpha_j\}$. We find that

$$\begin{aligned} F_1(\mathbf{x}, \mathbf{y}) &= 0, & G_1(\mathbf{x}, \mathbf{y}) &= \xi, \\ F_2(\mathbf{x}, \mathbf{y}) &= \sum_j e^{-ik_j x} \alpha_j, & G_2(\mathbf{x}, \mathbf{y}) &= 0. \end{aligned} \tag{4.43}$$

The only non-vanishing diffusion coefficient is, independent of the number of Fourier coefficients α_j, given by

$$a_{22}(x, \xi) = 2 \int_0^\infty \mathbb{E}_\infty \Big\{ \frac{\partial \mu}{\partial x_2}(0, x) \frac{\partial \mu}{\partial x_2}(s, x) \Big\} ds.$$

The only non-vanishing drift term is

$$b_1(x, \xi) = \xi.$$

Writing $a_{22}(x) = \sigma^2(x)$, we obtain that $(\mathbf{X}_\varepsilon(t), \mathbf{\Xi}_\varepsilon(t))$ converges (weakly) to the process solution of

$$\begin{aligned} \dot{X}(t) &= \Xi(t), & X(0) &= x_2, \\ d\Xi(t) &= \sigma(X(t)) dW_t, & \Xi(0) &= 0. \end{aligned} \tag{4.44}$$

Here W_t is the usual one-dimensional centered Brownian motion with variance $\mathbb{E}\{W_t^2\} = t$.

Chapter 5

Wigner Transforms

5.1 Definition of the Wigner transform

Let $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ be two m-dimensional vector fields on $\mathbb{R}^d$, and let $\varepsilon > 0$ be a real number tailored to represent the spatial scale at which phenomena occur, typically $\varepsilon \ll 1$. We define the **Wigner** transform of the two fields $\mathbf{u}$ and $\mathbf{v}$ as the $m \times m$ matrix-valued field on $\mathbb{R}^d \times \mathbb{R}^d$:

$$W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}\mathbf{u}(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\mathbf{v}^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})d\mathbf{y}. \tag{5.1}$$

Here, $\mathbf{v}^*$ represents the complex conjugate of the adjoint vector to $\mathbf{v}$.

With our definition of the Fourier transform we have

$$W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \mathcal{F}^{-1}_{\mathbf{y}\to\mathbf{k}}\Big(\mathbf{u}(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\mathbf{v}^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\Big)(\mathbf{x},\mathbf{k}), \tag{5.2}$$

which is obviously equivalent to the fact that

$$\tilde{W}_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{y}) = \mathcal{F}_{\mathbf{k}\to\mathbf{y}}[W_\varepsilon[\mathbf{u},\mathbf{v}]](\mathbf{x},\mathbf{y}) = \mathbf{u}(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\mathbf{v}^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}). \tag{5.3}$$

The Wigner transforms can thus be seen as a the Fourier transform in the fast spatial variations of the two-point correlation of the two fields. It is thus an object defined in the *phase space*, which tries to account for the rapid oscillations at reduced wavenumber $\mathbf{k}$ (physical wavenumber $\mathbf{k}/\varepsilon$) in the vicinity of a macroscopic scale point $\mathbf{x}$.

Scaling. We verify that

$$\varepsilon^d W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\varepsilon\mathbf{k}) = W_1[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}). \tag{5.4}$$

This is consistent with the fact that the reduced wavenumber $\mathbf{k}$ corresponds to oscillations with physical wavenumber $\mathbf{k}/\varepsilon$. We also have the natural relationship

$$W_\varepsilon[\mathbf{u}(\alpha\cdot),\mathbf{v}(\alpha\cdot)](\mathbf{x},\mathbf{k}) = W_\varepsilon[\mathbf{u},\mathbf{v}](\alpha\mathbf{x},\frac{\mathbf{k}}{\alpha}). \tag{5.5}$$

Noteworthy relationships. We verify directly from (5.1) and from the interpretation (5.2) that

$$\begin{aligned} W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) &= W^*_\varepsilon[\mathbf{v},\mathbf{u}](\mathbf{x},\mathbf{k}), \\ \int_{\mathbb{R}^d} W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})d\mathbf{k} &= (\mathbf{u}\mathbf{v}^*)(\mathbf{x}), \\ \int_{\mathbb{R}^d} \mathbf{k} W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})d\mathbf{k} &= \frac{i\varepsilon}{2}(\mathbf{u}\nabla\mathbf{v}^* - \nabla\mathbf{u}\mathbf{v}^*)(\mathbf{x}), \\ \int_{\mathbb{R}^{2d}} |\mathbf{k}|^2 W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})d\mathbf{k}d\mathbf{x} &= \varepsilon^2 \int_{\mathbb{R}^d} \nabla\mathbf{u}\cdot\nabla\mathbf{v}^* d\mathbf{x}. \end{aligned} \tag{5.6}$$

Exercise 5.1.1 Check the above properties.

Wigner transform in Fourier domain. Let us define the Fourier transform of the Wigner transform

$$\hat{W}_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{p},\mathbf{k}) = \mathcal{F}_{\mathbf{x}\to\mathbf{p}}[W_\varepsilon[\mathbf{u},\mathbf{v}]](\mathbf{p},\mathbf{k}). \tag{5.7}$$

Then we find that

$$\hat{W}_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{p},\mathbf{k}) = \frac{1}{(2\pi\varepsilon)^d}\hat{\mathbf{u}}\big(\frac{\mathbf{p}}{2}+\frac{\mathbf{k}}{\varepsilon}\big)\widehat{\mathbf{v}^*}\big(\frac{\mathbf{p}}{2}-\frac{\mathbf{k}}{\varepsilon}\big). \tag{5.8}$$

Exercise 5.1.2 Check the above properties. Recall that $\hat{\bar{f}}(\boldsymbol{\xi}) = \bar{\hat{f}}(-\boldsymbol{\xi})$.

5.2 Convergence properties

Let $\phi_\varepsilon(\mathbf{x})$ be a complex-valued (scalar to simplify) sequence of functions uniformly (in ε) bounded in $L^2(\mathbb{R}^d)$. We consider the Wigner transform of the sequence

$$W_\varepsilon(\mathbf{x},\mathbf{k}) \equiv W_\varepsilon[\phi_\varepsilon](\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}\phi_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\phi^*_\varepsilon(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})d\mathbf{y}. \tag{5.9}$$

The sequence of Wigner transforms defined above satisfies the following uniform bound. We introduce the space $\mathcal{A}$ of functions $\lambda(\mathbf{x},\mathbf{k})$ of $\mathbf{x}$ and $\mathbf{k}$ such that

$$\|\lambda\|_{\mathcal{A}} = \int_{\mathbb{R}^d} d\mathbf{y} \sup_{\mathbf{x}} |\tilde{\lambda}(\mathbf{x},\mathbf{y})| < \infty, \tag{5.10}$$

where

$$\tilde{\lambda}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d} d\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{y}}\lambda(\mathbf{x},\mathbf{k}), \tag{5.11}$$

is the Fourier transform of λ in $\mathbf{k}$. Then, as the following lemma shows, the distributions $W_\varepsilon(\mathbf{x},\mathbf{k})$ are uniformly bounded in $\mathcal{A}'$, the dual space to $\mathcal{A}$ when the sequence of fields $\phi_\varepsilon(\mathbf{x})$ are uniformly bounded in $L^2(\mathbb{R}^d)$.

Lemma 5.2.1 *Let $\phi_\varepsilon(\mathbf{x})$ be uniformly bounded in $L^2(\mathbb{R}^d)$ by Φ. The family $W_\varepsilon(\mathbf{x},\mathbf{k})$ is uniformly bounded in $\mathcal{A}'$, and more precisely,*

$$\|W_\varepsilon\|_{\mathcal{A}'} \le \|\phi_\varepsilon\|^2_{L^2(\mathbb{R}^d)} \le \Phi^2. \tag{5.12}$$

for all $\varepsilon > 0$.

Proof. Let $\lambda(\mathbf{x},\mathbf{k}) \in \mathcal{A}$. Then,

$$\begin{aligned}\langle W_\varepsilon, \lambda\rangle &\equiv \int_{\mathbb{R}^{2d}} W_\varepsilon(\mathbf{x},\mathbf{k})\lambda(\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k} = \int_{\mathbb{R}^{3d}} e^{i\mathbf{k}\cdot\mathbf{y}}\phi_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\phi_\varepsilon^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\lambda(\mathbf{x},\mathbf{k})\frac{d\mathbf{x}d\mathbf{k}d\mathbf{y}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{y}} \\ &= \int_{\mathbb{R}^{2d}} \phi_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\phi_\varepsilon^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\tilde{\lambda}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}.\end{aligned}$$

Therefore, using the Cauchy-Schwarz inequality in $\mathbf{x}$, we have

$$\begin{aligned}|\langle W_\varepsilon,\lambda\rangle| &\leq \int_{\mathbb{R}^d}\sup_{\mathbf{x}}|\tilde{\lambda}(\mathbf{x},\mathbf{y})|\Big(\int_{\mathbb{R}^d}\big|\phi_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\phi_\varepsilon^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\big|d\mathbf{x}\Big)d\mathbf{y} \\ &\leq \int_{\mathbb{R}^d}\sup_{\mathbf{x}}|\tilde{\lambda}(\mathbf{x},\mathbf{y})|\Big(\int_{\mathbb{R}^d}\big|\phi_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\big|^2 d\mathbf{x}\Big)^{1/2}\Big(\int_{\mathbb{R}^d}\big|\phi_\varepsilon(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\big|^2 d\mathbf{x}\Big)^{1/2}d\mathbf{y} \\ &\leq \int_{\mathbb{R}^d}\sup_{\mathbf{x}}|\tilde{\lambda}(\mathbf{x},\mathbf{y})|d\mathbf{y}\Big(\int_{\mathbb{R}^d}\big|\phi_\varepsilon(\mathbf{x})\big|^2 d\mathbf{x}\Big) \leq \|\phi_\varepsilon\|^2_{L^2(\mathbb{R}^d)}\|\lambda\|_{\mathcal{A}}.\end{aligned}$$

This gives (5.12) since by definition, we have

$$\|W_\varepsilon\|_{\mathcal{A}'} = \sup_{\lambda\in\mathcal{A}}\frac{|\langle W_\varepsilon,\lambda\rangle|}{\|\lambda\|_{\mathcal{A}}}. \tag{5.13}$$

□

This result shows that the sequence W_ε converges weakly * in $\mathcal{A}'$. The Banach-Alaoglu theorem (stating that the unit ball in a space E' is compact for the weak * topology $\sigma(E',E)$) then implies that for each sequence $\varepsilon_n \to 0$, we can extract a subsequence $\varepsilon_{n'}$ such that $W_{\varepsilon_{n'}}$ converges in the weak * topology to a limit $W \in \mathcal{A}'$. What this means is that

$$\lim_{\varepsilon_{n'}\to 0}\langle W_{\varepsilon_{n'}},\lambda\rangle = \langle W,\lambda\rangle, \qquad \text{for all } \lambda\in\mathcal{A}.$$

The space $\mathcal{A}'$ is a space of *distributions* on $\mathbb{R}^{2d}$, i.e., a subspace of $\mathcal{D}'(\mathbb{R}^{2d})$. However this is a big subspace that includes bounded measures $\mathcal{M}(\mathbb{R}^{2d})$. Therefore, Wigner transforms, which may be smooth at fixed ε (for instance when ϕ_ε is smooth), are no longer necessarily smooth in the limit $\varepsilon \to 0$.

The above results extend to matrix valued Wigner transforms. Let $\mathbf{u}_\varepsilon(\mathbf{x})$ and $\mathbf{v}_\varepsilon(\mathbf{x})$ be uniformly bounded in $(L^2(\mathbb{C}^d))^m$ by Φ. Then the Wigner transform

$$W_\varepsilon(\mathbf{x},\mathbf{k}) = W_\varepsilon[\mathbf{u}_\varepsilon,\mathbf{v}_\varepsilon](\mathbf{x},\mathbf{k}), \tag{5.14}$$

is uniformly bounded in $(\mathcal{A}^{m\times m})'$ and consequently admits converging subsequences in the same space for the weak * topology.

Exercise 5.2.1 Verify this claim.

Now let us restrict ourselves to the case where $\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon$. It turns out that the limit is more regular than $(\mathcal{A}^{m\times m})'$: it is in the space of bounded measures $\mathcal{M}^{m\times m}(\mathbb{R}^{2d})$. Moreover it is a nonnegative Hermitian matrix-valued measure. That it is Hermitian comes from the first property in (5.6). We refer to [17] for the proof that it is a nonnegative measure, i.e., that for all $\mathbf{e}\in\mathbb{C}^m$, the limit W^0 satisfies

$$\sum_{i,j=1}^m W^0_{ij}e_i e_j^* \geq 0. \tag{5.15}$$

Let us consider the matrix $\mathbf{u}_\varepsilon\mathbf{u}_\varepsilon^*$. Since $\mathbf{u}_\varepsilon$ is bounded in L^2, then each component in $\mathbf{u}_\varepsilon\mathbf{u}_\varepsilon^*$ is bounded in $L^1(\mathbb{R}^d)$, hence in $\mathcal{M}(\mathbb{R}^d)$, the space of bounded measures on $\mathbb{R}^d$, which is the

dual of the space of compactly supported continuous functions on $\mathbb{R}^d$ equipped with the sup norm $\|\varphi\|_\infty = \sup_{\mathbf{x}\in\mathbb{R}^d} |\varphi(\mathbf{x})|$. The same Banach-Alaoglu theorem used above then implies that $\mathbf{u}_\varepsilon \mathbf{u}_\varepsilon^*$ admits subsequences that converge weakly * in the space of bounded measures to a matrix ν:

$$\mathbf{u}_\varepsilon \mathbf{u}_\varepsilon^* \to \nu \quad \text{for the weak * topology in } \mathcal{M}(\mathbb{R}^d). \tag{5.16}$$

In many cases, ν is precisely the object we are interested in: when $\mathbf{u}_\varepsilon$ is a field, ν has the units of an energy. We have the following important properties of the Wigner transform.

Definition 5.2.2 *A bounded family $\mathbf{u}_\varepsilon(\mathbf{x})$ in L^2 is said to be* $\varepsilon-$oscillatory *as $\varepsilon \to 0$ if for every continuous compactly supported function φ on $\mathbb{R}^d$, we have*

$$\overline{\lim_{\varepsilon\to 0}} \int_{|\mathbf{k}|\geq R/\varepsilon} |\widehat{\varphi \mathbf{u}_\varepsilon}(\mathbf{k})|^2 d\mathbf{k} \to 0 \quad \text{as } R \to \infty. \tag{5.17}$$

A bounded family $\mathbf{u}_\varepsilon(\mathbf{x})$ in L^2 is said to be compact at infinity *as $\varepsilon \to 0$ if*

$$\overline{\lim_{\varepsilon\to 0}} \int_{|\mathbf{x}|\geq R} |\mathbf{u}_\varepsilon|^2(\mathbf{x}) d\mathbf{x} \to 0 \quad \text{as } R \to \infty. \tag{5.18}$$

Compactness at infinity means that the functions oscillate in the vicinity of the origin uniformly in ε. ε-oscillatory means that the typical frequency of oscillation of the functions is precisely ε^{-1}. We can verify that sufficient conditions for such a behavior is for instance:

$$\|(\varepsilon\nabla)\mathbf{u}_\varepsilon\|_{L^2} \leq C, \text{ independent of } \varepsilon. \tag{5.19}$$

Here $(\varepsilon\nabla)$ can be replaced by an arbitrary number (including real-valued) of derivatives, e.g. of the form $(\varepsilon^2\Delta)^s$. Then we have the following properties [17]:

Proposition 5.2.3 *Let $\mathbf{u}_\varepsilon$ be a bounded family in $(L^2(\mathbb{R}^d))^m$ with Wigner transform W_ε converging (up to extraction of a subsequence) to a limiting measure W_0. Let us denote by $w_0 = \operatorname{Tr} W_0$. Then w_0 is a bounded measure on $\mathbb{R}^{2d}$. Moreover we have*

$$W_0(A, \mathbb{R}^d) \leq \nu(A), \qquad A \text{ any Borel subset in } \mathbb{R}^d, \tag{5.20}$$

with equality if and only if $\mathbf{u}_\varepsilon$ is $\varepsilon-$oscillatory.

We also have that

$$w_0(\mathbb{R}^{2d}) \leq \overline{\lim_{\varepsilon\to 0}} \int_{\mathbb{R}^d} |\mathbf{u}_\varepsilon(\mathbf{x})|^2 d\mathbf{x}, \tag{5.21}$$

with equality if and only if $\mathbf{u}_\varepsilon$ is $\varepsilon-$oscillatory and compact at infinity.

The proof of this and similar results may be found in [17, 22].

The above results are important in the following sense. It states that for $\mathbf{u}_\varepsilon$ is $\varepsilon-$oscillatory and compact at infinity, we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} |\mathbf{u}_\varepsilon(\mathbf{x})|^2 d\mathbf{x} = \operatorname{Tr} \nu(\mathbb{R}^d) = w_0(\mathbb{R}^{2d}). \tag{5.22}$$

This means that all the energy there is in the system is captured in the limit $\varepsilon \to 0$ by the Wigner transform. The equality $W_0(A, \mathbb{R}^d) = \nu(A)$ states that the average of the Wigner transform over wavenumbers $\mathbf{k}$ does give the local energy density in the limit $\varepsilon \to 0$. This however only happens when the function $\mathbf{u}_\varepsilon$ oscillates at the right scale. If it oscillates at the scale ε^2, then the limiting measure ν will still capture those oscillations; however not the Wigner transform W_ε and (5.20) would become a strict inequality.

When $\mathbf{u}_\varepsilon$ converges weakly to 0, then $\nu(A)$ can be considered as a local defect measure, which measures by how much $\mathbf{u}_\varepsilon$ does not converge strongly to 0 on A. When $\nu(A) = 0$, then $\mathbf{u}_\varepsilon$ converges strongly to 0 on A. The limit W_0 is a *microlocal* defect measure, which measures the defect to compactness not only locally in space, but microlocally in the phase space. Whereas $\nu(A)$ tells that $\mathbf{u}_\varepsilon$ does not converge strongly to 0 on A, $W_0(\mathbf{x}, \mathbf{k})$ also says in which directions $\mathbf{k}$ it oscillates and what is the "strength" of such oscillations.

This justifies the terminology that W_0 is a phase-space energy density. It states how much energy oscillates at position $\mathbf{x}$ with reduced wavenumber $\mathbf{k}$. Note that $W_\varepsilon(\mathbf{x}, \mathbf{k})$ cannot quite be given this interpretation of phase-space energy density because nothing prevents it from begin locally negative. It nonetheless helps to intuitively consider $W_\varepsilon(\mathbf{x}, \mathbf{k})$ as a phase-space energy density-like object; all the more that $\int_{\mathbb{R}^d} W_\varepsilon(\mathbf{x}, \mathbf{k}) d\mathbf{k} = |\mathbf{u}_\varepsilon|^2(\mathbf{x})$ from the second line in (5.6).

Let us conclude this section by a remark on $W_\varepsilon[\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon](\mathbf{x}, \mathbf{k})$. Since it is uniformly bounded in $(\mathcal{A}^{m\times m})'$, it converges to a limit W_0 as well. However the limiting distribution W_0 need not be non-negative. Yet it is a measure on $\mathbb{R}^{2d}$. Indeed we have from the definition of the Wigner transform and from (5.6) that

$$\begin{aligned}(W_\varepsilon + W_\varepsilon^*)[\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon] &= W_\varepsilon[\mathbf{u}_\varepsilon + \mathbf{v}_\varepsilon, \mathbf{u}_\varepsilon + \mathbf{v}_\varepsilon] - W_\varepsilon[\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon] - W_\varepsilon[\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon],\\ (W_\varepsilon - W_\varepsilon^*)[\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon] &= i\big(W_\varepsilon[\mathbf{u}_\varepsilon + i\mathbf{v}_\varepsilon, \mathbf{u}_\varepsilon + i\mathbf{v}_\varepsilon] - W_\varepsilon[\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon] - W_\varepsilon[i\mathbf{v}_\varepsilon, i\mathbf{v}_\varepsilon]\big).\end{aligned} \tag{5.23}$$

All terms on both left hand sides converge to signed measures so that $W_\varepsilon[\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon](\mathbf{x}, \mathbf{k})$ also converges to an unsigned measure on $\mathbb{R}^{2d}$. The above formulas may also be used to translate the results stated in Proposition 5.2.3 to limiting correlations.

5.3 Equations for the Wigner transform

The Wigner transform introduced in the preceding section will be a useful tool in the analysis of the propagation of high frequency waves in random media. Let us now assume that $\mathbf{u}_\varepsilon^\varphi(t, \mathbf{x})$ for $\varphi = 1, 2$ are two wave field solutions of wave equations of the form

$$\varepsilon \frac{\partial \mathbf{u}_\varepsilon^\varphi}{\partial t} + A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi = 0, \quad \varphi = 1, 2, \tag{5.24}$$

with appropriate initial conditions. We thus explicitly assume that $\mathbf{u}_\varepsilon^\varphi$ solve a first-order equation in time. Typically, A_ε is a differential operator in the spatial variables, although more general operators and operators with coefficients that depend on time as well may be considered. Because it simplifies life a bit and it is true for classical wave equations, we assume that $\mathbf{u}_\varepsilon^\varphi$ is real-valued and that A_ε^φ also are real-valued operators.

When we expect that the fields $\mathbf{u}_\varepsilon^\varphi$ oscillate at the frequency ε^{-1}, the Wigner transform of the two fields will provide a tool to analyze their correlation function, or the energy density of a wave field when $\mathbf{u}_\varepsilon^1 = \mathbf{u}_\varepsilon^2$. One of the major advantages of the Wigner transform is that it satisfies a closed-form equation. This should not come too much as a surprise. Since we have an equation for $\mathbf{u}_\varepsilon^\varphi$, it is not difficult to find an equation for the correlation $\mathbf{u}_\varepsilon^1(\mathbf{x})\mathbf{u}_\varepsilon^2(\mathbf{y})$ for a large class of equations of the form (5.24). We have seen that the Wigner transform is then not much more that the Fourier transform of a two point correlation function of fields.

More specifically, an equation for the Wigner transform defined as

$$W_\varepsilon(t, \mathbf{x}, \mathbf{k}) = W[\mathbf{u}_\varepsilon^1(t, \cdot), \mathbf{u}_\varepsilon^2(t, \cdot)](\mathbf{x}, \mathbf{k}), \tag{5.25}$$

is obtained as follows. We verify from (5.1) and (5.24) that

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + W[A^1_\varepsilon \mathbf{u}^1_\varepsilon, \mathbf{u}^2_\varepsilon] + W[\mathbf{u}^1_\varepsilon, A^2_\varepsilon \mathbf{u}^2_\varepsilon] = 0. \tag{5.26}$$

It thus remains to find operators $\mathcal{A}^\varphi_\varepsilon$ such that $W[A^1_\varepsilon \mathbf{u}^1_\varepsilon, \mathbf{u}^2_\varepsilon] = \mathcal{A}^1_\varepsilon[W_\varepsilon]$ and $W[\mathbf{u}^1_\varepsilon, A^2_\varepsilon \mathbf{u}^2_\varepsilon] = \mathcal{A}^{2*}_\varepsilon[W_\varepsilon]$ to obtain a closed-form evolution equation for $W_\varepsilon(t, \mathbf{x}, \mathbf{k})$. The initial condition $W_\varepsilon(0, \mathbf{x}, \mathbf{k})$ is then the Wigner transform of the initial conditions $\mathbf{u}^1_\varepsilon(0, \mathbf{x})$ and $\mathbf{u}^2_\varepsilon(0, \mathbf{x})$ for the wave fields, which are supposed to be known. The derivation of such operators $\mathcal{A}^\varphi_\varepsilon$ is based on the following pseudo-differential calculus.

Pseudo-differential calculus. Let $P(\mathbf{x}, \varepsilon\mathbf{D})$ be a matrix-valued pseudo-differential operator, defined by

$$P(\mathbf{x}, \varepsilon\mathbf{D})\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\mathbf{k}} P(\mathbf{x}, i\varepsilon\mathbf{k})\hat{\mathbf{u}}(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^d}. \tag{5.27}$$

We assume that $P(\mathbf{x}, i\varepsilon\mathbf{k})$ is a smooth function and use the same mathematical symbol for the operator $P(\mathbf{x}, \varepsilon\mathbf{D})$ and its symbol $P(\mathbf{x}, i\varepsilon\mathbf{k})$. We also define $\mathbf{D} = \nabla$ as the gradient in the spatial variables, and not $-i$ times the gradient as is often the convention. Thus for us, $\mathbf{D}$ has symbol $i\mathbf{k}$. We verify that when $P(\mathbf{x}, i\varepsilon\mathbf{k})$ is polynomial in its second variable, then $P(\mathbf{x}, \varepsilon\mathbf{D})$ is a differential operator.

Let $u = u_R + iu_I$. For u real, we find $\bar{\hat{u}} = \hat{u}(-\xi)$. Applying to both u_R and u_I yields

$$\hat{\bar{u}}(\xi) = \bar{\hat{u}}(-\xi). \tag{5.28}$$

Now let $\hat{W}[\mathbf{u}, \mathbf{v}](\mathbf{p}, \mathbf{k})$ be the Fourier transform $\mathcal{F}_{\mathbf{x}\to\mathbf{p}}$ of the Wigner transform. We recall that

$$\hat{W}[\mathbf{u}, \mathbf{v}](\mathbf{p}, \mathbf{k}) = \frac{1}{(2\pi\varepsilon)^d}\hat{\mathbf{u}}\big(\frac{\mathbf{p}}{2} + \frac{\mathbf{k}}{\varepsilon}\big)\hat{\mathbf{v}}^*\big(\frac{\mathbf{p}}{2} - \frac{\mathbf{k}}{\varepsilon}\big) = \frac{1}{(2\pi\varepsilon)^d}\hat{\mathbf{u}}\big(\frac{\mathbf{p}}{2} + \frac{\mathbf{k}}{\varepsilon}\big)\hat{\mathbf{v}}^*\big(-\frac{\mathbf{p}}{2} + \frac{\mathbf{k}}{\varepsilon}\big), \tag{5.29}$$

thanks to (5.28) whether $\mathbf{v}$ is real-valued or not. This implies that for a homogeneous operator $P(\varepsilon\mathbf{D})$, we have

$$\hat{W}[P(\varepsilon\mathbf{D})\mathbf{u}, \mathbf{v}](\mathbf{p}, \mathbf{k}) = P\big(i\mathbf{k} + \frac{\varepsilon i\mathbf{p}}{2}\big)\hat{W}[\mathbf{u}, \mathbf{v}](\mathbf{p}, \mathbf{k}), \tag{5.30}$$

whence

$$W[P(\varepsilon\mathbf{D})\mathbf{u}, \mathbf{v}](\mathbf{x}, \mathbf{k}) = P(i\mathbf{k} + \frac{\varepsilon\mathbf{D}}{2})W[\mathbf{u}, \mathbf{v}](\mathbf{x}, \mathbf{k}). \tag{5.31}$$

The same calculation shows that when $\mathbf{v}$ and $P(\varepsilon\mathbf{D})\mathbf{v}$ are real-valued, we have

$$W[\mathbf{u}, P(\varepsilon\mathbf{D})\mathbf{v}](\mathbf{x}, \mathbf{k}) = \Big[W[\mathbf{u}, \mathbf{v}](\mathbf{x}, \mathbf{k})P^*\big(i\mathbf{k} - \frac{\varepsilon\mathbf{D}}{2}\big)\Big]. \tag{5.32}$$

In the latter right-hand side, we use the convention that the differential operator $\mathbf{D}$ acts on $W[\mathbf{u}, \mathbf{v}](\mathbf{x}, \mathbf{k})$, and thus should be interpreted as the inverse Fourier transform of the matrix $\hat{W}[\mathbf{u}, \mathbf{v}](\mathbf{p}, \mathbf{k})P^*(i\mathbf{k} - \frac{\varepsilon i\mathbf{p}}{2})$. Here, $P^*(u)$ denotes the complex conjugate of the transpose of the matrix P with coefficients $P_{ij}(u)$, i.e., the matrix with coefficients $\overline{P_{ji}(u)}$. Another way of stating this is that

$$\Big[W[\mathbf{u}, \mathbf{v}](\mathbf{x}, \mathbf{k})P^*\big(i\mathbf{k} - \frac{\varepsilon\mathbf{D}}{2}\big)\Big]_{jk} = \sum_{p=1}^m P^*_{pk}\big(i\mathbf{k} - \frac{\varepsilon\mathbf{D}}{2}\big)W_{jp}[\mathbf{u}, \mathbf{v}](\mathbf{x}, \mathbf{k}). \tag{5.33}$$

We use the notation $[\cdot]$ to represent such a convention.

We now generalize the above calculation to

$$W[P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \mathcal{L}_P W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}). \tag{5.34}$$

We verify that

$$\mathcal{F}[P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{u}](\mathbf{k}) = \int_{\mathbb{R}^d} \hat{P}(\mathbf{k}-\boldsymbol{\xi}, i\varepsilon\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi})\frac{d\boldsymbol{\xi}}{(2\pi)^d}.$$

Using (5.29), we thus obtain that

$$\hat{W}[P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{u},\mathbf{v}](\mathbf{p},\mathbf{k}) = \int_{\mathbb{R}^d} \hat{P}(\boldsymbol{\xi}, i\mathbf{k} + i\varepsilon(\frac{\mathbf{p}}{2}-\boldsymbol{\xi}))\hat{W}[\mathbf{u},\mathbf{v}](\mathbf{p}-\boldsymbol{\xi},\mathbf{k}-\frac{\varepsilon\boldsymbol{\xi}}{2})\frac{d\boldsymbol{\xi}}{(2\pi)^d}.$$

After Fourier transforms, we finally obtain that

$$\mathcal{L}_P W(\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \hat{P}(\boldsymbol{\xi}, i\mathbf{k} + i\varepsilon(\frac{\mathbf{p}}{2}-\boldsymbol{\xi}))\frac{d\mathbf{p}}{(2\pi)^d} \right) e^{i\boldsymbol{\xi}\cdot\mathbf{y}} W(\mathbf{y},\mathbf{k}-\frac{\varepsilon\boldsymbol{\xi}}{2})\frac{d\boldsymbol{\xi}d\mathbf{y}}{(2\pi)^d}. \tag{5.35}$$

We verify that (5.35) generalizes (5.31). A very similar expression similarly generalizes (5.32).

Exercise 5.3.1 Work out that generalization.

Asymptotic expansions. The operator $\mathcal{L}_P$ defined in (5.35) is amenable to asymptotic expansions. Throughout the text, we shall use the convention

$$P'(\mathbf{x}, i\mathbf{k}) = \nabla_{i\mathbf{k}} P(\mathbf{x}, i\mathbf{k}) = -i\nabla_{\mathbf{k}} P(\mathbf{x}, i\mathbf{k}). \tag{5.36}$$

For functions $W(\mathbf{x},\mathbf{k})$ that are sufficiently smooth in the $\mathbf{k}$ variable, we have the Taylor expansion

$$W(\mathbf{x},\mathbf{k}-\frac{\varepsilon\boldsymbol{\xi}}{2}) = W(\mathbf{x},\mathbf{k}) - \frac{\varepsilon\boldsymbol{\xi}}{2}\cdot\nabla_{\mathbf{k}} W(\mathbf{x},\mathbf{k}) + O(\varepsilon^2).$$

Similar asymptotic expansions in (5.35) yield that for smooth functions $W(\mathbf{x},\mathbf{k})$ we have

$$\begin{aligned}
\mathcal{L}_P W(\mathbf{x},\mathbf{k}) &= \mathcal{M}_\varepsilon W(\mathbf{x},\mathbf{k}) + \varepsilon\mathcal{N}_\varepsilon W(\mathbf{x},\mathbf{k}) + O(\varepsilon^2) \\
\mathcal{M}_\varepsilon W(\mathbf{x},\mathbf{k}) &= P(\mathbf{x}, i\mathbf{k} + \frac{\varepsilon\mathbf{D}}{2})W(\mathbf{x},\mathbf{k}) + \frac{i\varepsilon}{2}\nabla_{\mathbf{x}} P(\mathbf{x}, i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2})\cdot\nabla_{\mathbf{k}} W(\mathbf{x},\mathbf{k}) \\
\mathcal{N}_\varepsilon W(\mathbf{x},\mathbf{k}) &= i\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{k}} P(\mathbf{x}, i\mathbf{k}+\frac{\varepsilon\mathbf{D}_{\mathbf{x}}}{2})W(\mathbf{x},\mathbf{k}).
\end{aligned} \tag{5.37}$$

The above calculations allow us to deduce that for functions $W(\mathbf{x},\mathbf{k})$ that are sufficiently smooth in both variables $\mathbf{x}$ and $\mathbf{k}$, we have

$$\begin{aligned}
\mathcal{L}_P W(\mathbf{x},\mathbf{k}) &= \mathcal{L}_\varepsilon W(\mathbf{x},\mathbf{k}) + \varepsilon\mathcal{N}_\varepsilon W(\mathbf{x},\mathbf{k}) + O(\varepsilon^2) \\
\mathcal{L}_\varepsilon W(\mathbf{x},\mathbf{k}) &= P(\mathbf{x}, i\mathbf{k})W(\mathbf{x},\mathbf{k}) + \frac{\varepsilon}{2i}\{P,W\}(\mathbf{x},\mathbf{k}),
\end{aligned} \tag{5.38}$$

where we have defined the Poisson bracket

$$\{P,W\}(\mathbf{x},\mathbf{k}) = (\nabla_{\mathbf{k}} P\cdot\nabla_{\mathbf{x}} W - \nabla_{\mathbf{x}} P\cdot\nabla_{\mathbf{k}} W)(\mathbf{x},\mathbf{k}). \tag{5.39}$$

Similarly, we define

$$W[\mathbf{u}, P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{v}](\mathbf{x},\mathbf{k}) = \mathcal{L}^* W(\mathbf{x},\mathbf{k}). \tag{5.40}$$

We verify that when $\mathbf{v}$ and $P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{v}$ are real-valued,

$$\begin{aligned}
\mathcal{L}^*W(\mathbf{x},\mathbf{k}) &= (\mathcal{M}^*_\varepsilon+\varepsilon\mathcal{N}^*_\varepsilon)W(\mathbf{x},\mathbf{k})+O(\varepsilon^2)=(\mathcal{L}^*_\varepsilon+\varepsilon\mathcal{N}^*_\varepsilon)W(\mathbf{x},\mathbf{k})+O(\varepsilon^2)\\
\mathcal{M}^*_\varepsilon W(\mathbf{x},\mathbf{k}) &= [W(\mathbf{x},\mathbf{k})P^*(\mathbf{x},i\mathbf{k}-\frac{\varepsilon\mathbf{D}}{2})]-\frac{i\varepsilon}{2}[\nabla_{\mathbf{k}}W(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{x}}P^*(\mathbf{x},i\mathbf{k}-\frac{\varepsilon\mathbf{D}}{2})]\\
\mathcal{L}^*_\varepsilon W(\mathbf{x},\mathbf{k}) &= [W(\mathbf{x},\mathbf{k})P^*(\mathbf{x},i\mathbf{k})]+\frac{\varepsilon}{2i}\{W,P^*\}(\mathbf{x},\mathbf{k}),\\
\mathcal{N}^*_\varepsilon W(\mathbf{x},\mathbf{k}) &= -i[W(\mathbf{x},\mathbf{k})\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{k}}P^*(\mathbf{x},i\mathbf{k}+\frac{\varepsilon\mathbf{D}_{\mathbf{x}}}{2})].
\end{aligned}\tag{5.41}$$

When $W=W[\mathbf{u},\mathbf{v}]$ with $\mathbf{u}$ and $\mathbf{v}$ bounded in L^2, and $P(\mathbf{x},i\mathbf{k})$ is a smooth function, then we verify that the above $O(\varepsilon^2)$ terms are of the form $\varepsilon^2R_\varepsilon$ with R_ε uniformly bounded in $(\mathcal{S}')^{(m\times m)}$, the space of matrix-valued Schwartz distributions.

Exercise 5.3.2 Verify that R_ε is uniformly bounded in $(\mathcal{S}')^{(m\times m)}$.

With the above hypotheses, we thus find the following useful result

$$\begin{aligned}
W_\varepsilon[P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{u},\mathbf{v}] &= PW_\varepsilon[\mathbf{u},\mathbf{v}]+\frac{\varepsilon}{2i}\{P,W_\varepsilon[\mathbf{u},\mathbf{v}]\}+i\varepsilon\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{k}}P(\mathbf{x},i\mathbf{k})W_\varepsilon[\mathbf{u},\mathbf{v}]+\varepsilon^2R_\varepsilon,\\
W_\varepsilon[\mathbf{u},P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{v}] &= [W_\varepsilon[\mathbf{u},\mathbf{v}]P^*]+\frac{\varepsilon}{2i}\{W_\varepsilon[\mathbf{u},\mathbf{v}],P^*\}-i\varepsilon W_\varepsilon[\mathbf{u},\mathbf{v}]\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{k}}P^*(\mathbf{x},i\mathbf{k})+\varepsilon^2S_\varepsilon,
\end{aligned}\tag{5.42}$$

where R_ε and S_ε are bounded in $(\mathcal{S}')^{(m\times m)}$.

Highly oscillatory coefficients. The above pseudo-differential calculus was obtained for smooth pseudo-differential operators P. When these operators involve highly oscillatory coefficients, different asymptotic expansions are necessary. We consider here the case where the operator involves multiplication by a highly oscillatory matrix-valued coefficient. Let $V(\mathbf{x})$ be a real-valued matrix-valued function. Then we find that

$$\begin{aligned}
W[V(\frac{\mathbf{x}}{\varepsilon})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) &= \int_{\mathbb{R}^d}e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}\hat{V}(\mathbf{p})W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d},\\
W[\mathbf{u},V(\frac{\mathbf{x}}{\varepsilon})\mathbf{v}](\mathbf{x},\mathbf{k}) &= \int_{\mathbb{R}^d}e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\hat{V}^t(\mathbf{p})\frac{d\mathbf{p}}{(2\pi)^d}.
\end{aligned}\tag{5.43}$$

Here $\hat{V}(\mathbf{p})$ is the Fourier transform of $V(\mathbf{x})$ component by component. This may be generalized as follows. Let $V(\mathbf{x},\mathbf{y})$ be a real-valued matrix function, with Fourier transform $\check{V}(\mathbf{q},\mathbf{p})$ and Fourier transform with respect to the second variable $\hat{V}(\mathbf{x},\mathbf{p})$. We then find that

$$\begin{aligned}
W[V(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) &= \int_{\mathbb{R}^{2d}}e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}e^{i\mathbf{x}\cdot\mathbf{q}}\check{V}(\mathbf{q},\mathbf{p})W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}-\frac{\varepsilon\mathbf{q}}{2})\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^{2d}},\\
&= \int_{\mathbb{R}^d}e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}\hat{V}(\mathbf{x},\mathbf{p})W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}+O(\varepsilon),
\end{aligned}\tag{5.44}$$

provided that $W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})$ is sufficiently smooth in the $\mathbf{k}$ variable.

Exercise 5.3.3 Work out the formula for $W[V(\mathbf{x})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})$. Show that it is asymptotically equivalent to (5.37).

Multiple scale expansion. The error terms in (5.37) and (5.38), although both deduced from Taylor expansions, have different expressions. While the former involves second-order derivatives in $\mathbf{k}$ of $W(\mathbf{x},\mathbf{k})$, the latter involves second-order derivatives in both the $\mathbf{k}$ and $\mathbf{x}$ variables. When $W(\mathbf{x},\mathbf{k})$ has bounded second-order derivatives in $\mathbf{x}$ and $\mathbf{y}$, then $(\mathcal{L}_P-\mathcal{L}_\varepsilon)W=$

$O(\varepsilon^2)$ and $(\mathcal{L}_P - \mathcal{M}_\varepsilon)W = O(\varepsilon^2)$.In the sequel however, we will need to apply the operator $\mathcal{M}_\varepsilon$ to functions that oscillate in the $\mathbf{x}$ variable and are smooth in the $\mathbf{k}$ variable. Such functions will have the form $W(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, \mathbf{k})$. The differential operator $\mathbf{D}$ acting on such functions then takes the form

$$\mathbf{D} = \mathbf{D}_\mathbf{x} + \frac{1}{\varepsilon}\mathbf{D}_\mathbf{y}.$$

We then verify that

$$\mathcal{M}_\varepsilon[W(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, \mathbf{k})](\mathbf{x}, \mathbf{k}) = P(\mathbf{x}, i\mathbf{k} + \frac{\mathbf{D}_\mathbf{y}}{2})W(\mathbf{x}, \mathbf{y}, \mathbf{k})|_{\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}} + O(\varepsilon). \tag{5.45}$$

We will not need higher-order terms. Note that on such functions, $(\mathcal{L}_P - \mathcal{L}_\varepsilon)W = O(1)$, which implies that $\mathcal{L}_\varepsilon$ cannot be used as an approximation of $\mathcal{L}_P$.

5.4 Examples of Wigner transforms

Compact sequence. Consider a sequence of functions $u_\varepsilon(\mathbf{x})$ converging to u strongly in $L^2(\mathbb{R}^d)$ as $\varepsilon \to$. Then one verifies that

$$u_\varepsilon(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2})u_\varepsilon^*(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}) \to |u(\mathbf{x})|^2,$$

weakly in $\mathcal{S}'(\mathbb{R}^{2d})$ (to simplify) as $\varepsilon \to 0$. This implies that the limiting Wigner transform

$$W_0[u_\varepsilon, u_\varepsilon](\mathbf{x}, \mathbf{k}) = |u(\mathbf{x})|^2\delta(\mathbf{k}). \tag{5.46}$$

Exercise 5.4.1 Verify the above statements.

Oscillatory sequence. Consider a sequence of the form $u_\varepsilon(\mathbf{x}) = \phi_{\boldsymbol{\xi}}(\mathbf{x})e^{i\mathbf{x}\cdot\boldsymbol{\xi}/\varepsilon}$ for $\boldsymbol{\xi} \in \mathbb{R}^d$ fixed and $\phi_{\boldsymbol{\xi}}$ a smooth function. Then the limiting Wigner transform is given by

$$W_0[u_\varepsilon, u_\varepsilon](\mathbf{x}, \mathbf{k}) = |\phi_{\boldsymbol{\xi}}(\mathbf{x})|^2\delta(\mathbf{k} - \boldsymbol{\xi}). \tag{5.47}$$

We thus see that the Wigner transform of a plane wave is a delta function. The above result extends to a superposition of plane waves. Let

$$u_\varepsilon(\mathbf{x}) = \sum_{m=1}^{M} \phi_m(\mathbf{x})e^{i\mathbf{x}\cdot\boldsymbol{\xi}_m/\varepsilon}, \tag{5.48}$$

where the momenta $\boldsymbol{\xi}_m$ are mutually distinct. Then the limiting Wigner transform is given by

$$W_0[u_\varepsilon, u_\varepsilon](\mathbf{x}, \mathbf{k}) = \sum_{m=1}^{M} |\phi_m(\mathbf{x})|^2\delta(\mathbf{k} - \boldsymbol{\xi}_m). \tag{5.49}$$

This result is fundamental in applications: it states that the Wigner transform of two plane waves with different wavenumbers tends to 0 in the limit $\varepsilon \to 0$:

$$W[e^{i\mathbf{x}\cdot\boldsymbol{\xi}_1/\varepsilon}, e^{i\mathbf{x}\cdot\boldsymbol{\xi}_2/\varepsilon}](\mathbf{x}, \mathbf{k}) = 0, \tag{5.50}$$

provided that $\boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2$. Only those plane waves propagating in the same direction with the same wavenumber interact coherently in the limit $\varepsilon \to 0$. If $\boldsymbol{\xi}_p = \boldsymbol{\xi}_q$ in (5.48), then the coefficient $|\phi_p + \phi_q|^2$ would appear in (5.49) rather than $|\phi_p|^2 + |\phi_q|^2$. This example also shows

that the Wigner transform converges only weakly to its limit. Indeed the product of two plane waves with different directions certainly does not converge strongly to 0 as $\varepsilon \to 0$.

Warning: the above result holds for a finite number of plane waves. When the number of plane waves becomes infinite, the above result may not hold. The reason is that if two plane waves with wavenumbers differing by $O(\varepsilon)$ are present, we cannot conclude that they are uncorrelated in the limit. Only plane waves with sufficiently different directions (i.e., much larger than $O(\varepsilon)$) are uncorrelated in the limit $\varepsilon \to 0$. A typical example where the above limit does not hold is that of the Wigner transform of a Bessel function; where the Bessel function is defined for $d = 2$ as the superposition of all plane waves with wavenumber of modulus $|\mathbf{k}| = 1$. I leave this as a (quite difficult) exercise.

Point concentration sequence. Consider now the sequence

$$u_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^{d/2}}\phi(\frac{\mathbf{x}}{\varepsilon}). \tag{5.51}$$

Then the limiting Wigner measure is given by

$$W[u_\varepsilon, u_\varepsilon](\mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^d}\delta(\mathbf{x})|\hat{\phi}(\mathbf{k})|^2. \tag{5.52}$$

The energy concentrates at one point in space and is radiated in each wavenumber $\mathbf{k}$ according to the Fourier transform of the waveform $\phi(\mathbf{x})$.

Coherent state. The last example may be generalized as follows

$$u_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^{d/2}}\phi(\frac{\mathbf{x} - \mathbf{x}_0}{\varepsilon})e^{i\mathbf{x}\cdot\boldsymbol{\xi}/\varepsilon}. \tag{5.53}$$

Then we find that the limiting Wigner measure is given by

$$W[u_\varepsilon, u_\varepsilon](\mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^d}\delta(\mathbf{x} - \mathbf{x}_0)|\hat{\phi}(\mathbf{k} - \boldsymbol{\xi})|^2. \tag{5.54}$$

For different scalings, we obtain the following results. Assume that

$$u_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^{d/2}}\phi(\frac{\mathbf{x} - \mathbf{x}_0}{\varepsilon^\alpha})e^{i\mathbf{x}\cdot\boldsymbol{\xi}/\varepsilon}. \tag{5.55}$$

When $\alpha = 1$, we have the result (5.54). When $\alpha > 1$, we verify that $W \equiv 0$. This is because all oscillations occur at a frequency $\varepsilon^{-\alpha} \gg \varepsilon^{-1}$ that the scaled Wigner transform cannot capture. When $0 < \alpha < 1$, we verify that

$$W[u_\varepsilon, u_\varepsilon](\mathbf{x}, \mathbf{k}) = \|\phi\|^2_{L^2(\mathbb{R}^d)}\delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{k} - \boldsymbol{\xi}). \tag{5.56}$$

When $\alpha = 0$, we recall that this is an *oscillatory sequence*, treated in (5.47).

WKB states. Let us now suppose that

$$u_\varepsilon(\mathbf{x}) = \phi(\mathbf{x})e^{iS(\mathbf{x})/\varepsilon^\alpha}, \tag{5.57}$$

where ϕ and S are sufficiently smooth. When $\alpha < 1$, we verify that the high oscillations do not play any role in the limit and

$$W[u_\varepsilon, u_\varepsilon](\mathbf{x}, \mathbf{k}) = |\phi(\mathbf{x})|^2\delta(\mathbf{k}). \tag{5.58}$$

When $\alpha = 1$, we have the limiting Wigner transform

$$W[u_\varepsilon, u_\varepsilon](\mathbf{x}, \mathbf{k}) = |\phi(\mathbf{x})|^2 \delta(\mathbf{k} - \nabla S(\mathbf{x})). \tag{5.59}$$

Proof. We calculate that

$$\begin{aligned}(W_\varepsilon, a) &= \int_{\mathbb{R}^{3d}} e^{i\mathbf{k}\cdot\mathbf{y}} e^{\frac{i}{\varepsilon}S(\mathbf{x}-\varepsilon\frac{\mathbf{y}}{2})} \phi(\mathbf{x} - \varepsilon\frac{\mathbf{y}}{2}) \phi^*(\mathbf{x} + \varepsilon\frac{\mathbf{y}}{2}) e^{-\frac{i}{\varepsilon}S(\mathbf{x}+\varepsilon\frac{\mathbf{y}}{2})} a(\mathbf{x}, \mathbf{k}) \frac{d\mathbf{x}d\mathbf{y}d\mathbf{k}}{(2\pi)^d} \\ &= \int_{\mathbb{R}^{2d}} e^{-i\mathbf{y}\cdot\nabla\mathbf{S}(\mathbf{x})} e^{i\varepsilon r_\varepsilon(\mathbf{x},\mathbf{y})} \phi(\mathbf{x} - \varepsilon\frac{\mathbf{y}}{2}) \phi^*(\mathbf{x} + \varepsilon\frac{\mathbf{y}}{2}) \hat{a}(\mathbf{x}, \mathbf{y}) d\mathbf{x}d\mathbf{y},\end{aligned}$$

where $\hat{a}(\mathbf{x}, \mathbf{y})$ is the inverse Fourier transform $\mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{y}}$ of $a(\mathbf{x}, \mathbf{k})$, and r_ε is real-valued and uniformly bounded for sufficiently smooth functions $S(\mathbf{x})$. This implies that above term is uniformly (in ε) integrable in $(\mathbf{x}, \mathbf{y})$ for $\phi \in L^2$ and $a \in \mathcal{A}$ (for then $\hat{a}(\mathbf{x}, \mathbf{y})$ is uniformly bounded in $\mathbf{x}$ and integrable in $\mathbf{y}$). By the dominated Lebesgue convergence theorem, we obtain that the above term converges to

$$(W_0, a) = \int_{\mathbb{R}^{2d}} e^{-i\mathbf{y}\cdot\nabla S(\mathbf{x})} |\phi(\mathbf{x})|^2 \hat{a}(\mathbf{x}, \mathbf{y}) d\mathbf{x}d\mathbf{y},$$

which is nothing but

$$(W_0, a) = \int_{\mathbb{R}^{2d}} \delta(\mathbf{k} - \nabla S(\mathbf{x})) |\phi(\mathbf{x})|^2 a(\mathbf{x}, \mathbf{k}) d\mathbf{x}d\mathbf{k},$$

whence the result. □

The case $\alpha > 1$ is more delicate. When $S(\mathbf{x})$ has no critical points, i.e., $\nabla S \neq 0$ for all $\mathbf{x} \in \mathbb{R}^d$, then the limiting Wigner measure is $W \equiv 0$ as in the case of plane waves. In the presence of critical points, a more refined analysis is necessary.

Limiting Liouville equation. Let us assume that the phase $S(t, \mathbf{x})$ and the amplitude $\phi(t, \mathbf{x})$ solve the following eikonal and transport equations:

$$\frac{\partial S}{\partial t} + \omega(\mathbf{x}, \nabla S) = 0, \quad \frac{\partial |\phi|^2}{\partial t} + \nabla \cdot (|\phi|^2 (\nabla_{\mathbf{k}} \omega)(\mathbf{x}, \nabla S)) = 0, \tag{5.60}$$

where $\omega(\mathbf{x}, \mathbf{k})$ is a Hamiltonian. Then the Wigner transform defined by (5.59), namely

$$W(t, \mathbf{x}, \mathbf{k}) = |\phi(t, \mathbf{x})|^2 \delta(\mathbf{k} - \nabla S(t, \mathbf{x})), \tag{5.61}$$

as it turns out, solves the following Liouville equation,

$$\frac{\partial W}{\partial t} + \{\omega, W\} = 0, \tag{5.62}$$

where the Poisson bracket is defined in (5.39).

The proof is an exercise in distribution theory. We find that

$$\begin{aligned}\frac{\partial W}{\partial t} &= \frac{\partial}{\partial t}|\phi(t)|^2 \delta(\mathbf{k} - \nabla S(t, \mathbf{x})) + |\phi|^2 \frac{\partial}{\partial t}(\delta(\mathbf{k} - \nabla S(t, \mathbf{x}))) \\ &= \frac{\partial}{\partial t}|\phi(t)|^2 \delta(\mathbf{k} - \nabla S(t, \mathbf{x})) - |\phi|^2 (\nabla_{\mathbf{k}}\delta)(\mathbf{k} - \nabla S(t, \mathbf{x}))) \cdot \nabla_{\mathbf{x}} \frac{\partial}{\partial t} S(t, \mathbf{x}) \\ &= -\nabla \cdot (|\phi|^2 (\nabla_{\mathbf{k}}\omega)(\mathbf{x}, \nabla S)) \delta(\mathbf{k} - \nabla S(t, \mathbf{x})) + |\phi|^2 (\nabla_{\mathbf{k}}\delta)(\mathbf{k} - \nabla S(t, \mathbf{x}))) \cdot \nabla_{\mathbf{x}}(\omega(\mathbf{x}, \nabla S)) \\ &= -\nabla_{\mathbf{x}}|\phi|^2 \cdot \nabla_{\mathbf{k}}\omega(\mathbf{x}, \nabla S)\delta(\mathbf{k} - \nabla S) - |\phi|^2 \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{k}}\omega(\mathbf{x}, \nabla S)\delta(\mathbf{k} - \nabla S) \\ &\quad -|\phi|^2 \nabla^2_{\mathbf{k}}\omega(\mathbf{x}, \nabla S) \cdot \nabla^2_{\mathbf{x}} S\delta(\mathbf{k} - \nabla S) + |\phi|^2 \nabla_{\mathbf{k}}\delta(\mathbf{k} - \nabla S) \cdot \nabla_{\mathbf{x}}\omega(\mathbf{x}, \nabla S) \\ &\quad +|\phi|^2 \nabla_{\mathbf{k}}\delta(\mathbf{k} - \nabla S)\nabla_{\mathbf{k}}\omega \cdot \nabla^2_{\mathbf{x}} S.\end{aligned}$$

We verify that

$$\nabla_{\mathbf{k}} W \cdot \nabla_{\mathbf{x}} \omega(\mathbf{x},\mathbf{k}) = |\phi|^2 \nabla_{\mathbf{k}} \delta(\mathbf{k}-\nabla S)\cdot \nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k})$$
$$= |\phi|^2 \nabla_{\mathbf{k}} \delta(\mathbf{k}-\nabla S)\cdot \nabla_{\mathbf{x}}\omega(\mathbf{x},\nabla S(t,\mathbf{x})) - |\phi|^2 \delta(\mathbf{k}-\nabla S)\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\omega(\mathbf{x},\nabla S(t,\mathbf{x})).$$

Indeed, let $a(\mathbf{k})$ be a test function. Then

$$(\nabla\delta(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{F},a) = -(\delta(\mathbf{k}-\mathbf{k}_0),\nabla_{\mathbf{k}}\cdot(\mathbf{F}a)) = -\nabla\cdot\mathbf{F}(\mathbf{k}_0)(\delta(\mathbf{k}-\mathbf{k}_0),a)+\mathbf{F}(\mathbf{k}_0)\cdot(\nabla\delta(\mathbf{k}-\mathbf{k}_0),a).$$

Similarly, we find that

$$\nabla_{\mathbf{x}} W\cdot\nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k}) = (\nabla_{\mathbf{x}}|\phi|^2)\delta(\mathbf{k}-\nabla S)\cdot\nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})$$
$$+|\phi|^2\nabla_{\mathbf{k}}\delta(\mathbf{k}-\nabla S(t,\mathbf{x}))\nabla^2_{\mathbf{x}}S\cdot\nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})$$
$$= (\nabla_{\mathbf{x}}|\phi|^2)\delta(\mathbf{k}-\nabla S)\cdot\nabla_{\mathbf{k}}\omega(\mathbf{x},\nabla S)$$
$$+|\phi|^2\nabla_{\mathbf{k}}\delta(\mathbf{k}-\nabla S)\nabla^2_{\mathbf{x}}S\cdot\nabla_{\mathbf{k}}\omega(\mathbf{x},\nabla S) + |\phi|^2\delta(\mathbf{k}-\nabla S)\nabla^2_{\mathbf{x}}S\nabla^2_{\mathbf{k}}\omega(\mathbf{x},\nabla S).$$

Upon careful inspection, one verifies that the above equalities yield

$$\frac{\partial W}{\partial t} = \nabla_{\mathbf{k}} W\cdot\nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k}) - \nabla_{\mathbf{x}} W\cdot\nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k}).$$

This is the Liouville equation (5.62). This important equation states that in "low frequency" media, i.e., heterogeneous media with slow spatial variations so that (5.60) make sense, the energy density satisfies a *linear* partial differential equation. It is moreover straightforward to solve the linear equation by the method of characteristics. Indeed let us define

$$W(t,\mathbf{x},\mathbf{k}) = W_0(\mathbf{X}(-t),\mathbf{K}(-t)), \tag{5.63}$$

with $(\mathbf{X}(t),\mathbf{K}(t))$ solution of the Hamiltonian system

$$\dot{\mathbf{X}}(t) = \nabla_{\mathbf{k}}\omega(\mathbf{X}(t),\mathbf{K}(t)), \quad \dot{\mathbf{K}}(t) = -\nabla_{\mathbf{x}}\omega(\mathbf{X}(t),\mathbf{K}(t)), \qquad \mathbf{X}(0)=\mathbf{x}, \quad \mathbf{K}(0)=\mathbf{k}. \tag{5.64}$$

Then one verifies that $W(t,\mathbf{x},\mathbf{k})$ in (5.63) solves (5.62) with initial conditions $W(0,\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k})$.

That the limit Wigner transform satisfies a Liouville equation is no surprise. In the limit of vanishing wavelength, the wave energy density follows the trajectories of classical mechanics. We have obtained this result in the framework of the WKB, or geometric optics, approximation. We'll see that this result holds in more general situations where the geometric optics approximation is not valid.

5.5 Semiclassical limit for Schrödinger equations

The high frequency Schrödinger equation (after the usual change of variables $\mathbf{x}\to\mathbf{x}/\varepsilon$ and $t\to t/\varepsilon$) is given by

$$i\varepsilon\frac{\partial\phi_\varepsilon}{\partial t} + \frac{\varepsilon^2}{2}\Delta\phi_\varepsilon - V(\mathbf{x})\phi_\varepsilon = 0, \tag{5.65}$$

with ε-oscillatory initial conditions $\phi_{0\varepsilon}(\mathbf{x})$. The potential $V(\mathbf{x})$ is slowly varying. This is thus the problem of high frequency waves in low frequency media that was handled by geometric optics in an earlier chapter.

If we look for geometric optics solutions of the form

$$\phi_\varepsilon(t,\mathbf{x}) \approx A(t,\mathbf{x})e^{iS(t,\mathbf{x})/\varepsilon}, \tag{5.66}$$

then the evolution equations for the phase S and the amplitude A take the form of the following eikonal and transport equations:

$$\frac{\partial S}{\partial t} + H(\mathbf{x}, \nabla S(t,\mathbf{x})) = 0, \qquad \frac{\partial |A|^2}{\partial t} + \nabla \cdot (|A|^2 \nabla_{\mathbf{k}} H(\mathbf{x}, \nabla S(\mathbf{x}))) = 0, \tag{5.67}$$

where the Hamiltonian has the familiar expression

$$H(\mathbf{x},\mathbf{k}) = \frac{1}{2}|\mathbf{k}|^2 + V(\mathbf{x}). \tag{5.68}$$

Exercise 5.5.1 Derive the equations in (5.67).

The results in the preceding section show that the Wigner transform of (5.66) converges to the solution of a Liouville equation as $\varepsilon \to 0$. We will now re-derive this result without using the geometric optics approximation.

Let $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ be the Wigner transform of the solution to (5.65). Note that

$$\int_{\mathbb{R}^d} W_\varepsilon(t,\mathbf{x},\mathbf{k}) d\mathbf{k} = |\phi_\varepsilon(t,\mathbf{x})|^2,$$

so that the Wigner transform allows us to reconstruct the probability density of the quantum waves.

Following the steps recalled earlier in the chapter, namely (5.26) and (5.35), we find the following closed form equation for W_ε:

$$\frac{\partial W_\varepsilon}{\partial t} + \mathbf{k} \cdot \nabla_{\mathbf{x}} W_\varepsilon + \mathcal{L}_\varepsilon W_\varepsilon = 0, \tag{5.69}$$

where

$$\mathcal{L}_\varepsilon W(\mathbf{x},\mathbf{k}) = \frac{i}{\varepsilon} \int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{x}} \hat{V}(\mathbf{p}) \Big(W(\mathbf{x}, \mathbf{k} - \frac{\varepsilon \mathbf{p}}{2}) - W(\mathbf{x}, \mathbf{k} + \frac{\varepsilon \mathbf{p}}{2}) \Big) \frac{d\mathbf{p}}{(2\pi)^d}. \tag{5.70}$$

Let us assume that $W(\mathbf{x},\mathbf{k})$ is sufficiently smooth in its $\mathbf{k}$ variable. Then one relatively easily finds that

$$\mathcal{L}_\varepsilon W(\mathbf{x},\mathbf{k}) \to -\nabla_{\mathbf{x}} V(\mathbf{x}) \cdot \nabla_{\mathbf{k}} W(\mathbf{x},\mathbf{k}), \quad \text{as} \quad \varepsilon \to 0. \tag{5.71}$$

Exercise 5.5.2 Verify the latter statement. Recall that the Fourier transform of $\nabla_{\mathbf{x}} V$ is $i\mathbf{k}\cdot\hat{V}(\mathbf{k})$.

This shows formally that the limiting equation for W is the Liouville equation

$$\frac{\partial W}{\partial t} + \mathbf{k} \cdot \nabla_{\mathbf{x}} W - \nabla_{\mathbf{x}} V \cdot \nabla_{\mathbf{k}} W = 0. \tag{5.72}$$

This is nothing but (5.62) for the specific choice of a Hamiltonian in (5.68).

The above formal derivation is nice, but does not treat the geometric optics case: in the latter case, $W(t,\mathbf{x},\mathbf{k})$ is not smooth in $\mathbf{k}$ as one verifies from (5.61).

Since $\psi_\varepsilon(t,\mathbf{x})$ is bounded in $L^2(\mathbb{R}^d)$, we know that $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ belongs to $\mathcal{A}'$ for all times and thus converges weakly in that space to $W(t,\mathbf{x},\mathbf{k})$. Let now $a(\mathbf{x},\mathbf{k})$ be a smooth test

function in $\mathcal{C}_c^\infty(\mathbb{R}^{2d})$, which we may take as real-valued because we know that $W(t,\mathbf{x},\mathbf{k})$ is real-valued. We verify that

$$(\mathcal{L}_\varepsilon W_\varepsilon, a) = \int_{\mathbb{R}^{3d}} e^{i\mathbf{p}\cdot\mathbf{x}} i\hat{V}(\mathbf{p}) W_\varepsilon(t,\mathbf{x},\mathbf{k}) \frac{1}{\varepsilon}\Big(a(\mathbf{x},\mathbf{k}+\frac{\varepsilon\mathbf{p}}{2}) - a(\mathbf{x},\mathbf{k}-\frac{\varepsilon\mathbf{p}}{2})\Big) \frac{d\mathbf{p}d\mathbf{k}d\mathbf{x}}{(2\pi)^d}.$$

We verify that

$$\frac{1}{\varepsilon}\Big(a(\mathbf{x},\mathbf{k}+\frac{\varepsilon\mathbf{p}}{2}) - a(\mathbf{x},\mathbf{k}-\frac{\varepsilon\mathbf{p}}{2})\Big) = \mathbf{p}\cdot\nabla_{\mathbf{k}} a(\mathbf{x},\mathbf{k}) + \varepsilon r_\varepsilon(\mathbf{x},\mathbf{k},\mathbf{p}), \tag{5.73}$$

where $(\mathbf{x},\mathbf{k}) \to r_\varepsilon(\mathbf{x},\mathbf{k},\mathbf{p})$ is bounded in $\mathcal{A}$ uniformly in $\mathbf{p}\in\mathbb{R}^d$ and so is $(\mathbf{x},\mathbf{k}) \to \mathbf{p}\cdot\nabla_{\mathbf{k}} a(\mathbf{x},\mathbf{k})$. This shows that

$$\int_{\mathbb{R}^{3d}} e^{i\mathbf{p}\cdot\mathbf{x}} i\hat{V}(\mathbf{p}) W_\varepsilon(t,\mathbf{x},\mathbf{k}) \varepsilon r_\varepsilon(\mathbf{x},\mathbf{k},\mathbf{p}) \frac{d\mathbf{p}d\mathbf{k}d\mathbf{x}}{(2\pi)^d} \to 0$$

as $\varepsilon\to 0$. Since $e^{i\mathbf{p}\cdot\mathbf{x}} i\mathbf{p}\hat{V}(\mathbf{p})$ integrates to $\nabla V(\mathbf{x})$ (assuming that V is sufficiently regular), we deduce from the weak convergence of W_ε to W in $\mathcal{A}'$ that

$$(\mathcal{L}_\varepsilon W_\varepsilon, a) \to (W, \nabla V\cdot\nabla_{\mathbf{k}} a) = -(\nabla V\cdot\nabla_{\mathbf{k}} W, a) \text{ in } \mathcal{D}'(\mathbb{R}^{2d}). \tag{5.74}$$

This shows that the left-hand side in (5.69) converges to the left-hand side in (5.72) in $\mathcal{D}'(\mathbb{R}^{2d})$ uniformly in time as $\varepsilon\to 0$. Since the right-hand side in (5.69) converges to that in (5.72), the limiting equation (5.72) for $W(t,\mathbf{x},\mathbf{k})$ is established.

We have thus proved the following

Proposition 5.5.1 *Let $\psi_{0\varepsilon}(\mathbf{x})$ be an ε-oscillatory sequence of functions with Wigner transform converging weakly in $\mathcal{A}'$ to $W_0(\mathbf{x},\mathbf{k})$. Let $\psi_\varepsilon(t,\mathbf{x})$ be the solution of the Schrödinger equation* (5.65) *and $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ its Wigner transform. Then $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ converges weakly in $\mathcal{A}'$ uniformly in time to a (uniquely defined accumulation point) $W(t,\mathbf{x},\mathbf{k})$, which is a weak solution of the Liouville equation* (5.72) *with initial conditions $W_0(\mathbf{x},\mathbf{k})$.*

Note that the whole sequence $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ converges to $W(t,\mathbf{x},\mathbf{k})$: we do not need to extract subsequences. The reason is that there is a unique solution in the space of bounded measures to the Liouville equation (5.72) with initial conditions $W_0(\mathbf{x},\mathbf{k})$, which by assumption is the only accumulation point of $W_{\varepsilon 0}(\mathbf{x},\mathbf{k})$. This uniquely defines the possible accumulation points of $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ as $\varepsilon\to 0$.

The result stated in the proposition is much more general than the one obtained by geometric optics. Much more general initial conditions than those of WKB type (5.66) can be considered. All we need really is bounded initial conditions in the L^2 sense that are ε oscillatory so that no energy is lost when passing to the limit $\varepsilon\to 0$. Note also that the Liouville equation is defined for all times, unlike the eikonal equation. Bicharacteristics, unlike their spatial projections the rays, never cross in the phase space, so that the Liouville equation (a linear equation) is not limited to sufficiently small times so that no caustics appear. The Wigner transform and its limiting Liouville equation allow us to avoid the problem of caustics inherent to the geometric optics formulation. The price to pay is that the Wigner transform is defined in the phase space, which is much bigger than the physical domain (since the Hamiltonian is an invariant of Hamilton's equations, the system in the phase space is $2d-1$ dimensional rather than d dimensional in the physical domain).

Chapter 6

Radiative transfer equations

High frequency wave propagation in highly heterogeneous media has long been modeled by radiative transfer equations in many fields: quantum waves in semiconductors, electromagnetic waves in turbulent atmospheres and plasmas, underwater acoustic waves, elastic waves in the Earth's crust. These kinetic models account for the wave energy transport in the phase space, i.e., in the space of positions and momenta.

Such kinetic models account for the multiple interactions of wave fields with the fluctuations of the underlying medium. We saw in Chapter 3 the interaction of high frequency waves with low frequency media and in 2 the interaction of low frequency waves with high frequency media. Radiative transfer equations model the interaction of high frequency waves in high frequency media. The latter description encompasses many regimes of wave propagation. We consider here the so-called **weak coupling** regime, where the correlation length of the underlying medium is comparable to the typical wavelength in the system. In order for energy to propagate, this forces the fluctuations to be of small amplitude, whence the term "weak".

A systematic method to derive kinetic equations from symmetric first-order hyperbolic systems, including systems of acoustics and elastic equations, in the weak-coupling limit has been presented in [28] and extended in various forms in [4, 5, 6, 8]. In these papers, the energy density of waves is captured by the spatial *Wigner transform*, which was introduced in Chapter 5. The method is based on *formal* multiple-scale asymptotic expansions in the Wigner transform and extends to fairly general equations the kinetic models rigorously derived in [14, 29] for the Schrödinger equation. Mathematically rigorous methods of derivation of macroscopic models for wave propagation in heterogeneous media are postponed to later chapters.

We focus here on a non-symmetric two-by-two first-order system and on the scalar wave equation to model acoustic wave propagation.

6.1 Non-symmetric two-by-two system

We recall the system introduced in section 1.2.1 for pressure $p(t,\mathbf{x})$ and the rescaled time derivative of pressure $q(t,\mathbf{x}) = c^{-2}(\mathbf{x})p_t(t,\mathbf{x})$, so that $\mathbf{u} = (p,q)^t$ solves the following 2×2 system

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + A\mathbf{u} &= 0, & t>0,\ \mathbf{x}\in\mathbb{R}^d,\\ \mathbf{u}(0,\mathbf{x}) &= (g(\mathbf{x}), c^{-2}(\mathbf{x})h(\mathbf{x}))^t, & \mathbf{x}\in\mathbb{R}^d. \end{aligned} \tag{6.1}$$

where

$$A = -\begin{pmatrix} 0 & c^2(\mathbf{x}) \\ \Delta & 0 \end{pmatrix} = J\ \Lambda(\mathbf{x}), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda(\mathbf{x}) = \begin{pmatrix} -\Delta & 0 \\ 0 & c^2(\mathbf{x}) \end{pmatrix}. \tag{6.2}$$

Note that J is a skew-symmetric matrix ($J^t = -J$) and that Λ is a symmetric matrix-valued operator for the usual L^2 scalar product. We also recall that energy conservation takes the form

$$\mathcal{E}(t) = \frac{1}{2\rho_0} \int_{\mathbb{R}^d} \mathbf{u}\Lambda\mathbf{u} d\mathbf{x} = \mathcal{E}(0). \tag{6.3}$$

High frequency limit. Kinetic models arise in the high frequency limit of wave propagation. We thus rescale $t \to \varepsilon^{-1}t$ and $\mathbf{x} \to \varepsilon^{-1}\mathbf{x}$ and obtain the following equation for $\mathbf{u}_\varepsilon$:

$$\varepsilon\frac{\partial \mathbf{u}_\varepsilon}{\partial t} + A_\varepsilon \mathbf{u}_\varepsilon = 0, \qquad A_\varepsilon = -\begin{pmatrix} 0 & c_\varepsilon^2(\mathbf{x}) \\ \varepsilon^2\Delta & 0 \end{pmatrix}, \tag{6.4}$$

with initial conditions of the form $\mathbf{u}_\varepsilon(0,\mathbf{x}) = \mathbf{u}_{0\varepsilon}(\varepsilon^{-1}\mathbf{x})$. We verify that acoustic energy conservation implies that

$$\mathcal{E}(t) = \frac{1}{2\rho_0} \int_{\mathbb{R}^d} \big(|\varepsilon\nabla p_\varepsilon|^2(t,\mathbf{x}) + c_\varepsilon^2(\mathbf{x}) q_\varepsilon^2(t,\mathbf{x})\big) d\mathbf{x} = \mathcal{E}(0), \tag{6.5}$$

is independent of time.

The above energy conservation is governed by quantities of the form $|\varepsilon\nabla p_\varepsilon|^2$ and q_ε^2. Whereas such quantities do not solve closed-form equations in the high frequency limit $\varepsilon \to 0$, they can be decomposed in the phase space into a quantity that solves a transport equation. The role of kinetic models is to derive such a transport equation from the wave equations. The Wigner transform is perfectly adapted to such a derivation.

Two by two hyperbolic systems. In the weak coupling regime, the medium is characterized by the sound speed:

$$c_\varepsilon^2(\mathbf{x}) = c_0^2 - \sqrt{\varepsilon} V(\frac{\mathbf{x}}{\varepsilon}), \tag{6.6}$$

where c_0 is the background speed assumed to be constant to simplify and $V(\mathbf{x})$ accounts for the random fluctuations. The correlation length of the random heterogeneities of order ε is here to ensure maximum interaction between the waves and the underlying media. The scaling $\sqrt{\varepsilon}$ is the unique scaling that allows the energy to be significantly modified by the fluctuations while still solving a transport equation. Larger fluctuations lead to other regimes, such as localization, which cannot be accounted for by kinetic models. Since the localization length is always smaller than the diffusive (kinetic) length in spatial dimension $d = 1$, we restrict ourselves to the case $d \geq 2$.

Let us consider the correlation of two fields $\mathbf{u}_\varepsilon^1$ and $\mathbf{u}_\varepsilon^2$ propagating in random media with the same background velocity c_0 but possibly different heterogeneities modeled by V^φ, $\varphi = 1, 2$. We also replace the Laplacian in (6.4) by the more general smooth, real-valued, positive Fourier multiplier operator $p(\varepsilon\mathbf{D})$, which may account for (spatial) dispersive effects. We assume moreover that $p(-i\mathbf{k}) = p(i\mathbf{k})$. We retrieve $p(\varepsilon\mathbf{D}) = \Delta$ for $p(i\boldsymbol{\xi}) = (i\boldsymbol{\xi})\cdot(i\boldsymbol{\xi}) = -|\boldsymbol{\xi}|^2$.

We thus consider the equation

$$\varepsilon\frac{\partial \mathbf{u}_\varepsilon^\varphi}{\partial t} + A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi = 0, \qquad \varphi = 1, 2, \tag{6.7}$$

and assume the following structure for A_ε^φ:

$$A_\varepsilon^\varphi = -\begin{pmatrix} 0 & c_0^2 \\ p(\varepsilon\mathbf{D}) & 0 \end{pmatrix} + \sqrt{\varepsilon} V^\varphi(\frac{\mathbf{x}}{\varepsilon})K, \qquad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{6.8}$$

The correlation of two signals propagating in two different media may be of interest in probing the temporal variations in the statistics of random media and has also found recent applications in the analysis of time reversed waves.

6.2 Structure of the random fluctuations

The random inhomogeneities of the underlying media are modeled by the functions $V^\varphi(\mathbf{x})$. We assume that $V^\varphi(\mathbf{x})$ for $\varphi = 1, 2$ is a statistically homogeneous mean-zero random field. Because higher-order statistical moments of the heterogeneous fluctuations do not appear in kinetic models, all we need to know about the statistics of the random media in the high frequency limit are the two-point correlation functions, or equivalently their Fourier transform the power spectra, defined by

$$c_0^4 R^{\varphi\psi}(\mathbf{x}) = \langle V^\varphi(\mathbf{y}) V^\psi(\mathbf{y}+\mathbf{x})\rangle, \qquad 1 \le \varphi, \psi \le 2, \tag{6.9}$$

$$(2\pi)^d c_0^4 \hat{R}^{\varphi\psi}(\mathbf{p})\delta(\mathbf{p}+\mathbf{q}) = \langle \hat{V}^\varphi(\mathbf{p}) \hat{V}^\psi(\mathbf{q})\rangle. \tag{6.10}$$

Here $\langle\cdot\rangle$ means ensemble average (mathematical expectation). We verify that $\hat{R}^{\varphi\psi}(-\mathbf{p}) = \hat{R}^{\varphi\psi}(\mathbf{p})$.

We can also consider more general random fluctuations of the form $V^\varphi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$, where for each $\mathbf{x} \in \mathbb{R}^d$, $V^\varphi(\mathbf{x}, \mathbf{y})$ is a statistically homogeneous mean-zero random field.

6.3 Equation for the Wigner transform

We define the Wigner transform of the two fields as

$$W_\varepsilon(t, \mathbf{x}, \mathbf{k}) = W[\mathbf{u}_\varepsilon^1(t, \cdot), \mathbf{u}_\varepsilon^2(t, \cdot)](\mathbf{x}, \mathbf{k}), \tag{6.11}$$

and deduce from (5.24) and (6.11) that

$$\varepsilon \frac{\partial W_\varepsilon}{\partial t} + W[A_\varepsilon^1 \mathbf{u}_\varepsilon^1, \mathbf{u}_\varepsilon^2] + W[\mathbf{u}_\varepsilon^1, A_\varepsilon^2 \mathbf{u}_\varepsilon^2] = 0. \tag{6.12}$$

The pseudo-differential calculus recalled in Chapter 5 allows us to obtain the following equation for the Wigner transform:

$$\varepsilon \frac{\partial W_\varepsilon}{\partial t} + P(i\mathbf{k} + \frac{\varepsilon\mathbf{D}}{2})W_\varepsilon + W_\varepsilon P^*(i\mathbf{k} - \frac{\varepsilon\mathbf{D}}{2}) + \sqrt{\varepsilon}\big(\mathcal{K}_\varepsilon^1 K W_\varepsilon + \mathcal{K}_\varepsilon^{2*} W_\varepsilon K^*\big) = 0, \tag{6.13}$$

$$P(i\mathbf{k} + \frac{\varepsilon\mathbf{D}}{2}) = -\begin{pmatrix} 0 & c_0^2 \\ p(i\mathbf{k} + \frac{\varepsilon\mathbf{D}}{2}) & 0 \end{pmatrix}, \qquad \mathcal{K}_\varepsilon^\varphi W = \int_{\mathbb{R}^d} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}} \hat{V}^\varphi(\mathbf{p}) W(\mathbf{k} - \frac{\mathbf{p}}{2}) \frac{d\mathbf{p}}{(2\pi)^d}. \tag{6.14}$$

Note that (6.13) is an exact evolution equation for the two-by-two Wigner transform $W_\varepsilon(t, \mathbf{x}, \mathbf{k})$. Its initial conditions are obtained by evaluating (6.11) at $t = 0$, and thus depend on the initial conditions for $\mathbf{u}_\varepsilon^1$ and $\mathbf{u}_\varepsilon^2$.

6.4 Multiple scale expansion

We are now interested in the high-frequency limit as $\varepsilon \to 0$ of W_ε. Because of the presence of a highly-oscillatory phase $\exp(\varepsilon^{-1} i\mathbf{x}\cdot\mathbf{k})$ in the operator $\mathcal{K}_\varepsilon^\varphi$, direct asymptotic expansions on W_ε and (6.13) cannot provide the correct limit. Rather, as is classical in the homogenization of equations in highly oscillatory media, as we have seen in Chapter 2, we introduce the following two-scale version of W_ε:

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = W_\varepsilon(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}), \tag{6.15}$$

and still use the symbol W_ε for the function on $\mathbb{R}^{3d+1}$ in the new variables $(t,\mathbf{x},\mathbf{y},\mathbf{k})$. We then find that the differential operator $\mathbf{D}$ acting on the spatial variables should be replaced by $\mathbf{D_x} + \frac{1}{\varepsilon}\mathbf{D_y}$. The equation for W_ε thus becomes

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + P(i\mathbf{k}+\frac{\mathbf{D_y}}{2}+\frac{\varepsilon\mathbf{D_x}}{2})W_\varepsilon + W_\varepsilon P^*(i\mathbf{k}-\frac{\mathbf{D_y}}{2}-\frac{\varepsilon\mathbf{D_x}}{2}) + \sqrt{\varepsilon}\big(\mathcal{K}^1 K W_\varepsilon + \mathcal{K}^{2*} W_\varepsilon K^*\big) = 0, \tag{6.16}$$

where we have defined

$$\mathcal{K}^\varphi W = \int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{p}}\hat{V}^\varphi(\mathbf{p}) W(\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}. \tag{6.17}$$

Asymptotic expansions in the new set of variables can now account for the fast oscillations of the heterogeneous medium. Using the asymptotic expansion $P = P_0 + \varepsilon P_1 + O(\varepsilon^2)$ in (6.14) and

$$W_\varepsilon(t,\mathbf{x},\mathbf{y},\mathbf{k}) = W_0(t,\mathbf{x},\mathbf{k}) + \sqrt{\varepsilon}W_1(t,\mathbf{x},\mathbf{y},\mathbf{k}) + \varepsilon W_2(t,\mathbf{x},\mathbf{y},\mathbf{k}), \tag{6.18}$$

we equate like powers of ε in (6.16) to obtain a sequence of three equations.

6.5 Leading-order equation and dispersion relation

The leading equation in the above expansion yields

$$\mathcal{L}_0 W_0 \equiv P_0(i\mathbf{k})W_0 + W_0 P_0^*(i\mathbf{k}) = 0; \quad P_0 = -J\Lambda_0,\ \Lambda_0 = \begin{pmatrix} -p(i\mathbf{k}) & 0 \\ 0 & c_0^2 \end{pmatrix},\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.19}$$

Let us define $q_0(i\mathbf{k}) = \sqrt{-p(i\mathbf{k})}$. The diagonalization of the dispersion matrix P_0 yields

$$\lambda_\pm(\mathbf{k}) = \pm i c_0 q_0(i\mathbf{k}),\ \mathbf{b}_\pm(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm i q_0^{-1}(i\mathbf{k}) \\ c_0^{-1} \end{pmatrix},\ \mathbf{c}_\pm(\mathbf{k}) = \Lambda_0\mathbf{b}_\pm(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm i q_0(i\mathbf{k}) \\ c_0 \end{pmatrix}. \tag{6.20}$$

The vectors are normalized such that $\mathbf{b}_\pm^*\Lambda_0\mathbf{b}_\pm = \mathbf{b}_\pm^*\mathbf{c}_\pm = 1$ and we verify the spectral decomposition $P_0 = \lambda_+\mathbf{b}_+\mathbf{c}_+^* + \lambda_-\mathbf{b}_-\mathbf{c}_-^*$. Since $(\mathbf{b}_+(\mathbf{k}),\mathbf{b}_-(\mathbf{k}))$ forms a basis of $\mathbb{R}^2$ for any $\mathbf{k}\in\mathbb{R}^d_*$, any matrix W may thus be decomposed as $W = \sum_{i,j=\pm}\alpha_{ij}\mathbf{b}_i\mathbf{b}_j^*$ where $\alpha_{ij} = \mathbf{c}_i^* W\mathbf{c}_j = \mathrm{tr}(W\mathbf{c}_i\mathbf{c}_j^*)$, and a straightforward calculation shows that $\mathbf{c}_k^*(P_0W + WP_0^*)\mathbf{c}_m = \alpha_{km}(\lambda_k + \overline{\lambda_m})$. Using the above decomposition for the matrix $W_0 = \sum_{i,j=\pm} a_{ij}\mathbf{b}_i\mathbf{b}_j^*$, equation (6.19) implies that $a_{+-} = a_{-+} = 0$ so that

$$W_0 = a_+\mathbf{b}_+\mathbf{b}_+^* + a_-\mathbf{b}_-\mathbf{b}_-^*; \qquad a_\pm = \mathbf{c}_\pm^* W_0\mathbf{c}_\pm. \tag{6.21}$$

Because all the components of $\mathbf{u}_\varepsilon^\varphi$ are real-valued, we verify that $\bar{W}(-\mathbf{k}) = W(\mathbf{k})$. Here $\bar{}$ means complex conjugation component by component. From the above expression for $a_\pm$ and the fact that $\mathbf{c}(-\mathbf{k}) = \mathbf{c}(\mathbf{k})$, we deduce that

$$\bar{a}_\pm(-\mathbf{k}) = a_\mp(\mathbf{k}). \tag{6.22}$$

It is thus sufficient to find an equation for $a_+(\mathbf{k})$. We verify that $\int_{\mathbb{R}^d} a_+ d\mathbf{k} = \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{tr}(\Lambda_0 W_0) d\mathbf{k}$, so that in the case where (5.24) is (6.4) and W_ε is the Wigner transform of $\mathbf{u}_\varepsilon$, we have

$$\frac{1}{2}\int_{\mathbb{R}^{2d}} \mathrm{tr}(\Lambda_0 W_0) d\mathbf{k} d\mathbf{x} = \mathcal{E}(t) = \int_{\mathbb{R}^{2d}} a_+(t,\mathbf{x},\mathbf{k}) d\mathbf{k} d\mathbf{x}, \tag{6.23}$$

at least in the limit $\varepsilon \to 0$, where $\mathcal{E}$ is defined in (6.5). Thus a_+ can be given the interpretation of an energy density in the phase-space.

6.6 First-order corrector

The next equation in the asymptotic expansion in powers of ε is

$$P_0(i\mathbf{k} + \frac{\mathbf{D_y}}{2})W_1 + W_1 P_0^*(i\mathbf{k} - \frac{\mathbf{D_y}}{2}) + \theta W_1 + \mathcal{K}^1 K W_0 + \mathcal{K}^{2*} W_0 K^* = 0. \tag{6.24}$$

The parameter $0 < \theta \ll 1$ is a regularization (limiting absorption) parameter that will be sent to 0 at the end. It is required to ensure the causality of wave propagation [28]. We denote by $\hat{W}_1(t,\mathbf{x},\mathbf{p},\mathbf{k})$ the Fourier transform $\mathbf{y} \to \mathbf{p}$ of W_1,

$$\hat{W}_1(t,\mathbf{x},\mathbf{p},\mathbf{k}) = \mathcal{F}_{\mathbf{y}\to\mathbf{p}}[W_1(t,\mathbf{x},\mathbf{y},\mathbf{k})](t,\mathbf{x},\mathbf{p},\mathbf{k}). \tag{6.25}$$

Since W_0 is independent of $\mathbf{y}$ so that its Fourier transform $\mathbf{y} \to \mathbf{p}$: $\hat{W}_0 = (2\pi)^d \delta(\mathbf{p}) W_0$, we verify that $\hat{W}_1$ satisfies the equation

$$P_0(i\mathbf{k}+i\frac{\mathbf{p}}{2})\hat{W}_1 + \hat{W}_1 P_0^*(i\mathbf{k}-i\frac{\mathbf{p}}{2}) + \theta\hat{W}_1 + \hat{V}^1(\mathbf{p}) K W_0(\mathbf{k}-\frac{\mathbf{p}}{2}) + \hat{V}^2(\mathbf{p}) W_0(\mathbf{k}+\frac{\mathbf{p}}{2}) K^* = 0. \tag{6.26}$$

Since the vectors $\mathbf{b}_i(\mathbf{k})$ form a complete basis of $\mathbb{R}^2$ for all $\mathbf{k}$, we can decompose $\hat{W}_1$ as

$$\hat{W}_1(\mathbf{p},\mathbf{k}) = \sum_{i,j=\pm} \alpha_{ij}(\mathbf{p},\mathbf{k}) \mathbf{b}_i(\mathbf{k}+\frac{\mathbf{p}}{2}) \mathbf{b}_j^*(\mathbf{k}-\frac{\mathbf{p}}{2}). \tag{6.27}$$

Multiplying (6.26) by $\mathbf{c}_m^*(\mathbf{k}+\frac{\mathbf{p}}{2})$ on the left and by $\mathbf{c}_n(\mathbf{k}-\frac{\mathbf{p}}{2})$ on the right, recalling $\lambda_n^* = -\lambda_n$, and calculating

$$\mathbf{b}_n^*(\mathbf{p}) K^* \mathbf{c}_m(\mathbf{q}) = \frac{1}{2c_0^2}\lambda_m(\mathbf{q}), \quad \mathbf{c}_m^*(\mathbf{p}) K \mathbf{b}_n(\mathbf{q}) = \frac{-1}{2c_0^2}\lambda_m(\mathbf{p}), \tag{6.28}$$

we get

$$\alpha_{mn}(\mathbf{p},\mathbf{k}) = \frac{1}{2c_0^2} \frac{\hat{V}^1(\mathbf{p})\lambda_m(\mathbf{k}+\frac{\mathbf{p}}{2}) a_n(\mathbf{k}-\frac{\mathbf{p}}{2}) - \hat{V}^2(\mathbf{p})\lambda_n(\mathbf{k}-\frac{\mathbf{p}}{2}) a_m(\mathbf{k}+\frac{\mathbf{p}}{2})}{\lambda_m(\mathbf{k}+\frac{\mathbf{p}}{2}) - \lambda_n(\mathbf{k}-\frac{\mathbf{p}}{2}) + \theta}. \tag{6.29}$$

Note that W_1 is linear in the random fields V^φ. As in earlier multiple scale expansions, we obtain W_1 as a function of W_0. However this still does not provide us with any equation for the leading-order term W_0.

6.7 Transport equation

Finally the third equation in the expansion in powers of ε yields

$$\begin{aligned} &P_0(i\mathbf{k}+\frac{\mathbf{D_y}}{2})W_2+W_2P_0^*(i\mathbf{k}-\frac{\mathbf{D_y}}{2})+\mathcal{K}_1KW_1+\mathcal{K}_2^*W_1K^* \\ &+\frac{\partial W_0}{\partial t}+P_1(i\mathbf{k})W_0+W_0P_1^*(i\mathbf{k})=0. \end{aligned} \tag{6.30}$$

We consider ensemble averages in the above equation and thus look for an equation for $\langle a_+\rangle$, which we still denote by a_+. We may assume that W_2 is orthogonal to W_0 in order to justify the expansion in ε, so that $\langle \mathbf{c}_+^*\mathcal{L}_0W_2\mathbf{c}_+\rangle=0$. This part *cannot* be justified rigorously and may be seen as a reasonable **closure argument**. Such a closure is known to provide the correct limit as $\varepsilon\to 0$ in cases that can be analyzes rigorously [14].

We multiply the above equation on the left by $\mathbf{c}_+^*(\mathbf{k})$ and on the right by $\mathbf{c}_+(\mathbf{k})$. Recalling the convention in (5.36), we obtain that since $p(i\mathbf{k})=-q_0^2(i\mathbf{k})$, we have $p'(i\mathbf{k})=-i\nabla_{\mathbf{k}}p(i\mathbf{k})=2iq_0(i\mathbf{k})\nabla_{\mathbf{k}}q_0(i\mathbf{k})$. This implies that

$$\mathbf{c}_+^*P_1W_0\mathbf{c}_+=\mathbf{c}_+^*W_0P_1^*\mathbf{c}_+=\frac{c_0}{2}\nabla_{\mathbf{k}}q_0(i\mathbf{k})\cdot\nabla_{\mathbf{x}}a_+(\mathbf{x},\mathbf{k}).$$

Let us define

$$\omega_+(\mathbf{k})=c_0q_0(i\mathbf{k})=-i\lambda_+(i\mathbf{k}). \tag{6.31}$$

Our convention for the frequencies in the acoustic case are then $\omega_\pm\pm c_0|\mathbf{k}|=0$. We thus find that

$$\mathbf{c}_+^*\mathcal{L}_2W_0\mathbf{c}_+=\{\omega_+,a_+\}(\mathbf{x},\mathbf{k}),$$

where the Poisson bracket is defined in (5.39). When $p(i\boldsymbol{\xi})=(i\boldsymbol{\xi})^2$, we obtain that $c_0\nabla_{\mathbf{k}}q_0(i\mathbf{k})=c_0\hat{\mathbf{k}}$. Upon taking ensemble averages and still denoting by a_+ the ensemble average $\langle a_+\rangle$, we get the equation

$$\frac{\partial a_+}{\partial t}+\{\omega_+,a_+\}(\mathbf{x},\mathbf{k})+\langle\mathbf{c}_+^*\mathcal{L}_1W_1\mathbf{c}_+\rangle=0.$$

Here we have defined

$$\mathcal{L}_1W=\mathcal{K}^1KW+\mathcal{K}^{2*}WK^*. \tag{6.32}$$

In the absence of scattering ($\mathcal{L}_1\equiv 0$), we thus observe that the phase-space energy density $a(t,\mathbf{x},\mathbf{k})$ solves the **Liouville** equation, which was introduced in (5.62).

Let us define $\hat{W}_1(\mathbf{p},\mathbf{k})=\hat{V}^1(\mathbf{p})W_1^1(\mathbf{p},\mathbf{k})+\hat{V}^2(\mathbf{p})W_1^2(\mathbf{p},\mathbf{k})$ with obvious notation. Using the symmetry $\hat{R}^{ij}(-\mathbf{p})=\hat{R}^{ij}(\mathbf{p})$, we deduce that

$$\begin{aligned} \widehat{\langle\mathcal{L}_1W_1(\mathbf{p},\mathbf{k})\rangle}= \quad &\delta(\mathbf{p})c_0^4\int_{\mathbb{R}^d}\big(\hat{R}^{11}(\mathbf{k}-\mathbf{q})KW_1^1(\mathbf{q}-\mathbf{k},\frac{\mathbf{k}+\mathbf{q}}{2})+\hat{R}^{12}(\mathbf{k}-\mathbf{q})KW_1^2(\mathbf{q}-\mathbf{k},\frac{\mathbf{k}+\mathbf{q}}{2}) \\ &+\hat{R}^{21}(\mathbf{k}-\mathbf{q})W_1^1(\mathbf{k}-\mathbf{q},\frac{\mathbf{k}+\mathbf{q}}{2})K^*+\hat{R}^{22}(\mathbf{k}-\mathbf{q})W_1^2(\mathbf{k}-\mathbf{q},\frac{\mathbf{k}+\mathbf{q}}{2})K^*\big)d\mathbf{q}. \end{aligned}$$

Using the convention of summation over repeated indices, we obtain after some algebra that

$$\begin{aligned} \langle\mathbf{c}_+^*(\mathbf{k})\mathcal{L}_1W_1(\mathbf{k})\mathbf{c}_+(\mathbf{k})\rangle=\frac{\lambda_+(\mathbf{k})}{4(2\pi)^d}\int_{\mathbb{R}^d}\big(&\frac{-\hat{R}^{11}(\mathbf{k}-\mathbf{q})\lambda_i(\mathbf{q})a_+(\mathbf{k})}{\lambda_i(\mathbf{q})-\lambda_+(\mathbf{k})+\theta}+\frac{\hat{R}^{12}(\mathbf{k}-\mathbf{q})\lambda_+(\mathbf{k})a_i(\mathbf{q})}{\lambda_i(\mathbf{q})-\lambda_+(\mathbf{k})+\theta} \\ &+\frac{\hat{R}^{12}(\mathbf{k}-\mathbf{q})\lambda_+(\mathbf{k})a_j(\mathbf{q})}{\lambda_+(\mathbf{k})-\lambda_j(\mathbf{q})+\theta}+\frac{-\hat{R}^{22}(\mathbf{k}-\mathbf{q})\lambda_j(\mathbf{q})a_+(\mathbf{k})}{\lambda_+(\mathbf{k})-\lambda_j(\mathbf{q})+\theta}\big)d\mathbf{q}. \end{aligned} \tag{6.33}$$

Since $\lambda_j(\mathbf{k})$ is purely imaginary, we deduce from the relation

$$\frac{1}{ix+\varepsilon}\to\frac{1}{ix}+\pi\mathrm{sign}(\varepsilon)\delta(x),\ \text{as}\ \varepsilon\to 0,$$

which holds in the sense of distributions, that

$$\lim_{0<\theta\to 0}\big(\frac{1}{\lambda_j(\mathbf{q})-\lambda_+(\mathbf{k})+\theta}+\frac{1}{\lambda_+(\mathbf{q})-\lambda_j(\mathbf{k})+\theta}\big)=2\pi\delta(i\lambda_j(\mathbf{q})-i\lambda_+(\mathbf{k})).$$

This implies that $j=+$ in order for the delta function not to be restricted to the point $\mathbf{k}=0$ (we assume $\lambda_+(\mathbf{k})=0$ implies $\mathbf{k}=0$). So using (6.31), we obtain that

$$\langle \mathbf{c}_+^*(\mathbf{k})\mathcal{L}_1 W_1(\mathbf{k})\mathbf{c}_+(\mathbf{k})\rangle=(\Sigma(\mathbf{k})+i\Pi(\mathbf{k}))a_+(\mathbf{k})-\int_{\mathbb{R}^d}\sigma(\mathbf{k},\mathbf{q})a_+(\mathbf{q})\delta(\omega_+(\mathbf{q})-\omega_+(\mathbf{k}))d\mathbf{q},$$

where we have defined the scattering coefficients:

$$\begin{aligned}
\Sigma(\mathbf{k}) &= \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{R}^{11}+\hat{R}^{22}}{2}(\mathbf{k}-\mathbf{q})\delta(\omega_+(\mathbf{q})-\omega_+(\mathbf{k}))d\mathbf{q},\\
i\Pi(\mathbf{k}) &= \frac{1}{4(2\pi)^d}\ \text{p.v.}\int_{\mathbb{R}^d}(\hat{R}^{11}-\hat{R}^{22})(\mathbf{k}-\mathbf{q})\sum_{i=\pm}\frac{\lambda_+(\mathbf{k})\lambda_i(\mathbf{q})}{\lambda_+(\mathbf{k})-\lambda_i(\mathbf{q})}d\mathbf{q},\\
\sigma(\mathbf{k},\mathbf{q}) &= \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\hat{R}^{12}(\mathbf{k}-\mathbf{q}).
\end{aligned}\tag{6.34}$$

The radiative transfer equation for a_+ is thus

$$\frac{\partial a_+}{\partial t}+\{\omega_+,a_+\}(\mathbf{x},\mathbf{k})+(\Sigma(\mathbf{k})+i\Pi(\mathbf{k}))a_+=\int_{\mathbb{R}^d}\sigma(\mathbf{k},\mathbf{q})a_+(\mathbf{q})\delta(\omega_+(\mathbf{q})-\omega_+(\mathbf{k}))d\mathbf{q}.\tag{6.35}$$

In the case where the two media are identical and $p(i\mathbf{k})=-|\mathbf{k}|^2$ so that $q_0(i\mathbf{k})=|\mathbf{k}|$ and $\omega_+(\mathbf{k})=c_0|\mathbf{k}|$, we retrieve the classical radiative transfer equation for acoustic wave propagation [28], whereas (6.35) generalizes the kinetic model obtained in [9].

The radiative transfer equation for the energy density of acoustic waves thus takes the form

$$\frac{\partial a_+}{\partial t}+c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}a_+(\mathbf{x},\mathbf{k})=\int_{\mathbb{R}^d}\sigma(\mathbf{k},\mathbf{q})(a_+(\mathbf{q})-a_+(\mathbf{k}))\delta(c_0|\mathbf{q}|-c_0|\mathbf{k}|)d\mathbf{q},\tag{6.36}$$

where $\hat{R}(\mathbf{p})=\hat{R}^{11}(\mathbf{p})=\hat{R}^{12}(\mathbf{p})=\hat{R}^{22}(\mathbf{p})$ and

$$\sigma(\mathbf{k},\mathbf{q})=\frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\hat{R}(\mathbf{k}-\mathbf{q}).\tag{6.37}$$

The latter form shows one of the main properties of the radiative transfer equation, namely that the scattering operator is conservative (its integral over wavenumbers vanishes) and that it is elastic, i.e., the wavenumber $|\mathbf{k}|$ of waves after scattering equals that $|\mathbf{p}|$ before scattering. Moreover defining the total scattering cross section as

$$\Sigma(|\mathbf{k}|)=\int_{\mathbb{R}^d}\sigma(\mathbf{k},\mathbf{q})\delta(\omega_+(\mathbf{q})-\omega_+(\mathbf{k}))d\mathbf{q},$$

we can interpret $\sigma(\mathbf{k},\mathbf{p})/\Sigma(|\mathbf{k}|)$ as the probability of scattering from wavenumber $\mathbf{p}$ into wavenumber $\mathbf{k}$ such that $\omega=c_0|\mathbf{k}|=c_0|\mathbf{p}|$ is preserved.

Chapter 7

Parabolic regime

7.1 Derivation of the parabolic wave equation

Let us consider the scalar wave equation for the pressure field $p(z, \mathbf{x}, t)$:

$$\frac{1}{c^2(z,\mathbf{x})}\frac{\partial^2 p}{\partial t^2} - \Delta p = 0. \tag{7.1}$$

Here $c(z, \mathbf{x})$ is the local wave speed that we will assume to be random, and the Laplacian operator includes both direction of propagation, z, and the transverse variable $\mathbf{x} \in \mathbb{R}^d$. In the physical setting, we have $d = 2$. We consider dimensions $d \geq 1$ to stress that the analysis of the problem is independent of the number of transverse dimensions. If we assume that at time $t = 0$, the wave field has a "beam-like" structure in the z direction, and if back-scattering may be neglected, we can replace the wave equation by its parabolic (also known as paraxial) approximation [30]. More precisely, the pressure p may be approximated as

$$p(z, \mathbf{x}, t) \approx \int_{\mathbb{R}} e^{i(-c_0\kappa t + \kappa z)}\psi(z, \mathbf{x}, \kappa)c_0 d\kappa, \tag{7.2}$$

where ψ satisfies the Schrödinger equation

$$\begin{aligned} &2i\kappa\frac{\partial\psi}{\partial z}(z, \mathbf{x}, \kappa) + \Delta_{\mathbf{x}}\psi(z, \mathbf{x}, \kappa) + \kappa^2(n^2(z, \mathbf{x}) - 1)\psi(z, \mathbf{x}, \kappa) = 0, \\ &\psi(z = 0, \mathbf{x}, \kappa) = \psi_0(\mathbf{x}, \kappa), \end{aligned} \tag{7.3}$$

with $\Delta_{\mathbf{x}}$ the transverse Laplacian in the variable $\mathbf{x}$. The index of refraction $n(z, \mathbf{x}) = c_0/c(z, \mathbf{x})$, and c_0 in (7.2) is a reference speed.

A formal justification for the above approximation goes as follows. We start with the reduced wave equation

$$\Delta\hat{p} + \kappa^2 n^2(z, \mathbf{x})\hat{p} = 0, \tag{7.4}$$

and look for solutions of (7.4) in the form $\hat{p}(z, \mathbf{x}) = e^{i\kappa z}\psi(z, \mathbf{x})$. We obtain that

$$\frac{\partial^2\psi}{\partial z^2} + 2i\kappa\frac{\partial\psi}{\partial z} + \Delta_{\mathbf{x}}\psi + \kappa^2(n^2 - 1)\psi = 0. \tag{7.5}$$

The index of refraction $n(z, \mathbf{x})$ is fluctuating in both the axial z and transversal $\mathbf{x}$ variables and thus has the form

$$n^2(z, \mathbf{x}) = 1 - 2\sigma V\left(\frac{z}{l_z}, \frac{\mathbf{x}}{l_x}\right),$$

where V is a mean-zero random field, and where l_x and l_z are the correlation lengths of V in the transverse and longitudinal directions, respectively. The small parameter σ measures the strength of the fluctuations.

We now introduce two macroscopic distances of wave propagation: L_x in the $\mathbf{x}$-plane and L_z in the z-direction. We also introduce a carrier wave number κ_0 and replace $\kappa \to \kappa_0\kappa$, κ now being a non-dimensional wavenumber. The physical parameters determined by the medium are the length scales l_x, l_z and the non-dimensional parameter $\sigma \ll 1$.

We present the relationship between the various scalings introduced above that need be satisfied so that wave propagation occurs in a regime close to that of radiative transfer. Equation (7.5) in the non-dimensional variables $z \to z/L_z$, $\mathbf{x} \to \mathbf{x}/L_x$ becomes

$$\frac{1}{L_z^2}\frac{\partial^2\psi}{\partial z^2} + \frac{2i\kappa\kappa_0}{L_z}\frac{\partial\psi}{\partial z} + \frac{1}{L_x^2}\Delta_{\mathbf{x}}\psi - 2\kappa^2\kappa_0^2\sigma V\left(\frac{zL_z}{l_z}, \frac{\mathbf{x}L_x}{l_x}\right)\psi = 0. \tag{7.6}$$

Let us introduce the following parameters

$$\delta_x = \frac{l_x}{L_x}, \quad \delta_z = \frac{l_z}{L_z}, \quad \gamma_x = \frac{1}{\kappa_0 l_x}, \quad \gamma_z = \frac{1}{\kappa_0 l_z}, \tag{7.7}$$

and recast (7.6) as

$$\gamma_z\delta_z\frac{\partial^2\psi}{\partial z^2} + 2i\kappa\frac{\partial\psi}{\partial z} + \frac{\delta_x^2\gamma_x^2}{\delta_z\gamma_z}\Delta_{\mathbf{x}}\psi - \frac{2\kappa^2\sigma}{\gamma_z\delta_z}V\left(\frac{z}{\delta_z}, \frac{\mathbf{x}}{\delta_x}\right)\psi = 0. \tag{7.8}$$

Let us now assume the following relationships among the various parameters

$$\delta_x = \delta_z \ll 1, \quad \gamma_z = \gamma_x^2 \ll 1, \quad \sigma = \gamma_z\sqrt{\delta_x}, \quad \varepsilon = \delta_x. \tag{7.9}$$

Then (7.8), after multiplication by $\varepsilon/2$, becomes

$$\frac{\gamma_z\varepsilon^2}{2}\frac{\partial^2\psi}{\partial z^2} + i\kappa\varepsilon\frac{\partial\psi}{\partial z} + \frac{\varepsilon^2}{2}\Delta_{\mathbf{x}}\psi - \kappa^2\sqrt{\varepsilon}V\left(\frac{z}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}\right)\psi = 0. \tag{7.10}$$

We now observe that, when $\kappa = O(1)$ and $\gamma_z \ll 1$, the first term in (7.10) is small and may be neglected in the leading order since $|\varepsilon^2\psi_{zz}| = O(1)$. Then (7.10) becomes

$$i\kappa\varepsilon\frac{\partial\psi}{\partial z} + \frac{\varepsilon^2}{2}\Delta_{\mathbf{x}}\psi - \kappa^2\sqrt{\varepsilon}V\left(\frac{z}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}\right)\psi = 0 \tag{7.11}$$

which is the parabolic wave equation (7.3) in the radiative transfer scaling. The rigorous passage to the parabolic approximation in a three-dimensional layered random medium in a similar scaling is discussed in [1].

Exercise 7.1.1 (i) Show that the above choices imply that

$$l_x \ll l_z.$$

Therefore the correlation length in the longitudinal direction z should be much longer than in the transverse plane $\mathbf{x}$.
(ii) Show that

$$L_x = l_x\frac{l_x^4}{\sigma^2 l_z^4}, \qquad L_z = l_z\frac{l_x^4}{\sigma^2 l_z^4}, \qquad L_x \ll L_z.$$

The latter is the usual constraint for the validity of the parabolic approximation (beam-like structure of the wave).

In the above scalings, there remains one free parameter, namely $\gamma_z = l_z^2/l_x^2$, as one can verify, or equivalently

$$\frac{L_x}{L_z} = \frac{l_x}{l_z} \equiv \varepsilon^\eta, \qquad \eta > 0, \tag{7.12}$$

where $\eta > 0$ is necessary since $L_x \ll L_z$. Note that as $\eta \to 0$, we recover an isotropic random medium (with $l_z \equiv l_x$) and the usual regime of radiative transfer derived in the preceding chapter. The parabolic (or paraxial) regime thus shares some of the features of the radiative transfer regime, and because the fluctuations depend on the variable z, which plays a similar role to the time variable in the radiative transfer theory, the mathematical analysis is much simplified.

7.2 Wigner Transform and mixture of states

We want to analyze the energy density of the solution to the paraxial wave equation in the limit $\varepsilon \to 0$. As in the preceding chapter, the Wigner transform is a useful tool. Let us recast the above paraxial wave equation as the following Cauchy problem

$$\begin{aligned} &i\varepsilon\kappa\frac{\partial\psi_\varepsilon}{\partial z} + \frac{\varepsilon^2}{2}\Delta\psi_\varepsilon - \kappa^2\sqrt{\varepsilon}V\left(\frac{z}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}\right)\psi_\varepsilon = 0 \\ &\psi_\varepsilon(0,\mathbf{x}) = \psi_\varepsilon^0(\mathbf{x};\zeta). \end{aligned} \tag{7.13}$$

Here, the initial data depend on an additional random variable ζ defined over a state space S with a probability measure $d\varpi(\zeta)$. Its use will become clear very soon.

Let us define the Wigner transform as the usual Wigner transform of the field ψ_ε averaged over the space $(S, d\varpi(\zeta))$:

$$W_\varepsilon(z,\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d\times S} e^{i\mathbf{k}\cdot\mathbf{y}}\psi_\varepsilon\left(z,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2};\zeta\right)\bar{\psi}_\varepsilon\left(z,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2};\zeta\right)\frac{d\mathbf{y}}{(2\pi)^d}d\varpi(\zeta). \tag{7.14}$$

We *assume* that the initial data $W_\varepsilon(0,\mathbf{x},\mathbf{k})$ converges strongly in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ to a limit $W_0(\mathbf{x},\mathbf{k})$. This is possible thanks to the introduction of a mixture of states, i.e., an integration against the measure $\varpi(d\xi)$. This is the main reason why the space $(S, d\varpi(\zeta))$ is introduced.

Note that the Wigner transform of a *pure* state (e.g. when $\varpi(d\xi)$ concentrates at one point in S) is not bounded in $L^2(\mathbb{R}^{2d})$ uniformly in ε.

Exercise 7.2.1 Show that with the definition (5.1) and u and v scalar functions, we have:

$$\int_{\mathbb{R}^{2d}} |W_\varepsilon[u,v]|^2(\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k} = \frac{1}{(2\pi\varepsilon)^d}\|u\|^2_{L^2(\mathbb{R}^d)}\|v\|^2_{L^2(\mathbb{R}^d)}.$$

We thus need to *regularize* the Wigner transform if we want a uniform bound in a smaller space than $\mathcal{A}'$. We assume the existence of $(S, d\varpi(\zeta))$ such that $W_\varepsilon(0,\mathbf{x},\mathbf{k})$ above converges strongly in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ to a limit $W_0(\mathbf{x},\mathbf{k})$. We will come back to the effect of not regularizing the Wigner transform at the end of the chapter.

Using the calculus recalled in chapter 5, we verify that the Wigner transform satisfies the following evolution equation

$$\frac{\partial W_\varepsilon}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}}W_\varepsilon = \frac{\kappa}{i\sqrt{\varepsilon}}\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon}\Big(W_\varepsilon(\mathbf{k}-\frac{\mathbf{p}}{2}) - W_\varepsilon(\mathbf{k}+\frac{\mathbf{p}}{2})\Big)\frac{d\tilde{V}\big(\frac{z}{\varepsilon},\mathbf{p}\big)}{(2\pi)^d}. \tag{7.15}$$

Here, $\tilde{V}(z,\mathbf{p})$ is the partial Fourier transform of $V(z,\mathbf{x})$ in the variable $\mathbf{x}$. The above evolution equation *preserves* the $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ norm of $W_\varepsilon(t,\cdot,\cdot)$:

Lemma 7.2.1 *Let $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ be the solution of (7.15) with initial conditions $W_\varepsilon(0,\mathbf{x},\mathbf{k})$. Then we have*

$$\|W_\varepsilon(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} = \|W_\varepsilon(0,\cdot,\cdot)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}, \qquad \text{for all } t>0. \tag{7.16}$$

Proof. This can be obtained by integrations by parts in (7.15), in a way that is similar to showing that (7.13) preserves the L^2 norm. This can also be obtained from the definition of the Wigner transform and from Exercise 7.2.1. □

7.3 Hypotheses on the randomness

We describe here the construction of the random potential $V(z,\mathbf{x})$. Our main hypothesis is to assume that $V(z,\mathbf{x})$ is a Markov process in the z variable. This gives us access to a whole machinery relatively similar to the one used in the diffusion Markov approximation in Chapter 4 and Appendix A. The Markovian hypothesis is crucial to simplify the mathematical analysis because it allows us to treat the process $z\mapsto(V(z/\varepsilon,\mathbf{x}/\varepsilon),W_\varepsilon(z,\mathbf{x},\mathbf{k}))$ as jointly Markov.

In addition to being Markovian, $V(z,\mathbf{x})$ is assumed to be stationary in $\mathbf{x}$ and z, mean zero, and is constructed in the Fourier space as follows. Let $\mathcal{V}$ be the set of measures of bounded total variation with support inside a ball $B_L=\{|\mathbf{p}|\le L\}$

$$\mathcal{V}=\left\{\hat{V}:\ \int_{\mathbb{R}^d}|d\hat{V}|\le C,\ \text{supp }\hat{V}\subset B_L,\ \hat{V}(\mathbf{p})=\hat{V}^*(-\mathbf{p})\right\}, \tag{7.17}$$

and let $\tilde{V}(z)$ be a mean-zero Markov process on $\mathcal{V}$ with infinitesimal generator Q. The random potential $V(z,\mathbf{x})$ is given by

$$V(z,\mathbf{x})=\int_{\mathbb{R}^d}\frac{d\tilde{V}(z,\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{7.18}$$

It is real-valued and uniformly bounded; $|V(z,\mathbf{x})|\le C$. The correlation function $R(z,\mathbf{x})$ of $V(z,\mathbf{x})$ is

$$R(z,\mathbf{x})=\mathbb{E}\left\{V(s,\mathbf{y})V(z+s,\mathbf{x}+\mathbf{y})\right\} \qquad \text{for all } \mathbf{x},\mathbf{y}\in\mathbb{R}^d, \text{ and } z,s\in\mathbb{R}. \tag{7.19}$$

In the Fourier domain, this is equivalent to the following statement:

$$\mathbb{E}\left\{\langle\tilde{V}(s),\hat{\phi}\rangle\langle\tilde{V}(z+s),\hat{\psi}\rangle\right\}=(2\pi)^d\int_{\mathbb{R}^d}d\mathbf{p}\tilde{R}(z,\mathbf{p})\hat{\phi}(\mathbf{p})\hat{\psi}(-\mathbf{p}), \tag{7.20}$$

where $\langle\cdot,\cdot\rangle$ is the usual duality product on $\mathbb{R}^d\times\mathbb{R}^d$, and the power spectrum $\tilde{R}$ is the Fourier transform of $R(z,\mathbf{x})$ in $\mathbf{x}$:

$$\tilde{R}(z,\mathbf{p})=\int_{\mathbb{R}^d}d\mathbf{x}e^{-i\mathbf{p}\cdot\mathbf{x}}R(z,\mathbf{x}). \tag{7.21}$$

We assume that $\tilde{R}(z,\mathbf{p})\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, the space of Schwartz functions, for simplicity and define $\hat{R}(\omega,\mathbf{p})$ as

$$\hat{R}(\omega,\mathbf{p})=\int_{\mathbb{R}}dze^{-i\omega z}\tilde{R}(z,\mathbf{p}), \tag{7.22}$$

which is the space-time Fourier transform of R.

We now make additional assumptions on the infinitesimal generator so that the Fredholm alternative holds for the Poisson equation. Namely, we assume that the generator Q is a bounded operator on $L^\infty(\mathcal{V})$ with a unique invariant measure $\pi(\hat{V})$, i.e. a unique normalized

measure such that $Q^*\pi = 0$, and assume the existence of a constant $\alpha > 0$ such that if $\langle g, \pi\rangle = 0$, then

$$\|e^{rQ}g\|_{L^\infty_{\hat{\mathcal{V}}}} \leq C\|g\|_{L^\infty_{\hat{\mathcal{V}}}} e^{-\alpha r}. \tag{7.23}$$

The simplest example of a generator with gap in the spectrum and invariant measure π is a jump process on $\mathcal{V}$ where

$$Qg(\hat{V}) = \int_{\mathcal{V}} g(\hat{V}_1)d\pi(\hat{V}_1) - g(\hat{V}), \qquad \int_{\mathcal{V}} d\pi(\hat{V}) = 1.$$

Given the above hypotheses, the Fredholm alternative holds for the Poisson equation

$$Qf = g, \tag{7.24}$$

provided that g satisfies $\langle \pi, g\rangle = 0$. It has a unique solution f with $\langle \pi, f\rangle = 0$ and $\|f\|_{L^\infty_{\hat{V}}} \leq C\|g\|_{L^\infty_{\hat{V}}}$. The solution f is given explicitly by

$$f(\hat{V}) = -\int_0^\infty dr e^{rQ} g(\hat{V}), \tag{7.25}$$

and the integral converges absolutely thanks to (7.23).

7.4 The Main result

Let us summarize the hypotheses. We define $W_\varepsilon(z, \mathbf{x}, \mathbf{k})$ in (7.14) as a mixture of states of solutions to the paraxial wave equation (7.13). The mixture of state is such that $(\mathbf{x}, \mathbf{k}) \to W_\varepsilon(0, \mathbf{x}, \mathbf{k})$, whence $(\mathbf{x}, \mathbf{k}) \to W_\varepsilon(z, \mathbf{x}, \mathbf{k})$ for all $t > 0$ is uniformly bounded in $L^2(\mathbb{R}^{2d})$. We assume that $W_\varepsilon(0, \mathbf{x}, \mathbf{k})$ converges *strongly* in $L^2(\mathbb{R}^{2d})$ to its limit $W_0(0, \mathbf{x}, \mathbf{k})$. We further assume that the random field $V(z, \mathbf{x})$ satisfies the hypotheses described in section 7.3.

Then we have the following convergence result.

Theorem 7.4.1 *Under the above assumptions, the Wigner distribution W_ε converges in probability and weakly in $L^2(\mathbb{R}^{2d})$ to the solution $\overline{W}$ of the following transport equation*

$$\kappa\frac{\partial \overline{W}}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\overline{W} = \kappa^2\mathcal{L}\overline{W}, \tag{7.26}$$

where the scattering kernel has the form

$$\mathcal{L}W(\mathbf{x}, \mathbf{k}) = \int_{\mathbb{R}^d} \hat{R}\Big(\frac{|\mathbf{p}|^2 - |\mathbf{k}|^2}{2}, \mathbf{p} - \mathbf{k}\Big)\Big(W(\mathbf{x}, \mathbf{p}) - W(\mathbf{x}, \mathbf{k})\Big)\frac{d\mathbf{p}}{(2\pi)^d}. \tag{7.27}$$

More precisely, for any test function $\lambda \in L^2(\mathbb{R}^{2d})$ the process $\langle W_\varepsilon(z), \lambda\rangle$ converges to $\langle \overline{W}(z), \lambda\rangle$ in probability as $\varepsilon \to 0$, uniformly on finite intervals $0 \leq z \leq Z$.

Note that the whole process W_ε, and not only its average $\mathbb{E}\{W_\varepsilon\}$ converges to the (deterministic) limit $\overline{W}$. This means that the process W_ε is *statistically stable* in the limit $\varepsilon \to 0$. The process $W_\varepsilon(z, \mathbf{x}, \mathbf{k})$ does not converge pointwise to the deterministic limit: averaging against a test function $\lambda(\mathbf{x}, \mathbf{k})$ is necessary. However, the deterministic limit is in sharp contrast with the results obtained in the Markov diffusion limit in Chapter 4 and Appendix A.

The next section is devoted to a proof of the theorem. The main ingredients of the proof are now summarized as follows. Recall that the main mathematical assumption is that $V(z, \mathbf{x})$ is Markov in the z variable. Let us set $Z > 0$ and consider $z \in [0, Z]$. This allows us to show that $(V(z/\varepsilon, \mathbf{x}/\varepsilon), W_\varepsilon(z, \mathbf{x}, \mathbf{k}))$ is jointly Markov in the space $\mathcal{V} \times \mathcal{X}$, where $\mathcal{X} = \mathcal{C}([0, L]; B_W)$, where $B_W = \{\|W\|_2 \leq C\}$ is an appropriate ball in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Evolution equation and random process. Since κ plays no role in the derivation, we set $\kappa = 1$ to simplify. Recall that W_ε satisfies the Cauchy problem

$$\frac{\partial W_\varepsilon}{\partial z} + \mathbf{k} \cdot \nabla_{\mathbf{x}} W_\varepsilon = \mathcal{L}_\varepsilon W_\varepsilon,$$

with $W_\varepsilon(0, \mathbf{x}, \mathbf{k}) = W_\varepsilon^0(\mathbf{x}, \mathbf{k})$, where

$$\mathcal{L}_\varepsilon W_\varepsilon = \frac{1}{i\sqrt{\varepsilon}} \int_{\mathbb{R}^d} \frac{d\tilde{V}(\frac{z}{\varepsilon}, \mathbf{p})}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon} \left[W_\varepsilon(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}) - W_\varepsilon(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2}) \right]. \tag{7.28}$$

The solution to the above Cauchy problem is understood in the sense that for every smooth test function $\lambda(z, \mathbf{x}, \mathbf{k})$, we have

$$\langle W_\varepsilon(z), \lambda(z) \rangle - \langle W_\varepsilon^0, \lambda(0) \rangle = \int_0^z \langle W_\varepsilon(s), \left(\frac{\partial}{\partial s} + \mathbf{k} \cdot \nabla_{\mathbf{x}} + \mathcal{L}_\varepsilon \right) \lambda(s) \rangle ds.$$

Here, we have used that $\mathcal{L}_\varepsilon$ is a self-adjoint operator for $\langle \cdot, \cdot \rangle$. Therefore, for a smooth function $\lambda_0(\mathbf{x}, \mathbf{k})$, we obtain $\langle W_\varepsilon(z), \lambda_0 \rangle = \langle W_\varepsilon^0, \lambda_\varepsilon(0) \rangle$, where $\lambda_\varepsilon(s)$ is the solution of the backward problem

$$\frac{\partial \lambda_\varepsilon}{\partial s} + \mathbf{k} \cdot \nabla_{\mathbf{x}} \lambda_\varepsilon + \mathcal{L}_\varepsilon \lambda_\varepsilon(s) = 0, \quad 0 \le s \le z,$$

with the terminal condition $\lambda_\varepsilon(z, \mathbf{x}, \mathbf{k}) = \lambda_0(\mathbf{x}, \mathbf{k})$.

Tightness of the family of $\varepsilon-$measures. The above construction defines the process $W_\varepsilon(z)$ in $L^2(\mathbb{R}^{2d})$ and generates a corresponding measure P_ε on the space $C([0, L]; L^2(\mathbb{R}^{2d}))$ of continuous functions in time with values in L^2. The measure P_ε is actually supported on paths inside $\mathcal{X}$ defined above, which is the state space for the random process $W_\varepsilon(z)$. With its natural topology and the Borel $\sigma-$algebra $\mathcal{F}$, $(\mathcal{X}, \mathcal{F}, P_\varepsilon)$ defines a probability space on which $W_\varepsilon(z)$ is a random variable. Then $\mathcal{F}_s$ is defined as the filtration of the process $W_\varepsilon(z)$, i.e., the filtration generated by $\{W_\varepsilon(\tau), \tau < s\}$. We recall that intuitively, the filtration renders the past $\tau < s$ measurable, i.e., "known", and the future $\tau > s$ non-measurable, i.e., not known yet.

The family P_ε parameterized by $\varepsilon_0 > \varepsilon > 0$ will be shown to be *tight*. This in turns implies that P_ε converges weakly to P. More precisely, we can extract a subsequence of P_ε, still denoted by P_ε, such that for all continuous function f defined on $\mathcal{X}$, we have

$$\mathbb{E}^{P_\varepsilon}\{f\} \equiv \int_{\mathcal{X}} f(\omega) dP_\varepsilon(\omega) \to \int_{\mathcal{X}} f(\omega) dP(\omega) \equiv \mathbb{E}^P\{f\}, \qquad \text{as} \quad \varepsilon \to 0. \tag{7.29}$$

Construction of a first approximate martingale. Once tightness is ensured, the proof of convergence of W_ε to its deterministic limit is obtained in two steps. Let us fix a deterministic test function $\lambda(z, \mathbf{x}, \mathbf{k})$. We use the Markovian property of the random field $V(z, \mathbf{x})$ in z to construct a first functional $G_\lambda \colon \mathcal{X} \to C[0, L]$ by

$$G_\lambda[W](z) = \langle W, \lambda \rangle(z) - \int_0^z \langle W, \frac{\partial \lambda}{\partial z} + \mathbf{k} \cdot \nabla_{\mathbf{x}} \lambda + \mathcal{L}\lambda \rangle(\zeta) d\zeta. \tag{7.30}$$

Here, $\mathcal{L}$ is the limiting scattering kernel defined in (7.27). We will show that G_λ is an approximate P_ε-martingale (with respect to the filtration $\mathcal{F}_s$), and more precisely that

$$\left| \mathbb{E}^{P_\varepsilon} \{ G_\lambda[W](z) | \mathcal{F}_s \} - G_\lambda[W](s) \right| \le C_{\lambda, L} \sqrt{\varepsilon} \tag{7.31}$$

uniformly for all $W \in \mathcal{X}$ and $0 \leq s < z \leq L$. Choosing $s = 0$ above, the two convergences (7.29) and (7.31) (weak against strong) show that

$$\mathbb{E}^P \{G_\lambda[W](z)|\} - G_\lambda[W](0) = 0. \tag{7.32}$$

We thus obtain the transport equation (7.26) for $\overline{W} = \mathbb{E}^P \{W(z)\}$ in its weak formulation.

Exercise 7.4.1 Verify this statement.

Construction of a second approximate martingale and convergence of the full family of $\varepsilon-$measures. So far, we have characterized the convergence of the first moment of P_ε. We now consider the convergence of the second moment and show that the variance of the limiting process vanishes, whence the convergence to a deterministic process.

We will show that for every test function $\lambda(z, \mathbf{x}, \mathbf{k})$, the new functional

$$G_{2,\lambda}[W](z) = \langle W, \lambda \rangle^2(z) - 2 \int_0^z \langle W, \lambda \rangle(\zeta) \langle W, \frac{\partial \lambda}{\partial z} + \mathbf{k} \cdot \nabla_{\mathbf{x}} \lambda + \mathcal{L}\lambda \rangle(\zeta) d\zeta \tag{7.33}$$

is also an approximate P_ε-martingale. We then obtain that

$$\mathbb{E}^{P_\varepsilon} \left\{ \langle W, \lambda \rangle^2 \right\} \to \langle \overline{W}, \lambda \rangle^2. \tag{7.34}$$

This crucial convergence implies convergence in probability. It follows that the limit measure P is unique and deterministic, and that the whole sequence P_ε converges.

7.5 Proof of Theorem 7.4.1

The proof of tightness of the family of measures P_ε is postponed to the end of the section as it requires estimates that are developed in the proofs of convergence of the approximate martingales. We thus start with the latter proofs.

7.5.1 Convergence in expectation

To obtain the approximate martingale property (7.31), one has to consider the conditional expectation of functionals $F(W, \hat{V})$ with respect to the probability measure $\tilde{P}_\varepsilon$ on $D([0, L]; B_W \times \mathcal{V})$, the space of right-continuous paths with left-side limits [10] generated by the process (W, V). Note that W is a continuous function of z thanks to the evolution equation it solves. The process V however need not be continuous, whence the above space, sometimes referred to in the probabilistic literature as the space of *càd-làg* functions (which stands for the French *continu à droite, limite à gauche*). The only functions we need consider are in fact of the form $F(W, \hat{V}) = \langle W, \lambda(\hat{V}) \rangle$ with $\lambda \in L^\infty(\mathcal{V}; C^1([0, L]; \mathcal{S}(\mathbb{R}^{2d})))$. Given a function $F(W, \hat{V})$ let us define the conditional expectation

$$\mathbb{E}^{\tilde{P}_\varepsilon}_{W,\hat{V},z} \left\{ F(W, \hat{V}) \right\} (\tau) = \mathbb{E}^{\tilde{P}_\varepsilon} \left\{ F(W(\tau), \tilde{V}(\tau))|\ W(z) = W, \tilde{V}(z) = \hat{V} \right\}, \quad \tau \geq z.$$

The weak form of the infinitesimal generator of the Markov process generated by $\tilde{P}_\varepsilon$ is then given by

$$\frac{d}{dh} \mathbb{E}^{\tilde{P}_\varepsilon}_{W,\hat{V},z} \left\{ \langle W, \lambda(\hat{V}) \rangle \right\} (z + h) \Big|_{h=0} = \frac{1}{\varepsilon} \langle W, Q\lambda \rangle + \left\langle W, \left(\frac{\partial}{\partial z} + \mathbf{k} \cdot \nabla_{\mathbf{x}} + \frac{1}{\sqrt{\varepsilon}} \mathcal{K}[\hat{V}, \frac{\mathbf{x}}{\varepsilon}] \right) \lambda \right\rangle. \tag{7.35}$$

Exercise 7.5.1 Derive the above formula in detail using the definition of the Markov process V with infinitesimal generator Q and the evolution equation for W_ε written in the form

$$\frac{\partial W_\varepsilon}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}} W_\varepsilon = \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\tilde{V}(z/\varepsilon), \mathbf{x}/\varepsilon]W_\varepsilon, \tag{7.36}$$

where the operator $\mathcal{K}$ is defined by

$$\mathcal{K}[\hat{V}, \boldsymbol{\eta}]\psi(\mathbf{x}, \boldsymbol{\eta}, \mathbf{k}, \hat{V}) = \frac{1}{i}\int_{\mathbb{R}^d} \frac{d\hat{V}(\mathbf{p})}{(2\pi)^d} e^{i\mathbf{p}\cdot\boldsymbol{\eta}} \left[\psi(\mathbf{x}, \boldsymbol{\eta}, \mathbf{k} - \frac{\mathbf{p}}{2}) - \psi(\mathbf{x}, \boldsymbol{\eta}, \mathbf{k} + \frac{\mathbf{p}}{2})\right]. \tag{7.37}$$

The above equality implies that

$$G^\varepsilon_\lambda = \langle W, \lambda(\hat{V})\rangle(z) - \int_0^z \left\langle W, \left(\frac{1}{\varepsilon}Q + \frac{\partial}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}} + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V}, \frac{\mathbf{x}}{\varepsilon}]\right)\lambda\right\rangle(s)ds \tag{7.38}$$

is a $\tilde{P}_\varepsilon$-martingale since the drift term has been subtracted.

Given a test function $\lambda(z, \mathbf{x}, \mathbf{k}) \in C^1([0, L]; \mathcal{S})$ we construct a function

$$\lambda_\varepsilon(z, \mathbf{x}, \mathbf{k}, \hat{V}) = \lambda(z, \mathbf{x}, \mathbf{k}) + \sqrt{\varepsilon}\lambda^\varepsilon_1(z, \mathbf{x}, \mathbf{k}, \hat{V}) + \varepsilon\lambda^\varepsilon_2(z, \mathbf{x}, \mathbf{k}, \hat{V}), \tag{7.39}$$

with $\lambda^\varepsilon_{1,2}(t)$ bounded in $L^\infty(\mathcal{V}; L^2(\mathbb{R}^{2d}))$ uniformly in $z \in [0, L]$. This is the method of **perturbed test function**. Rather than performing asymptotic expansions on the Wigner transform itself, which is not sufficiently smooth to justify Taylor expansions, we perform the expansion on smooth test functions.

The functions $\lambda^\varepsilon_{1,2}$ will be chosen to remove all high-order terms in the definition of the martingale (7.38), i.e., so that

$$\|G^\varepsilon_{\lambda_\varepsilon}(z) - G_\lambda(z)\|_{L^\infty(\mathcal{V})} \le C_\lambda\sqrt{\varepsilon} \tag{7.40}$$

for all $z \in [0, L]$. Here $G^\varepsilon_{\lambda_\varepsilon}$ is defined by (7.38) with λ replaced by λ_ε, and G_λ is defined by (7.30). The approximate martingale property (7.31) follows from this.

The functions λ^ε_1 and λ^ε_2 are as follows. Let $\lambda_1(z, \mathbf{x}, \boldsymbol{\eta}, \mathbf{k}, \hat{V})$ be the mean-zero solution of the Poisson equation

$$\mathbf{k}\cdot\nabla_{\boldsymbol{\eta}}\lambda_1 + Q\lambda_1 = -\mathcal{K}\lambda. \tag{7.41}$$

It is given explicitly by

$$\lambda_1(z, \mathbf{x}, \boldsymbol{\eta}, \mathbf{k}, \hat{V}) = \frac{1}{i}\int_0^\infty dr e^{rQ}\int_{\mathbb{R}^d}\frac{d\hat{V}(\mathbf{p})}{(2\pi)^d}e^{ir(\mathbf{k}\cdot\mathbf{p})+i(\boldsymbol{\eta}\cdot\mathbf{p})}\left[\lambda(z, \mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}) - \lambda(z, \mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2})\right]. \tag{7.42}$$

Exercise 7.5.2 Prove the above formula. *Hint*: Go to the Fourier domain $\boldsymbol{\eta} \to \mathbf{p}$.

Then we let $\lambda^\varepsilon_1(z, \mathbf{x}, \mathbf{k}, \hat{V}) = \lambda_1(z, \mathbf{x}, \mathbf{x}/\varepsilon, \mathbf{k}, \hat{V})$. Furthermore, the second order corrector is given by $\lambda^\varepsilon_2(z, \mathbf{x}, \mathbf{k}, \hat{V}) = \lambda_2(z, \mathbf{x}, \mathbf{x}/\varepsilon, \mathbf{k}, \hat{V})$ where $\lambda_2(z, \mathbf{x}, \boldsymbol{\eta}, \mathbf{k}, \hat{V})$ is the mean-zero solution of

$$\mathbf{k}\cdot\nabla_{\boldsymbol{\eta}}\lambda_2 + Q\lambda_2 = \mathcal{L}\lambda - \mathcal{K}\lambda_1, \tag{7.43}$$

which exists because

$$\mathbb{E}\{\mathcal{K}\lambda_1\} = \mathcal{L}\lambda. \tag{7.44}$$

Exercise 7.5.3 Verify the above equality using the definition of the power spectrum of the potential V.

The explicit expression for λ_2 is given by

$$\lambda_2(z,\mathbf{x},\boldsymbol{\eta},\mathbf{k},\hat{V}) = -\int_0^\infty dr e^{rQ}\left[\mathcal{L}\lambda(z,\mathbf{x},\mathbf{k}) - [\mathcal{K}\lambda_1](z,\mathbf{x},\boldsymbol{\eta}+r\mathbf{k},\mathbf{k},\hat{V})\right].$$

Exercise 7.5.4 Verify this.

Using (7.41) and (7.43) we have

$$\begin{aligned}\frac{d}{dh}\mathbb{E}^{\tilde{P}_\varepsilon}_{W,\hat{V},z}\left\{\langle W,\lambda_\varepsilon\rangle\right\}(z+h)\Big|_{h=0} &= \left\langle W,\left(\frac{\partial}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}+\frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V},\frac{\mathbf{x}}{\varepsilon}]+\frac{1}{\varepsilon}Q\right)\left(\lambda+\sqrt{\varepsilon}\lambda_1^\varepsilon+\varepsilon\lambda_2^\varepsilon\right)\right\rangle \\ &= \left\langle W,\left(\frac{\partial}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\right)\lambda+\mathcal{L}\lambda\right\rangle + \left\langle W,\left(\frac{\partial}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\right)\left(\sqrt{\varepsilon}\lambda_1^\varepsilon+\varepsilon\lambda_2^\varepsilon\right)+\sqrt{\varepsilon}\mathcal{K}[\hat{V},\frac{\mathbf{x}}{\varepsilon}]\lambda_2^\varepsilon\right\rangle \\ &= \left\langle W,\left(\frac{\partial}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\right)\lambda+\mathcal{L}\lambda\right\rangle + \sqrt{\varepsilon}\langle W,\zeta_\varepsilon^\lambda\rangle,\end{aligned}$$

with

$$\zeta_\varepsilon^\lambda = \left(\frac{\partial}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\right)\lambda_1^\varepsilon + \sqrt{\varepsilon}\left(\frac{\partial}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\right)\lambda_2^\varepsilon + \mathcal{K}[\hat{V},\frac{\mathbf{x}}{\varepsilon}]\lambda_2^\varepsilon.$$

The terms $\mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda_{1,2}^\varepsilon$ above are understood as differentiation with respect to the slow variable $\mathbf{x}$ only, and not with respect to $\boldsymbol{\eta}=\mathbf{x}/\varepsilon$. It follows that $G^\varepsilon_{\lambda_\varepsilon}$ is given by

$$G^\varepsilon_{\lambda_\varepsilon}(z) = \langle W(z),\lambda_\varepsilon\rangle - \int_0^z ds\left\langle W,\left(\frac{\partial}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}+\mathcal{L}\right)\lambda\right\rangle(s) - \sqrt{\varepsilon}\int_0^z ds\langle W,\zeta_\varepsilon^\lambda\rangle(s) \quad (7.45)$$

and is a martingale with respect to the measure $\tilde{P}_\varepsilon$ defined on $D([0,L];B_W\times\mathcal{V})$. The estimate (7.31) follows from the following two lemmas.

Lemma 7.5.1 *Let $\lambda\in C^1([0,L];\mathcal{S}(\mathbb{R}^{2d}))$. Then there exists a constant $C_\lambda>0$ independent of $z\in[0,L]$ so that the correctors $\lambda_1^\varepsilon(z)$ and $\lambda_2^\varepsilon(z)$ satisfy the uniform bounds*

$$\|\lambda_1^\varepsilon(z)\|_{L^\infty(\mathcal{V};L^2)} + \|\lambda_2^\varepsilon(z)\|_{L^\infty(\mathcal{V};L^2)} \le C_\lambda \quad (7.46)$$

and

$$\big\|\frac{\partial\lambda_1^\varepsilon(z)}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda_1^\varepsilon(z)\big\|_{L^\infty(\mathcal{V};L^2)} + \big\|\frac{\partial\lambda_2^\varepsilon(z)}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda_2^\varepsilon(z)\big\|_{L^\infty(\mathcal{V};L^2)} \le C_\lambda. \quad (7.47)$$

Lemma 7.5.2 *There exists a constant C_λ such that*

$$\|\mathcal{K}[\hat{V},\mathbf{x}/\varepsilon]\|_{L^2\to L^2}\le C$$

for any $\hat{V}\in\mathcal{V}$ and all $\varepsilon\in(0,1]$.

Indeed, (7.46) implies that $|\langle W,\lambda\rangle - \langle W,\lambda_\varepsilon\rangle|\le C\sqrt{\varepsilon}$ for all $W\in X$ and $\hat{V}\in\mathcal{V}$, while (7.47) and Lemma 7.5.2 imply that for all $z\in[0,L]$

$$\|\zeta_\varepsilon^\lambda(z)\|_{L^2}\le C, \quad (7.48)$$

for all $\hat{V}\in\mathcal{V}$ so that (7.31) follows.

Proof of Lemma 7.5.2. Lemma 7.5.2 follows immediately from the definition of $\mathcal{K}$, the bound (7.17) and the Cauchy-Schwarz inequality.

We now prove Lemma 7.5.1. We will omit the z-dependence of the test function λ to simplify the notation.

Proof of Lemma 7.5.1. We only prove (7.46). Since $\lambda \in \mathcal{S}(\mathbb{R}^{2d})$, there exists a constant C_λ so that

$$|\lambda(\mathbf{x},\mathbf{k})| \leq \frac{C_\lambda}{(1+|\mathbf{x}|^{5d})(1+|\mathbf{k}|^{5d})}.$$

The value of the exponents $5d$ is by no means optimal, and is sufficient in what follows. Then we obtain using (7.17) and (7.23)

$$\begin{aligned}
&|\lambda_1^\varepsilon(z,\mathbf{x},\mathbf{k},\hat{V})| = C\left|\int_0^\infty dr e^{rQ}\int_{\mathbb{R}^d} d\hat{V}(\mathbf{p}) e^{ir(\mathbf{k}\cdot\mathbf{p})+i(\mathbf{x}\cdot\mathbf{p})/\varepsilon}\left[\lambda(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})-\lambda(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\right]\right| \\
&\leq C\int_0^\infty dr e^{-\alpha r}\sup_{\hat{V}}\int_{\mathbb{R}^d}|d\hat{V}(\mathbf{p})|\left[|\lambda(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})|+|\lambda(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})|\right] \\
&\leq \frac{C}{(1+|\mathbf{x}|^{5d})(1+(|\mathbf{k}|-L)^{5d}\chi_{|\mathbf{k}|\geq 5L}(\mathbf{k}))},
\end{aligned}$$

and the L^2-bound on λ_1 follows.

We show next that λ_2^ε is uniformly bounded. We have

$$\begin{aligned}
\lambda_2^\varepsilon(\mathbf{x},\mathbf{k},\hat{V}) = -\int_0^\infty dr e^{rQ}\Bigg[&\mathcal{L}\lambda(\mathbf{x},\mathbf{k}) - \frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot(\mathbf{x}/\varepsilon+r\mathbf{k})} \\
&\times\left[\lambda_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}+r\mathbf{k},\mathbf{k}-\frac{\mathbf{p}}{2},\hat{V})-\lambda_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}+r\mathbf{k},\mathbf{k}+\frac{\mathbf{p}}{2},\hat{V})\right]\Bigg].
\end{aligned}$$

The second term above may be written as

$$\begin{aligned}
&\frac{1}{i}\int_{\mathbb{R}^d}\frac{d\hat{V}(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot(\mathbf{x}/\varepsilon+r\mathbf{k})}\left[\lambda_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}+r\mathbf{k},\mathbf{k}-\frac{\mathbf{p}}{2},\hat{V})-\lambda_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}+r\mathbf{k},\mathbf{k}+\frac{\mathbf{p}}{2},\hat{V})\right] \\
&= -\int_{\mathbb{R}^d}\frac{d\hat{V}(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot(\mathbf{x}/\varepsilon+r\mathbf{k})}\int_0^\infty ds e^{sQ}\int_{\mathbb{R}^d}\frac{d\hat{V}(\mathbf{q})}{(2\pi)^d}e^{is(\mathbf{k}-\mathbf{p}/2)\cdot\mathbf{q}+i(\mathbf{x}/\varepsilon+r\mathbf{k})\cdot\mathbf{q}} \\
&\times\left[\lambda(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2})-\lambda(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}+\frac{\mathbf{q}}{2})\right] \\
&+\int_{\mathbb{R}^d}\frac{d\hat{V}(\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot(\mathbf{x}/\varepsilon+r\mathbf{k})}\int_0^\infty ds e^{sQ}\int_{\mathbb{R}^d}\frac{d\hat{V}(\mathbf{q})}{(2\pi)^d}e^{is(\mathbf{k}+\mathbf{p}/2)\cdot\mathbf{q}+i(\mathbf{x}/\varepsilon+r\mathbf{k})\cdot\mathbf{q}} \\
&\times\left[\lambda(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2})-\lambda(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}+\frac{\mathbf{q}}{2})\right].
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&|\lambda_2^\varepsilon(\mathbf{x},\mathbf{k},\hat{V})| \leq C\int_0^\infty dr e^{-\alpha r}\Bigg[|\mathcal{L}\lambda(\mathbf{x},\mathbf{k})|+\sup_{\hat{V}}\int_{\mathbb{R}^d}|d\hat{V}(\mathbf{p})|\int_0^\infty ds e^{-\alpha s}\sup_{\hat{V}_1}\int_{\mathbb{R}^d}|d\hat{V}_1(\mathbf{q})| \\
&\times\left(|\lambda(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2})|+|\lambda(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}+\frac{\mathbf{q}}{2})|+|\lambda(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2})|+\lambda(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}+\frac{\mathbf{q}}{2})\right)\Bigg] \\
&\leq C\left[|\mathcal{L}\lambda(\mathbf{x},\mathbf{k})|+\frac{1}{(1+|\mathbf{x}|^{5d})(1+(|\mathbf{k}|-L)^{5d}\chi_{|\mathbf{k}|\geq 5L}(\mathbf{k}))}\right],
\end{aligned}$$

and the L^2-bound on λ_2^ε in (7.46) follows because the operator $\mathcal{L}: L^2 \to L^2$ is bounded. The proof of (7.47) is very similar and is left as a painful exercise.

Lemma 7.5.1 and Lemma 7.5.2 together with (7.45) imply the bound (7.40). The tightness of measures P_ε given by Lemma 7.5.4 implies then that the expectation $\mathbb{E}\{W_\varepsilon(z,\mathbf{x},\mathbf{k})\}$ converges weakly in $L^2(\mathbb{R}^{2d})$ to the solution $\overline{W}(z,\mathbf{x},\mathbf{k})$ of the transport equation for each $z \in [0,L]$.

7.5.2 Convergence in probability

We now prove that for any test function λ the second moment $\mathbb{E}\left\{\langle W_\varepsilon, \lambda\rangle^2\right\}$ converges to $\langle \overline{W}, \lambda\rangle^2$. This will imply the convergence in probability claimed in Theorem 7.4.1. The proof is similar to that for $\mathbb{E}\left\{\langle W_\varepsilon, \lambda\rangle\right\}$ and is based on constructing an appropriate approximate martingale for the functional $\langle W \otimes W, \mu\rangle$, where $\mu(z, \mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2)$ is a test function, and $W \otimes W(z, \mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2) = W(z, \mathbf{x}_1, \mathbf{k}_1)W(z, \mathbf{x}_2, \mathbf{k}_2)$. We need to consider the action of the infinitesimal generator on functions of W and $\hat{V}$ of the form

$$F(W, \hat{V}) = \langle W(\mathbf{x}_1, \mathbf{k}_1)W(\mathbf{x}_2, \mathbf{k}_2), \mu(z, \mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, \hat{V})\rangle = \langle W \otimes W, \mu(\hat{V})\rangle$$

where μ is a given function. The infinitesimal generator acts on such functions as

$$\frac{d}{dh}\mathbb{E}^{\tilde{P}_\varepsilon}_{W,\hat{V},z}\left\{\langle W \otimes W, \mu(\hat{V})\rangle\right\}(z+h)\Big|_{h=0} = \frac{1}{\varepsilon}\langle W \otimes W, Q\lambda\rangle + \langle W \otimes W, \mathcal{H}^\varepsilon_2\mu\rangle, \quad (7.49)$$

where

$$\mathcal{H}^\varepsilon_2\mu = \sum_{j=1}^{2}\frac{1}{\sqrt{\varepsilon}}\mathcal{K}_j\left[\hat{V}, \frac{\mathbf{x}^j}{\varepsilon}\right]\mu + \mathbf{k}^j \cdot \nabla_{\mathbf{x}^j}\mu, \quad (7.50)$$

with

$$\mathcal{K}_1[\hat{V}, \boldsymbol{\eta}_1]\mu = \frac{1}{i}\int_{\mathbb{R}^d} d\hat{V}(\mathbf{p})e^{i(\mathbf{p}\cdot\boldsymbol{\eta}_1)}\left[\mu(\mathbf{k}_1 - \frac{\mathbf{p}}{2}, \mathbf{k}_2) - \mu(\mathbf{k}_1 + \frac{\mathbf{p}}{2}, \mathbf{k}_2)\right]$$

and

$$\mathcal{K}_2[\hat{V}, \boldsymbol{\eta}_2]\mu = \frac{1}{i}\int_{\mathbb{R}^d} d\hat{V}(\mathbf{p})e^{i(\mathbf{p}\cdot\boldsymbol{\eta}_2)}\left[\mu(\mathbf{k}_1, \mathbf{k}_2 - \frac{\mathbf{p}}{2}) - \mu(\mathbf{k}_1, \mathbf{k}_2 + \frac{\mathbf{p}}{2})\right].$$

Therefore the functional

$$G^{2,\varepsilon}_\mu = \langle W \otimes W, \mu(\hat{V})\rangle(z) \quad (7.51)$$
$$-\int_0^z \left\langle W \otimes W, \big(\frac{1}{\varepsilon}Q + \frac{\partial}{\partial z} + \mathbf{k}_1 \cdot \nabla_{\mathbf{x}_1} + \mathbf{k}_2 \cdot \nabla_{\mathbf{x}_2} + \frac{1}{\sqrt{\varepsilon}}(\mathcal{K}_1[\hat{V}, \frac{\mathbf{x}_1}{\varepsilon}] + \mathcal{K}_2[\hat{V}, \frac{\mathbf{x}_2}{\varepsilon}])\big)\mu\right\rangle(s)ds$$

is a $\tilde{P}^\varepsilon$ martingale. We let $\mu(z, \mathbf{x}, \mathbf{K}) \in \mathcal{S}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ be a test function independent of $\hat{V}$, where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, and $\mathbf{K} = (\mathbf{k}_1, \mathbf{k}_2)$. We define an approximation

$$\mu_\varepsilon(z, \mathbf{x}, \mathbf{K}) = \mu(z, \mathbf{x}, \mathbf{K}) + \sqrt{\varepsilon}\mu_1(z, \mathbf{x}, \mathbf{x}/\varepsilon, \mathbf{K}) + \varepsilon\mu_2(\mathbf{x}, \mathbf{x}/\varepsilon, \mathbf{K}).$$

We will use the notation $\mu^\varepsilon_1(z, \mathbf{x}, \mathbf{K}) = \mu_1(z, \mathbf{x}, \mathbf{x}/\varepsilon, \mathbf{K})$ and $\mu^\varepsilon_2(z, \mathbf{x}, \mathbf{K}) = \mu_2(z, \mathbf{x}, \mathbf{x}/\varepsilon, \mathbf{K})$. The functions μ_1 and μ_2 are to be determined. We now use (7.49) to get

$$D_\varepsilon := \frac{d}{dh}\big|_{h=0}\mathbb{E}_{W,\hat{V},z}(\langle W \otimes W, \mu_\varepsilon(\hat{V})\rangle)(z+h) = \frac{1}{\varepsilon}\left\langle W \otimes W, \left(Q + \sum_{j=1}^{2}\mathbf{k}^j \cdot \nabla_{\boldsymbol{\eta}^j}\right)\mu\right\rangle$$
$$+\frac{1}{\sqrt{\varepsilon}}\left\langle W \otimes W, \left(Q + \sum_{j=1}^{2}\mathbf{k}^j \cdot \nabla_{\boldsymbol{\eta}^j}\right)\mu_1 + \sum_{j=1}^{2}\mathcal{K}_j\left[\hat{V}, \boldsymbol{\eta}^j\right]\mu\right\rangle \quad (7.52)$$
$$+\left\langle W \otimes W, \left(Q + \sum_{j=1}^{2}\mathbf{k}^j \cdot \nabla_{\boldsymbol{\eta}^j}\right)\mu_2 + \sum_{j=1}^{2}\mathcal{K}_j\left[\hat{V}, \boldsymbol{\eta}^j\right]\mu_1 + \frac{\partial\mu}{\partial z} + \sum_{j=1}^{2}\mathbf{k}^j \cdot \nabla_{\mathbf{x}^j}\mu\right\rangle$$
$$+\sqrt{\varepsilon}\left\langle W \otimes W, \sum_{j=1}^{2}\mathcal{K}_j\left[\hat{V}, \boldsymbol{\eta}^j\right]\mu_2 + \left(\frac{\partial}{\partial z} + \sum_{j=1}^{2}\mathbf{k}^j \cdot \nabla_{\mathbf{x}^j}\right)(\mu_1 + \sqrt{\varepsilon}\mu_2)\right\rangle.$$

The above expression is evaluated at $\boldsymbol{\eta}_j = \mathbf{x}_j/\varepsilon$. The term of order ε^{-1} in D_ε vanishes since μ is independent of V and the fast variable $\boldsymbol{\eta}$. We cancel the term of order $\varepsilon^{-1/2}$ in the same way as before by defining μ_1 as the unique mean-zero (in the variables $\hat{V}$ and $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$) solution of

$$(Q + \sum_{j=1}^{2} \mathbf{k}^j \cdot \nabla_{\boldsymbol{\eta}^j})\mu_1 + \sum_{j=1}^{2} \mathcal{K}_j\big[\hat{V}, \boldsymbol{\eta}^j\big]\mu = 0. \tag{7.53}$$

It is given explicitly by

$$\begin{aligned}\mu_1(\mathbf{x}, \boldsymbol{\eta}, \mathbf{K}, \hat{V}) = \frac{1}{i}\int_0^\infty dr e^{rQ} \int_{\mathbb{R}^d} d\hat{V}(\mathbf{p}) e^{ir(\mathbf{k}_1\cdot\mathbf{p}) + i(\boldsymbol{\eta}_1\cdot\mathbf{p})} \Big[\mu(\mathbf{k}_1 - \frac{\mathbf{p}}{2}, \mathbf{k}_2) - \mu(\mathbf{k}_1 + \frac{\mathbf{p}}{2}, \mathbf{k}_2)\Big] \\ + \frac{1}{i}\int_0^\infty dr e^{rQ} \int_{\mathbb{R}^d} d\hat{V}(\mathbf{p}) e^{ir(\mathbf{k}_2\cdot\mathbf{p}) + i(\boldsymbol{\eta}_2\cdot\mathbf{p})} \Big[\mu(\mathbf{k}_1, \mathbf{k}_2 - \frac{\mathbf{p}}{2}) - \mu(\mathbf{k}_1, \mathbf{k}_2 + \frac{\mathbf{p}}{2})\Big].\end{aligned}$$

When μ has the form $\mu = \lambda \otimes \lambda$, then μ_1 has the form $\mu_1 = \lambda_1 \otimes \lambda + \lambda \otimes \lambda_1$ with the corrector λ_1 given by (7.42). Let us also define μ_2 as the mean zero with respect to π_V solution of

$$(Q + \sum_{j=1}^{2} \mathbf{k}^j \cdot \nabla_{\boldsymbol{\eta}^j})\mu_2 + \sum_{j=1}^{2} \mathcal{K}_j\big[\hat{V}, \boldsymbol{\eta}^j\big]\mu_1 = \overline{\sum_{j=1}^{2} \mathcal{K}_j\big[\hat{V}, \boldsymbol{\eta}^j\big]\mu_1}, \tag{7.54}$$

where $\overline{f} = \int d\pi_V f$. The function μ_2 is given by

$$\begin{aligned}\mu_2(\mathbf{x}, \boldsymbol{\eta}, \mathbf{K}, \hat{V}) = &- \int_0^\infty dr e^{rQ} \overline{[\mathcal{K}_1[\hat{V}, \boldsymbol{\eta}_1 + r\mathbf{k}_1]\mu_1(\mathbf{x}, \boldsymbol{\eta} + r\mathbf{K}, \mathbf{K})} \\ &- [\mathcal{K}_1[\hat{V}, \boldsymbol{\eta}_1 + r\mathbf{k}_1]\mu_1](\mathbf{x}, \boldsymbol{\eta} + r\mathbf{K}, \mathbf{K}, \hat{V})] \\ &- \int_0^\infty dr e^{rQ} \overline{[\mathcal{K}_2[\hat{V}, \mathbf{k}_2 + r\boldsymbol{\eta}_2]\mu_1(\mathbf{x}, \boldsymbol{\eta} + r\mathbf{K}, \mathbf{K})} \\ &- [\mathcal{K}_2[\hat{V}, \boldsymbol{\eta}_2 + r\mathbf{k}_2]\mu_1](\mathbf{x}, \boldsymbol{\eta} + r\mathbf{K}, \mathbf{K}, \hat{V})].\end{aligned} \tag{7.55}$$

Unlike the first corrector μ_1, the second corrector μ_2 may not be written as an explicit sum of tensor products even if μ has the form $\mu = \lambda \otimes \lambda$ because μ_1 depends on $\hat{V}$.

The $\tilde{P}^\varepsilon$-martingale $G^{2,\varepsilon}_{\mu_\varepsilon}$ is given by

$$\begin{aligned}G^{2,\varepsilon}_{\mu} = \langle W \otimes W, \mu(\hat{V})\rangle(z) - \int_0^z \left\langle W \otimes W, \big(\frac{\partial}{\partial z} + \mathbf{k}_1 \cdot \nabla_{\mathbf{x}_1} + \mathbf{k}_2 \cdot \nabla_{\mathbf{x}_2} + \mathcal{L}^\varepsilon_2\big)\mu \right\rangle (s) ds \\ -\sqrt{\varepsilon}\int_0^z \langle W \otimes W, \zeta^\mu_\varepsilon\rangle(s) ds,\end{aligned} \tag{7.56}$$

where ζ^μ_ε is given by

$$\zeta^\varepsilon_\mu = \sum_{j=1}^{2} \mathcal{K}_j\left[\hat{V}, \frac{\mathbf{x}_j}{\varepsilon}\right]\mu^\varepsilon_2 + \left(\frac{\partial}{\partial z} + \sum_{j=1}^{2} \mathbf{k}^j \cdot \nabla_{\mathbf{x}^j}\right)(\mu^\varepsilon_1 + \sqrt{\varepsilon}\mu^\varepsilon_2)$$

and the operator $\mathcal{L}_2^\varepsilon$ is defined by

$$\begin{aligned}
\mathcal{L}_2^\varepsilon\mu = &-\frac{1}{(2\pi)^d}\int_0^\infty dr\int_{\mathbb{R}^d} d\mathbf{p}\tilde{R}(r,\mathbf{p})\big[e^{ir(\mathbf{k}_1+\frac{\mathbf{p}}{2})\cdot\mathbf{p}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2)-\mu(z,\mathbf{x}_1,\mathbf{k}_1+\mathbf{p},\mathbf{x}_2,\mathbf{k}_2))\\
&-e^{ir(\mathbf{k}_1-\frac{\mathbf{p}}{2})\cdot\mathbf{p}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1-\mathbf{p},\mathbf{x}_2,\mathbf{k}_2)-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2))\big]\\
&+\big[e^{i\mathbf{p}\cdot\frac{\mathbf{x}_2-\mathbf{x}_1}{\varepsilon}+ir\mathbf{k}_2\cdot\mathbf{p}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2-\tfrac{\mathbf{p}}{2})-\mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2}))\\
&-e^{i\mathbf{p}\cdot\frac{\mathbf{x}_2-\mathbf{x}_1}{\varepsilon}+ir\mathbf{k}_2\cdot\mathbf{p}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2-\tfrac{\mathbf{p}}{2})-\mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2}))\big]\\
&+\big[e^{ir\mathbf{k}_1\cdot\mathbf{p}+i\frac{\mathbf{x}_1-\mathbf{x}_2}{\varepsilon}\cdot\mathbf{p}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2})-\mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2-\tfrac{\mathbf{p}}{2}))\\
&-e^{ir\mathbf{k}_1\cdot\mathbf{p}+i\frac{\mathbf{x}_1-\mathbf{x}_2}{\varepsilon}\cdot\mathbf{p}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2})-\mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2-\tfrac{\mathbf{p}}{2}))\big]\\
&+\big[e^{ir(\mathbf{k}_2+\frac{\mathbf{p}}{2})\cdot\mathbf{p}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2)-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2+\mathbf{p}))\\
&-e^{ir(\mathbf{k}_2-\frac{\mathbf{p}}{2})\cdot\mathbf{p}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2-\mathbf{p})-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2))\big].
\end{aligned}\tag{7.57}$$

We have used in the calculation of $\mathcal{L}_2^\varepsilon$ that for a sufficiently regular function f, we have

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\frac{d\hat{V}(\mathbf{q})}{(2\pi)^d}\int_0^\infty dr\, e^{rQ}\int_{\mathbb{R}^d}d\hat{V}(\mathbf{p})f(r,\mathbf{p},\mathbf{q})\right]=\int_0^\infty dr\int_{\mathbb{R}^d}\tilde{R}(r,\mathbf{p})f(r,\mathbf{p},-\mathbf{p})d\mathbf{p}.$$

The bound on ζ_ε^μ is similar to that on $\zeta_\varepsilon^\lambda$ obtained previously as the correctors μ_j^ε satisfy the same kind of estimates as the correctors λ_j:

Lemma 7.5.3 *There exists a constant $C_\mu>0$ so that the functions $\mu_{1,2}^\varepsilon$ obey the uniform bounds*

$$\|\mu_1^\varepsilon(z)\|_{L^2(\mathbb{R}^{2d})}+\|\mu_2^\varepsilon\|_{L^2(\mathbb{R}^{2d})}\le C_\mu,\tag{7.58}$$

and

$$\big\|\frac{\partial\mu_1^\varepsilon(z)}{\partial z}+\sum_{j=1}^2\mathbf{k}_j\cdot\nabla_{\mathbf{x}_j}\mu_1^\varepsilon(z)\big\|_{L^2(\mathbb{R}^{2d})}+\big\|\frac{\partial\mu_2^\varepsilon(z)}{\partial z}+\sum_{j=1}^2\mathbf{k}_j\cdot\nabla_{\mathbf{x}_j}\mu_2^\varepsilon(z)\big\|_{L^2(\mathbb{R}^{2d})}\le C_\mu,\tag{7.59}$$

for all $z\in[0,L]$ and $V\in\mathcal{V}$.

The proof of this lemma is very similar to that of Lemma 7.5.1 and is therefore omitted.

Unlike the first moment case, the averaged operator $\mathcal{L}_2^\varepsilon$ still depends on ε. We therefore do not have strong convergence of the $\tilde{P}^\varepsilon$-martingale $G_{\mu_\varepsilon}^{2,\varepsilon}$ to its limit yet. However, the a priori bound on W_ε in L^2 allows us to characterize the limit of $G_{\mu_\varepsilon}^{2,\varepsilon}$ and show strong convergence. This is shown as follows. The first and last terms in (7.57) that are independent of ε give the contribution:

$$\begin{aligned}
\mathcal{L}_2\mu=&\int_0^\infty dr\int_{\mathbb{R}^d}\frac{d\mathbf{p}}{(2\pi)^d}\Big[\tilde{R}(r,\mathbf{p}-\mathbf{k}_1)e^{ir\frac{p^2-k_1^2}{2}}(\mu(z,\mathbf{x}_1,\mathbf{p},\mathbf{x}_2,\mathbf{k}_2)-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2))\\
&+\tilde{R}(r,\mathbf{k}_1-\mathbf{p})e^{ir\frac{k_1^2-p^2}{2}}(\mu(z,\mathbf{x}_1,\mathbf{p}_1,\mathbf{x}_2,\mathbf{k}_2)-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2))\\
&+\tilde{R}(z,\mathbf{p}-\mathbf{k}_2)e^{ir\frac{p^2-k_2^2}{2}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{p})-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2))\\
&+\tilde{R}(z,\mathbf{k}_2-\mathbf{p})e^{ir\frac{k_2^2-p^2}{2}}(\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{p})-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2))\Big]\\
=&\int_{\mathbb{R}^d}\frac{d\mathbf{p}}{(2\pi)^d}\hat{R}(\frac{p^2-k_1^2}{2},\mathbf{p}-\mathbf{k}_1)(\mu(z,\mathbf{x}_1,\mathbf{p},\mathbf{x}_2,\mathbf{k}_2)-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2))\\
&+\hat{R}(\frac{p^2-k_2^2}{2},\mathbf{p}-\mathbf{k}_2)(\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{p})-\mu(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2)).
\end{aligned}$$

The two remaining terms give a contribution that tends to 0 as $\varepsilon \to 0$ for sufficiently smooth test functions. They are given by

$$(\mathcal{L}_2^\varepsilon - \mathcal{L}_2)\mu = \frac{1}{(2\pi)^d}\int_0^\infty dr \int_{\mathbb{R}^d} d\mathbf{p}\tilde{R}(r,\mathbf{p})\times$$
$$\big(e^{i\mathbf{p}\cdot\frac{\mathbf{x}_2-\mathbf{x}_1}{\varepsilon}+ir\mathbf{k}_2\cdot\mathbf{p}} + e^{ir\mathbf{k}_1\cdot\mathbf{p}+i\frac{\mathbf{x}_1-\mathbf{x}_2}{\varepsilon}\cdot\mathbf{p}}\big)\big(\mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2}) - \mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2-\tfrac{\mathbf{p}}{2})\big)$$
$$+\big(e^{i\mathbf{p}\cdot\frac{\mathbf{x}_2-\mathbf{x}_1}{\varepsilon}+ir\mathbf{k}_2\cdot\mathbf{p}} + e^{ir\mathbf{k}_1\cdot\mathbf{p}+i\frac{\mathbf{x}_1-\mathbf{x}_2}{\varepsilon}\cdot\mathbf{p}}\big)\big(\mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2-\tfrac{\mathbf{p}}{2}) - \mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2})\big).$$

We have

$$\tilde{R}(z,\mathbf{p}) = \tilde{R}(-z,-\mathbf{p}) \geq 0$$

by Bochner's theorem. Since $(\mathcal{L}_2^\varepsilon - \mathcal{L}_2)$ and λ are real quantities, we can take the real part of the above term and, after the change of variables $r \to -r$ and $\mathbf{p} \to -\mathbf{p}$, obtain

$$\begin{aligned}
(\mathcal{L}_2^\varepsilon - \mathcal{L}_2)\mu &= \frac{1}{(2\pi)^d}\int_{-\infty}^\infty dr \int_{\mathbb{R}^d} d\mathbf{p}\tilde{R}(r,\mathbf{p})\cos(\mathbf{p}\cdot\frac{\mathbf{x}_2-\mathbf{x}_1}{\varepsilon})(e^{ir\mathbf{k}_2\cdot\mathbf{p}} + e^{ir\mathbf{k}_1\cdot\mathbf{p}}) \\
&\quad\times\big(\mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2}) + \mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2-\tfrac{\mathbf{p}}{2}) \\
&\quad-\mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2-\tfrac{\mathbf{p}}{2}) - \mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2})\big) \\
&= \frac{2}{(2\pi)^d}\int_{\mathbb{R}^d} d\mathbf{p}(\hat{R}(-\mathbf{k}_1\cdot\mathbf{p},\mathbf{p}) + \hat{R}(-\mathbf{k}_2\cdot\mathbf{p},\mathbf{p}))\cos(\mathbf{p}\cdot\tfrac{\mathbf{x}_2-\mathbf{x}_1}{\varepsilon}) \\
&\quad\times\big(\mu(z,\mathbf{x}_1,\mathbf{k}_1+\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2}) - \mu(z,\mathbf{x}_1,\mathbf{k}_1-\tfrac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\tfrac{\mathbf{p}}{2})\big) \\
&= g_1 + g_2 + g_3 + g_4 + c.c.
\end{aligned}$$

We have (since μ is real-valued)

$$I = \int_{\mathbb{R}^{4d}} d\mathbf{x}_1 d\mathbf{k}_1 d\mathbf{x}_2 d\mathbf{k}_2 |g_1(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2)|^2 = C\int_{\mathbb{R}^{6d}} d\mathbf{x}_1 d\mathbf{k}_1 d\mathbf{x}_2 d\mathbf{k}_2 d\mathbf{p} d\mathbf{q}\hat{R}(-\mathbf{k}_1\cdot\mathbf{p},\mathbf{p})\hat{R}(-\mathbf{k}_1\cdot\mathbf{q},\mathbf{q})$$
$$\times e^{i(\mathbf{p}-\mathbf{q})\cdot\frac{\mathbf{x}_2-\mathbf{x}_1}{\varepsilon}}\mu(z,\mathbf{x}_1,\mathbf{k}_1-\frac{\mathbf{p}}{2},\mathbf{x}_2,\mathbf{k}_2+\frac{\mathbf{p}}{2})\mu(z,\mathbf{x}_1,\mathbf{k}_1-\frac{\mathbf{q}}{2},\mathbf{x}_2,\mathbf{k}_2+\frac{\mathbf{q}}{2}).$$

Using density arguments we may assume that μ has the form

$$\mu(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) = \mu_1(\mathbf{x}_1-\mathbf{x}_2)\mu_2(\mathbf{x}_1+\mathbf{x}_2)\mu_3(\mathbf{k}_1)\mu_4(\mathbf{k}_2).$$

Then we have

$$\begin{aligned}
I &= C\int_{\mathbb{R}^{6d}} d\mathbf{x}_1 d\mathbf{k}_1 d\mathbf{x}_2 d\mathbf{k}_2 d\mathbf{p} d\mathbf{q}\hat{R}(-\mathbf{k}_1\cdot\mathbf{p},\mathbf{p})\hat{R}(-\mathbf{k}_1\cdot\mathbf{q},\mathbf{q}) \\
&\times e^{-i(\mathbf{p}-\mathbf{q})\cdot\frac{\mathbf{x}_1}{\varepsilon}}\mu_1^2(\mathbf{x}_1)\mu_2^2(\mathbf{x}_2)\mu_3(\mathbf{k}_1-\frac{\mathbf{p}}{2})\mu_4(\mathbf{k}_2+\frac{\mathbf{p}}{2})\mu_3(\mathbf{k}_1-\frac{\mathbf{q}}{2})\mu_4(\mathbf{k}_2+\frac{\mathbf{q}}{2}) \\
&= C\|\mu_2\|_{L^2}^2\int_{\mathbb{R}^{4d}} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{p} d\mathbf{q}\hat{R}(-\mathbf{k}_1\cdot\mathbf{p},\mathbf{p})\hat{R}(-\mathbf{k}_1\cdot\mathbf{q},\mathbf{q})\hat{\nu}(\frac{\mathbf{p}-\mathbf{q}}{\varepsilon}) \\
&\times\mu_3(\mathbf{k}_1-\frac{\mathbf{p}}{2})\mu_4(\mathbf{k}_2+\frac{\mathbf{p}}{2})\mu_3(\mathbf{k}_1-\frac{\mathbf{q}}{2})\mu_4(\mathbf{k}_2+\frac{\mathbf{q}}{2})
\end{aligned}$$

where $\nu(\mathbf{x}) = \mu_1^2(\mathbf{x})$. We introduce $G(\mathbf{p}) = \sup_\omega \hat{R}(\omega,\mathbf{p})$ and use the Cauchy-Schwarz inequality in $\mathbf{k}_1$ and $\mathbf{k}_2$:

$$|I| \leq C\|\mu_2\|_{L^2}^2\|\mu_3\|_{L^2}^2\|\mu_4\|_{L^2}^2\int_{\mathbb{R}^{2d}} d\mathbf{p} d\mathbf{q} G(\mathbf{p})G(\mathbf{q})\left|\hat{\nu}(\frac{\mathbf{p}-\mathbf{q}}{\varepsilon})\right|.$$

We use again the Cauchy-Schwarz inequality, now in $\mathbf{p}$, to get

$$|I| \leq C\|\mu_2\|_{L^2}^2\|\mu_3\|_{L^2}^2\|\mu_4\|_{L^2}^2\|G\|_{L^2}\int_{\mathbb{R}^d} d\mathbf{q}G(\mathbf{q})\left(\int_{\mathbb{R}^d} d\mathbf{p}\left|\hat{\nu}(\frac{\mathbf{p}}{\varepsilon})\right|^2\right)^{1/2}$$
$$\leq C\varepsilon^{d/2}\|\mu_2\|_{L^2}^2\|\mu_3\|_{L^2}^2\|\mu_4\|_{L^2}^2\|G\|_{L^2}\|G\|_{L^1}\|\nu\|_{L^2}.$$

This proves that $\|(\mathcal{L}_2^\varepsilon - \mathcal{L}_2)\mu\|_{L^2} \to 0$ as $\varepsilon \to 0$. Note that oscillatory integrals of the form

$$\int_{\mathbb{R}^d} e^{i\frac{\mathbf{p}\cdot\mathbf{x}}{\varepsilon}}\mu(\mathbf{p})d\mathbf{p}, \tag{7.60}$$

are not small in the bigger space $\mathcal{A}'$, which is natural in the context of Wigner transforms. In this bigger space, we cannot control $(\mathcal{L}_2^\varepsilon - \mathcal{L}_2)\mu$ and actually suspect that the limit measure P may no longer be deterministic.

We therefore deduce that

$$G_\mu^2 = \langle W\otimes W, \mu(\hat{V})\rangle(z) - \int_0^z \left\langle W\otimes W, \big(\frac{\partial}{\partial z} + \mathbf{k}_1\cdot\nabla_{\mathbf{x}_1} + \mathbf{k}_2\cdot\nabla_{\mathbf{x}_2} + \mathcal{L}_2\big)\mu\right\rangle(s)ds$$

is an approximate $\tilde{P}_\varepsilon$ martingale. The limit of the second moment

$$W_2(z,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) = \mathbb{E}^P\left\{W(z,\mathbf{x}_1,\mathbf{k}_1)W(z,\mathbf{x}_2,\mathbf{k}_2)\right\}$$

thus satisfies (weakly) the transport equation

$$\frac{\partial W_2}{\partial t} + (\mathbf{k}_1\cdot\nabla_{\mathbf{x}_1} + \mathbf{k}_2\cdot\nabla_{\mathbf{x}_2})W_2 = \mathcal{L}_2W_2,$$

with initial data $W_2(0,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) = W_0(\mathbf{x}_1,\mathbf{k}_1)W_0(\mathbf{x}_2,\mathbf{k}_2)$. Moreover, the operator $\mathcal{L}_2$ acting on a tensor product $\lambda\otimes\lambda$ has the form

$$\mathcal{L}_2[\lambda\otimes\lambda] = \mathcal{L}\lambda\otimes\lambda + \lambda\otimes\mathcal{L}\lambda.$$

This implies that

$$\mathbb{E}^P\left\{W(z,\mathbf{x}_1,\mathbf{k}_1)W(z,\mathbf{x}_2,\mathbf{k}_2)\right\} = \mathbb{E}^P\left\{W(z,\mathbf{x}_1,\mathbf{k}_1)\right\}\mathbb{E}^P\left\{W(z,\mathbf{x}_2,\mathbf{k}_2)\right\}$$

by uniqueness of the solution to the above transport equation with initial conditions given by $W_0(\mathbf{x}_1,\mathbf{k}_1)W_0(\mathbf{x}_2,\mathbf{k}_2)$. This proves that the limiting measure P is deterministic and unique (because characterized by the transport equation) and that the sequence $W_\varepsilon(z,\mathbf{x},\mathbf{k})$ converges in probability to $W(z,\mathbf{x},\mathbf{k})$.

7.5.3 Tightness of P_ε

We now show tightness of the measures P_ε in $\mathcal{X}$. We have the lemma

Lemma 7.5.4 *The family of measures P_ε is weakly compact.*

The proof is as follows; see [11]. A theorem of Mitoma and Fouque [24, 16] implies that in order to verify tightness of the family P_ε it is enough to check that for each $\lambda \in C^1([0,L],\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d))$ the family of measures $\mathcal{P}_\varepsilon$ on $C([0,L];\mathbb{R})$ generated by the random processes $W_\lambda^\varepsilon(z) = \langle W_\varepsilon(z),\lambda\rangle$ is tight. Tightness of $\mathcal{P}_\varepsilon$ follows from the following two conditions. First, a Kolmogorov moment condition [10] in the form

$$E^{P_\varepsilon}\left\{|\langle W,\lambda\rangle(z) - \langle W,\lambda\rangle(z_1)|^\gamma\,|\langle W,\lambda\rangle(z_1) - \langle W,\lambda\rangle(s)|^\gamma\right\} \leq C_\lambda(z-s)^{1+\beta}, \quad 0\leq s\leq z\leq L \tag{7.61}$$

should hold with $\gamma > 0$, $\beta > 0$ and C_λ independent of ε. Second, we should have

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \mathrm{Prob}^{\mathcal{P}_\varepsilon} \left\{ \sup_{0\le z\le L} |\langle W, \lambda\rangle(z)| > R \right\} = 0.$$

The second condition holds automatically in our case since the process $W^\varepsilon_\lambda(z)$ is uniformly bounded for all $z > 0$ and $\varepsilon > 0$. In order to verify (7.61), note that we have

$$\langle W(z), \lambda\rangle = G^\varepsilon_{\lambda_\varepsilon}(z) - \sqrt{\varepsilon}\langle W, \lambda^\varepsilon_1\rangle - \varepsilon\langle W, \lambda^\varepsilon_2\rangle + \int_0^z ds\langle W, \frac{\partial\lambda}{\partial z} + k\cdot\nabla_x\lambda + \mathcal{L}\lambda\rangle(s) + \sqrt{\varepsilon}\int_0^z ds\langle W, \zeta^\lambda_\varepsilon\rangle(s).$$

The uniform bound (7.48) on $\zeta^\lambda_\varepsilon$ and the bounds on $\|\lambda^\varepsilon_{1,2}(z)\|_{L^2(\mathbb{R}^{2d})}$ in Lemma 7.5.1 imply that it suffices to check (7.61) for

$$x_\varepsilon(z) = G^\varepsilon_{\lambda_\varepsilon}(z) + \int_0^z ds\langle W, \frac{\partial\lambda}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda + \mathcal{L}\lambda\rangle(s).$$

We have

$$E\left\{ |x_\varepsilon(z) - x_\varepsilon(s)|^2 \Big| \mathcal{F}_s \right\} \le 2E\left\{ \left| \int_s^z d\tau\langle W, \frac{\partial\lambda}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda + \mathcal{L}\lambda\rangle(\tau)\right|^2 \Big| \mathcal{F}_s \right\}$$
$$+2E\left\{ \left|G^\varepsilon_{\lambda_\varepsilon}(z) - G^\varepsilon_{\lambda_\varepsilon}(s)\right|^2 \Big| \mathcal{F}_s \right\} \le C(z-s)^2 + 2E\left\{ \langle G^\varepsilon_{\lambda_\varepsilon}\rangle(z) - \langle G^\varepsilon_{\lambda_\varepsilon}\rangle(s) \big| \mathcal{F}_s \right\}.$$

Here $\langle G^\varepsilon_{\lambda_\varepsilon}\rangle$ is the increasing process associated with $G^\varepsilon_{\lambda_\varepsilon}$. We will now compute it explicitly. First we obtain that

$$\frac{d}{dh} E^{P_\varepsilon}_{W,\hat{V},t}\left\{ \langle W, \lambda_\varepsilon\rangle^2(z+h)\right\}\Big|_{h=0} = 2\langle W, \lambda_\varepsilon\rangle\langle W, \frac{\partial\lambda}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda_\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V}, \frac{\mathbf{x}}{\varepsilon}]\lambda_\varepsilon\rangle + \frac{1}{\varepsilon}Q\left[\langle W, \lambda_\varepsilon\rangle^2\right]$$

so that

$$\langle W, \lambda_\varepsilon\rangle^2(z) - \int_0^z \left(2\langle W, \lambda_\varepsilon\rangle(s)\langle W, \frac{\partial\lambda}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda_\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V}, \frac{\mathbf{x}}{\varepsilon}]\lambda_\varepsilon\rangle(s) + \frac{1}{\varepsilon}Q\left[\langle W, \lambda_\varepsilon\rangle^2\right](s)\right) ds$$

is a martingale. Therefore we have

$$\begin{aligned}\langle G^\varepsilon_{\lambda_\varepsilon}(z)\rangle &= \int_0^z ds\left[\frac{1}{\varepsilon}Q[\langle W, \lambda_\varepsilon\rangle^2] - \frac{2}{\varepsilon}\langle W, \lambda_\varepsilon\rangle\langle W, Q\lambda_\varepsilon\rangle\right](s) \\ &= \int_0^z ds\left(Q\left[\langle W, \lambda^\varepsilon_1\rangle^2\right] - \langle W, \lambda^\varepsilon_1\rangle\langle W, Q\lambda^\varepsilon_1\rangle(s)\right) + \sqrt{\varepsilon}\int_0^z ds H_\varepsilon(s)\end{aligned}$$

with

$$\begin{aligned}H_\varepsilon = 2\sqrt{\varepsilon}\,&(Q[\langle W, \lambda^\varepsilon_1\rangle\langle W, \lambda^\varepsilon_2\rangle] - \langle W, \lambda^\varepsilon_1\rangle\langle W, Q\lambda^\varepsilon_2\rangle - \langle W, \lambda^\varepsilon_2\rangle\langle W, Q\lambda^\varepsilon_1\rangle) \\ &+ \varepsilon\left(Q[\langle W, \lambda^\varepsilon_2\rangle^2] - 2\langle W, \lambda^\varepsilon_2\rangle\langle W, Q\lambda^\varepsilon_2\rangle\right).\end{aligned}$$

The boundedness of λ^ε_2 and that of Q on $L^\infty(\mathcal{V})$ imply that $|H_\varepsilon(s)| \le C$ for all $V \in \mathcal{V}$. This yields

$$E\left\{ \langle G^\varepsilon_{\lambda_\varepsilon}\rangle(z) - \langle G^\varepsilon_{\lambda_\varepsilon}\rangle(s) \big| \mathcal{F}_s \right\} \le C(z-s)$$

whence

$$E\left\{ |x_\varepsilon(z) - x_\varepsilon(s)|^2 \Big| \mathcal{F}_s \right\} \le C(z-s).$$

In order to obtain (7.61) we note that

$$
\begin{aligned}
&E^{P_\varepsilon}\left\{|x_\varepsilon(z)-x_\varepsilon(z_1)|^\gamma\,|x_\varepsilon(z_1)-x_\varepsilon(s)|^\gamma\right\}\\
&=E^{P_\varepsilon}\left\{E^{P_\varepsilon}\left\{\,|x_\varepsilon(z)-x_\varepsilon(z_1)|^\gamma\,|\,\mathcal{F}_{z_1}\right\}|x_\varepsilon(z_1)-x_\varepsilon(s)|^\gamma\right\}\\
&\le E^{P_\varepsilon}\left\{\left[E^{P_\varepsilon}\left\{\,|x_\varepsilon(z)-x_\varepsilon(z_1)|^2\Big|\,\mathcal{F}_{z_1}\right\}\right]^{\gamma/2}|x_\varepsilon(z_1)-x_\varepsilon(s)|^\gamma\right\}\\
&\le C(z-z_1)^{\gamma/2}E^{P_\varepsilon}\left\{|x_\varepsilon(z_1)-x_\varepsilon(s)|^\gamma\right\}\le C(z-z_1)^{\gamma/2}E^{P_\varepsilon}\left\{E^{P_\varepsilon}\left\{|x_\varepsilon(z_1)-x_\varepsilon(s)|^\gamma\,|\mathcal{F}_s\right\}\right\}\\
&\le C(z-z_1)^{\gamma/2}E^{P_\varepsilon}\left\{\left[E^{P_\varepsilon}\left\{|x_\varepsilon(z_1)-x_\varepsilon(s)|^2\Big|\,\mathcal{F}_s\right\}\right]^{\gamma/2}\right\}\le C(z-z_1)^{\gamma/2}(z_1-s)^{\gamma/2}\\
&\le C(z-s)^\gamma.
\end{aligned}
$$

Choosing now $\gamma>1$ we get (7.61) which finishes the proof of Lemma 7.5.4.

7.5.4 Remarks

Statistical stability and a priori bounds. As we have already mentioned, the uniform L^2 bound for the Wigner transform is crucial in the derivation of Thm. 7.4.1. In the absence of an a priori L^2 bound, we are not able to characterize the limiting measure P. However we can characterize its first moment. The derivation is done in [7]. Let us assume that W_ε is bounded in $\mathcal{A}'$, as is the case for the Wigner transform of a pure state ψ_ε uniformly bounded in $L^2(\mathbb{R}^d)$. Then we can show that $\mathbb{E}^{P_\varepsilon}\{W_\varepsilon\}$ converges weakly to $\overline{W}$, solution of (7.26), with appropriate initial conditions (the Wigner transform of the limit $\psi_\varepsilon(0,\mathbf{x})$). The proof is very similar to that obtained above, except that in the proof of convergence, as well as in the proof of tightness of the sequence of measures P_ε (now defined on a ball in $\mathcal{C}([0,L];\mathcal{A}')$), we need to show that the test functions $\lambda_{1,2}$ are bounded in $\mathcal{A}'$ rather than $L^2(\mathbb{R}^{2d})$.

However the proof of convergence of the second martingale in section 7.5.2 *does not* extend to the case of a uniform bound in $\mathcal{A}'$. Technically, the obstacle resides in the fact that the oscillatory integrals (7.60) are small in $L^2(\mathbb{R}^{2d})$ but *not* in $\mathcal{A}'$. Since $\mathcal{A}'$ includes bounded measures, any measure $\mu(d\mathbf{p})$ concentrating on the hyperplane orthogonal to $\mathbf{x}$ will render the integral (7.60) an order $O(1)$ quantity.

The above discussion does not provide proof that P_ε does not converge to a deterministic limit. However it strongly suggests that if W_ε is allowed to become quite singular in $\mathcal{A}'$, then on these paths P_ε may not become sufficiently self-averaging to converge to a deterministic limit. Actually, in the simplified regime of the Itô-Schrödinger equation (a further simplification compared to the paraxial wave equation), it is shown in [2] that the measure P_ε does not converge to a limiting deterministic measure when the initial Wigner measure is very singular (converges to a delta function in both $\mathbf{x}$ and $\mathbf{k}$). Instead, *scintillation* effects, which measure the distance between the second moment of W_ε and the square of its first moment, are shown to persist for all finite times (for an appropriate scaling). This does not characterize the limiting measure P either (this remains an open problem even in the Itô-Schrödinger framework), but at least shows that P is not deterministic.

Paraxial and radiative transfer regimes. Note that in the limit where the potential $V(z,\mathbf{x})$ oscillates very slowly in the z variable, so that $R(z,\mathbf{x})$ converges to a function that does not depend on z (because $V(z,\mathbf{x})$ becomes highly correlated in z), whence $\hat{R}(\omega,\mathbf{p})$ converges to a function of the form $\delta(\omega)\hat{R}(\mathbf{p})$, we obtain the limiting average transport equation

$$
\kappa\frac{\partial\overline{W}}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\overline{W}=\kappa^2\int_{\mathbb{R}^d}\hat{R}(\mathbf{p}-\mathbf{k})\delta\Big(\frac{|\mathbf{k}|^2}{2}-\frac{|\mathbf{p}|^2}{2}\Big)\Big(W(\mathbf{x},\mathbf{p})-W(\mathbf{x},\mathbf{k})\Big)\frac{d\mathbf{p}}{(2\pi)^d}. \tag{7.62}
$$

This is the radiative transfer equation for the Schrödinger equation (7.13) when the potential $V(\mathbf{x})$ is independent of the variable z. We do not recover the full radiative transfer equation as in Chapter 6 since we started with the paraxial approximation. However we recover the radiative transfer equation for the Schrödinger equation that can be derived formally using the same tools as those developed in 6.

Exercise 7.5.5 [long project] Derive (7.62) from the Wigner transform of the Schrödinger equation. Hint: see [28].

Note that the dispersion relation for wave equations $\omega = c_0|\mathbf{k}|$ is now replaced by its "paraxial" approximation $\omega = |\mathbf{k}|^2/2$, where $\mathbf{k}$ now is the transverse component of the wavevector only.

Appendix A

Notes on Diffusion Markov Processes

A.1 Markov Process and Infinitesimal Generator

A.1.1 Definitions and Kolmogorov equations

A process $\{X(t)\}_{t\geq 0}$ is called Markov with state space S if $X(t) \in S$ for all $t \geq 0$ and for all $A \in S$, $t \geq$, and $\tau \geq 0$, we have

$$P[X(t+\tau) \in A | X(s),\, s \leq t] = P[X(t+\tau) \in A | X(t)], \tag{A.1}$$

almost surely [12]. What this means is that the future of $X(u)$ for $u > t$ depends on its past $X(s)$ for $s \leq t$ only through its present $X(t)$. In other words, knowing $X(t)$ is enough to know $X(t+\tau)$. Markov processes forget about the past instantaneously.

Let us define the **Transition Probability Density** $p(t, y|s, x)$ as

$$p(t, y|s, x)dy = P[X(t) \in (y, y+dy) | X(s) = x] \quad \text{for } s \leq t. \tag{A.2}$$

The **Chapman-Kolmogorov** (CK) relation states that for $s < \tau < t$,

$$p(t, y|s, x) = \int p(\tau, \xi|s, x) p(t, y|\tau, \xi) d\xi \tag{A.3}$$

which is an integration over all intermediate states ξ taken at time τ. We can also see this as

$$(s, x) \longrightarrow (\tau, \xi) \longrightarrow (t, y).$$

For $s \leq t$, we define the $2-$**parameter family of solution operators** T_s^t as

$$(T_s^t f)(x) = \mathbb{E}\{f(X(t)) | X(s) = x\} = \int p(t, y|s, x) f(y) dy. \tag{A.4}$$

The properties of T_s^t, obtained from the CK relation, are

$$\begin{array}{lrcll} (1) & T_t^t & = & Id & \text{(Identity)} \\ (2) & T_s^t & = & T_s^\tau T_\tau^t & \text{for } s < \tau < t \\ (3) & \|T_s^t\|_{L^\infty} & \leq & 1 & \text{(contraction) .} \end{array} \tag{A.5}$$

We define the **Infinitesimal Generator** Q_t of the Markov process $X(t)$ as the operator

$$Q_t = \lim_{h \to 0} \frac{T_t^{t+h} - Id}{h} \tag{A.6}$$

The domain of definition of Q_t may be a dense subset of that of T_s^t.

Let us define a quantity whose calculation we are interested in

$$u(t, s, x) = T_s^t f(x) = \mathbb{E}\{f(X(t))|X(s) = x\}. \tag{A.7}$$

Here, s is the starting time, x the starting position, and t the final time. We are interested in the average of $f(X(t))$. We have that u satisfies the following **Kolmogorov Backward Equation** (KBE)

$$\begin{aligned} &\frac{\partial u}{\partial s} + Q_s u = 0 \quad \text{for} \quad s < t \\ &u(t, t, x) = f(x). \end{aligned} \tag{A.8}$$

Note that the differentiation is with respect to the backwards time s. We solve the PDE backwards in time from t to s. Q_s operates on the backwards starting position x. The transition probability density $p(t, y|s, x)$ satisfies the KBE with $f(x) = \delta(x - y)$:

$$\begin{aligned} &\frac{\partial p}{\partial s}(t, y|s, x) + (Q_s p(t, y|s, x))(x) = 0 \quad \text{for} \quad s < t \\ &p(t, y|t, x) = \delta(x - y). \end{aligned} \tag{A.9}$$

The transition probability density $p(t, y|s, x)$ also satisfies the following **Kolmogorov Forward Equation** (KFE)

$$\begin{aligned} &\frac{\partial p}{\partial t}(t, y|s, x) + (Q_t^* p(t, y|s, x))(y) = 0 \quad \text{for} \quad t > s \\ &p(s, y|s, x) = \delta(x - y). \end{aligned} \tag{A.10}$$

Here Q^* is the formal adjoint to Q, acting on the forward variable y. Note that this equation is solved forwards in time, from s to $t > s$.

A.1.2 Homogeneous Markov Processes

By definition, the statistics of a **homogeneous Markov process** do not change in time:

$$p(t, y|s, x) = p(t - s, y|x); \quad \text{probability depends on time difference only.} \tag{A.11}$$

The Chapman-Kolmogorov relation becomes

$$P(t + \tau, y|x) = \int P(t, \xi|x) P(\tau, y|\xi) d\xi.$$

We also now define the $1-$**parameter family of operators** T^t

$$(T^t f)(x) = \mathbb{E}\{f(X(t))|X(0) = x\} = \int p(t, y|x) f(y) dy. \tag{A.12}$$

with the properties

$$\begin{aligned} &(1) \qquad T_0^0 &&= Id \\ &(2) \qquad T^{t+\tau} &&= T^t T^\tau \\ &(3) \qquad \|T^t\|_{L^\infty} &&\leq 1. \end{aligned} \tag{A.13}$$

The family T^t forms a continuous contraction **semigroup** of operators. We also define the infinitesimal generator

$$Q = \lim_{t \to 0} \frac{T^t - Id}{t} \tag{A.14}$$

with no time dependence. We then verify that the **solution operator** T^t is given by

$$T^t = e^{tQ} \tag{A.15}$$

Indeed, if we define

$$u(t,x) = \mathbb{E}\{f(X(t))|X(0) = x\} = (T^t f)(x), \tag{A.16}$$

then we have the KBE with change of variables $t \to -t$

$$\begin{aligned} &\frac{\partial u}{\partial t} = Qu \quad \text{for} \quad t \geq 0 \\ &u(0,x) = f(x). \end{aligned} \tag{A.17}$$

This implies (A.15). We also have the KBE and KFE

$$\begin{aligned} &\frac{\partial p}{\partial t}(t,x) + Qp(t,x) = 0 \quad \text{for} \quad t \geq 0 \\ &p(0,y|x) = \delta(x-y), \end{aligned} \tag{A.18}$$

$$\begin{aligned} &\frac{\partial p}{\partial t}(t,y) + Q^* p(t,y) = 0 \quad \text{for} \quad t \geq 0 \\ &p(0,y|x) = \delta(x-y). \end{aligned} \tag{A.19}$$

In the KBE, y is a parameter, in KFE, x is a parameter.

A.1.3 Ergodicity for homogeneous Markov processes

By definition, a homogeneous Markov process is called **ergodic** if there exists a **unique normalized invariant measure** $\overline{p}(y)$ such that

$$Q^* \overline{p} = 0, \qquad \int \overline{p}(y) dy = 1. \tag{A.20}$$

Notice that ergodicity implies that the null space of Q is the space of constant functions, since $Q1 = 0$ as $T^t 1 = 1$. From (A.20) and (A.18), we deduce that

$$\overline{p}(x) = \int p(t_0, \xi|x) \overline{p}(\xi) d\xi, \tag{A.21}$$

for all time $t_0 \geq 0$. This justifies the notion of invariant measure. As $t \to \infty$, the density p converges to $\overline{p}$ exponentially. The spectrum of Q^* gives the rate of mixing, or of convergence.

We can now construct the inverse of an ergodic generator Q. We want to solve the problem

$$Qu = f \tag{A.22}$$

where f is a known source term. The Fredholm alternative states that this problem admits solutions if and only if

$$\mathbb{E}_\infty\{f(X)\} = \int f(y)\overline{p}(y)dy = 0, \qquad \text{i.e.} \quad f \perp \{\overline{p}\}. \tag{A.23}$$

Here, $\mathbb{E}_\infty$ is expectation with respect to the invariant measure. The solution u is defined up to a constant function. The solution orthogonal to $\overline{p}$, i.e. such that $\int \overline{p}(y)u(y)dy = 0$ is given by

$$u(y) = -\int_0^\infty e^{sQ} f ds = -\int_0^\infty T^s f ds. \tag{A.24}$$

We can summarize by saying that

$$-\int_0^\infty e^{sQ} ds : \quad D \to D$$

is the inverse of Q restricted to

$$D = \{\overline{p}\}^\perp = (\text{Null}\{Q^*\})^\perp.$$

A.2 Perturbation expansion and diffusion limit

Let us consider the ordinary differential equation

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \frac{1}{\sqrt{\varepsilon}} F\Big(\mathbf{X}_\varepsilon(t), \mathbf{Y}\big(\frac{t}{\varepsilon}\big)\Big) + G\Big(\mathbf{X}_\varepsilon(t), \mathbf{Y}\big(\frac{t}{\varepsilon}\big)\Big), \quad \mathbf{X}_\varepsilon(0) = \mathbf{X}_0 \in \mathbb{R}^d, \tag{A.25}$$

for some smooth functions $F(\mathbf{x}, \mathbf{y})$ and $G(\mathbf{x}, \mathbf{y})$. We set $\mathbf{Y}_\varepsilon(t) = \mathbf{Y}(\varepsilon^{-1}t)$. We assume that $\mathbf{Y}(t)$ is a homogeneous Markov process with infinitesimal generator Q bounded on $L^\infty(\mathcal{Y})$ and with a unique invariant measure $\pi(\mathbf{y})$ solution of

$$Q^*\pi(d\mathbf{y}) = 0. \tag{A.26}$$

Moreover we assume the existence of a spectral gap, or equivalently assume that the semigroup $T^r = e^{rQ}$ is strictly contracting on $\{\pi^\perp\}$, the set of functions f such that $\mathbb{E}_\infty\{f\} = (f, \pi) = 0$ for $(\cdot, \cdot)$ the usual inner product on $\mathcal{Y}$. The spectral gap $\alpha > 0$ is such that if $(g, \pi) = 0$, then

$$\|e^{rQ} g\|_{L^\infty(\mathcal{Y})} \leq C\|g\|_{L^\infty(\mathcal{Y})} e^{-\alpha r}.$$

This allows us to solve the following **Poisson** equation

$$Qf = g, \tag{A.27}$$

with the following *Fredholm alternative*. If $(\pi, g) \neq 0$, then the above equation admits no solution in $L^\infty(\mathcal{Y})$. If $(\pi, g) = 0$, then there exists a unique solution f such that $(\pi, f) = 0$, given by

$$f(\mathbf{y}) = -\int_0^\infty T^r g(\mathbf{y}) dr, \tag{A.28}$$

which moreover satisfies that $\|f\|_{L^\infty(\mathcal{Y})} \leq C\|g\|_{L^\infty(\mathcal{Y})}$.

Since $\mathbf{Y}(t)$ is a Markov process, then so are $\mathbf{Y}_\varepsilon(t)$ and $(\mathbf{X}_\varepsilon(t), \mathbf{Y}_\varepsilon(t))$ jointly. Let us consider moments of the form

$$u_\varepsilon(t, \mathbf{x}, \mathbf{y}) = \mathbb{E}_{(\mathbf{x},\mathbf{y})}\{f(\mathbf{X}_\varepsilon(t), \mathbf{Y}_\varepsilon(t))\}, \tag{A.29}$$

where the processes $(\mathbf{X}_\varepsilon(t), \mathbf{Y}_\varepsilon(t))$ start at $(\mathbf{x}, \mathbf{y})$ at $t = 0$, and where the function $f(\mathbf{x}, \mathbf{y})$ is smooth. We want to understand the limit of $u_\varepsilon(t, \mathbf{x}, \mathbf{y})$ as $\varepsilon \to 0$. This characterizes the limiting law of the joint process $(\mathbf{X}_\varepsilon(t), \mathbf{Y}_\varepsilon(t))$, thus of $\mathbf{X}_\varepsilon(t)$.

The analysis is carried out by using a perturbation expansion. The exposition closely follows that of [11]. The equation for u_ε is

$$\begin{aligned} &\frac{\partial}{\partial t}u_\varepsilon(t,\mathbf{x},\mathbf{y}) = \mathcal{L}_\varepsilon u_\varepsilon(t,\mathbf{x},\mathbf{y}), \\ &u_\varepsilon(0,\mathbf{x},\mathbf{y}) = f(\mathbf{x},\mathbf{y}), \end{aligned} \tag{A.30}$$

where the infinitesimal generator for $(\mathbf{X}_\varepsilon(t), \mathbf{Y}_\varepsilon(t))$ is

$$\begin{aligned} &\mathcal{L}_\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2, \\ &\mathcal{L}_0 = Q, \quad \mathcal{L}_1 = F(\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{x}}, \quad \mathcal{L}_2 = G(\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{x}}. \end{aligned} \tag{A.31}$$

We can then expand $u_\varepsilon = u_0 + \sqrt{\varepsilon}u_1 + \varepsilon u_2 + \zeta_\varepsilon$, plug this into (A.30), and equate like powers of ε. This yields

$$\begin{aligned} Qu_0 &= 0 \\ Qu_1 + \mathcal{L}_1 u_0 &= 0 \\ Qu_2 + \mathcal{L}_1 u_1 + \mathcal{L}_2 u_0 &= \frac{\partial u_0}{\partial t}. \end{aligned} \tag{A.32}$$

The first equation $Qu_0 = 0$ implies that $u_0(t,\mathbf{x})$ is independent of $\mathbf{y}$ since $Q1 = 0$. The second equation implies that

$$(\mathcal{L}_1 u_0, \pi(d\mathbf{y})) = 0,$$

which since $u_0(t,\mathbf{x})$ is independent of $\mathbf{y}$ necessitates that

$$(F(\mathbf{x},\mathbf{y}),\pi) = \int_{\mathcal{Y}} F(\mathbf{x},\mathbf{y})\pi(d\mathbf{y}) = \mathbb{E}_\infty\{F(\mathbf{x},\mathbf{y})\} = 0. \tag{A.33}$$

This is a constraint on $F(\mathbf{x},\mathbf{y})$ that we need to assume in order to obtain a limit for u_ε as $\varepsilon \to 0$. If the above average is not zero, this means that there is a drift at the scale ε that prevents one from having a non-trivial limit for u_ε as $\varepsilon \to 0$. When the constraint (A.33) is satisfied, the second equation in (A.32) admits a unique solution given by

$$u_1(t,\mathbf{x},\mathbf{y}) = \int_0^\infty e^{rQ}F(\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{x}}u_0(t,\mathbf{x})dr. \tag{A.34}$$

The third equation in (A.32) admits solutions only if the following compatibility condition holds:

$$\Big(\pi, \mathcal{L}_1 u_1 + \mathcal{L}_2 u_0 - \frac{\partial u_0}{\partial t}\Big) = 0. \tag{A.35}$$

More explicitly, we have

$$\frac{\partial u_0}{\partial t} = \int_{\mathcal{Y}} F(\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{x}}\int_0^\infty e^{rQ}F(\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{x}}u_0(t,\mathbf{x})dr\pi(d\mathbf{y}) + \int_{\mathcal{Y}} G(\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{x}}u_0(t,\mathbf{x})\pi(d\mathbf{y}).$$

Using the fact that $e^{rQ}F = T^r F = \mathbb{E}_\infty\{F(t+s)|F(t) = F\}$, this can be recast as

$$\begin{aligned} &\frac{\partial u_0}{\partial t} = \frac{1}{2}a_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j}u_0 + b_k(\mathbf{x})\frac{\partial}{\partial x_k}u_0, \\ &u_0(t,\mathbf{x}) = f(\mathbf{x}), \end{aligned} \tag{A.36}$$

with summation over repeated indices, where we have

$$\begin{aligned} \frac{1}{2}a_{ij}(\mathbf{x}) &= \int_0^\infty \mathbb{E}_\infty\big\{F_i(\mathbf{x},\mathbf{y}(0))F_j(\mathbf{x},\mathbf{y}(s))\big\}ds, \\ b_k(\mathbf{x}) &= \sum_{j=1}^d \int_0^\infty \mathbb{E}_\infty\big\{F_j(\mathbf{x},\mathbf{y}(0))\frac{\partial F_k}{\partial x_j}(\mathbf{x},\mathbf{y}(s))\big\}ds + \mathbb{E}_\infty\{G_k(\mathbf{x},\mathbf{y})\}. \end{aligned} \tag{A.37}$$

The limiting equation for $u_0(t,\mathbf{x})$ is thus a diffusion equation. The matrix a, which as we can verify is symmetric and positive definite, can be written as $a = \sigma^2$ (this expression is the reason for the factors 1/2 in (A.36) and (A.37)). Then the right-hand side in (A.36) is the infinitesimal generator of a diffusion Markov process satisfying the following *stochastic ordinary differential* equation (in the Itô sense):

$$d\mathbf{X} = b(\mathbf{x})dt + \sigma(\mathbf{x})d\mathbf{W}_t, \tag{A.38}$$

where $\mathbf{W}_t$ is $d-$ dimensional Brownian motion.

Theorem A.2.1 *For $f(\mathbf{x})$ smooth, let $u_\varepsilon(t,\mathbf{x},\mathbf{y})$ be the solution of* (A.30) *with initial condition $u_\varepsilon(0,\mathbf{x},\mathbf{y}) = f(\mathbf{x})$, and let $u(t,\mathbf{x})$ be the solution of the limiting equation* (A.36) *with the same initial conditions. Then provided that the functions F and G are sufficiently smooth, we have*

$$|u_\varepsilon(t,\mathbf{x},\mathbf{y}) - u(t,\mathbf{x})| = O(\sqrt{\varepsilon}), \quad 0 \le t \le T < \infty, \tag{A.39}$$

uniformly in $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \mathcal{Y}$.

Exercise A.2.1 Prove the above theorem assuming that the solutions to (A.36) and to (A.27) are sufficiently smooth and that the operator $\partial_t - \mathcal{L}_\varepsilon$ satisfies a maximum principle. Follow the steps of the proof of Thm. 2.1.1.

The above theorem shows that

$$\mathbb{E}_{\mathbf{x},\mathbf{y}}\{f(\mathbf{X}_\varepsilon(t))\} \to \mathbb{E}_{\mathbf{x}}\{f(\mathbf{X}(t)\}, \quad \varepsilon \to 0, \tag{A.40}$$

for all $0 \le t \le T < \infty$, where $\mathbf{X}(t)$ solves (A.38). It is not quite sufficient to show the weak convergence of $\mathbf{X}_\varepsilon$ to $\mathbf{X}$ as a measure on the space of paths $\mathcal{C}([0,T];\mathbb{R}^d)$, which requires a better control of the regularity in time of the convergence. The proof of weak convergence in $\mathcal{C}([0,T];\mathbb{R}^d)$ goes beyond the scope of these notes and we refer to [11] for additional details.

A.3 Remarks on stochastic integrals

In the preceding section we saw that the limiting process solved the following stochastic ordinary differential equation

$$d\mathbf{X} = b(\mathbf{x})dt + \sigma(\mathbf{x})d\mathbf{W}_t, \tag{A.41}$$

where $\mathbf{W}_t$ is $d-$ dimensional Brownian motion. Brownian motion W_t is defined as the stationary Markov process with transition probability density, which for any starting point $\mathbf{W}_0 = \mathbf{y} \in \mathbb{R}^d$, is given by

$$p(t,\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi t)^{d/2}} \exp\big(-\frac{|\mathbf{x}-\mathbf{y}|^2}{2t}\big), \qquad \mathbf{x} \in \mathbb{R}^d,\ t > 0. \tag{A.42}$$

This defines a stochastic process [12] and the process can be chosen to be continuous, i.e., in $\mathcal{C}([0,\infty);\mathbb{R}^d)$, in the sense that there is a *version* of $\mathbf{W}_t$ that is continuous. (X_t is a version of Y_t if $P(\{\omega; X_t(\omega) = Y_t(\omega)\}) = 1$).

Important properties of Brownian motion are:

(i) $\mathbf{W}_t$ is Gaussian, i.e., for all $(t_1, \cdots, t_k)$, $(\mathbf{W}_{t_1}, \cdots, \mathbf{W}_{t_k})$ has a multi-normal distribution. Among other, this shows that

$$\mathbb{E}^{\mathbf{x}}\{\mathbf{W}_t\} = \mathbf{x}, \qquad t > 0, \tag{A.43}$$

where $\mathbb{E}^{\mathbf{x}}$ denotes mathematical expectation for a Brownian motion stating at $\mathbf{W}_0 = \mathbf{x}$. Also we have

$$\mathbb{E}^{\mathbf{x}}\{|\mathbf{W}_t - \mathbf{x}|^2\} = dt, \qquad \mathbb{E}^{\mathbf{x}}\{(\mathbf{W}_t - \mathbf{x}) \cdot (\mathbf{W}_s - \mathbf{x})\} = d \min(s,t). \tag{A.44}$$

The above relations imply that

$$\mathbb{E}^{\mathbf{x}}\{|\mathbf{W}_t - \mathbf{W}_s|^2\} = n(t-s), \qquad \text{if } t \geq s. \tag{A.45}$$

(ii) $\mathbf{W}_t$ has independent increments, i.e.

$$\mathbf{W}_{t_1},\ \mathbf{W}_{t_2} - \mathbf{W}_{t_1},\ \cdots,\ \mathbf{W}_{t_k} - \mathbf{W}_{t_{k-1}}, \tag{A.46}$$

are independent for all $0 \leq t_1 < t_2 < \ldots < t_k$. This is a consequence of the Gaussian character of $\mathbf{W}_t$.

Note that $\mathbf{W}_t$ can be chosen continuous. However it can be shown that the continuous version is nowhere differentiable (almost surely). The notion of $d\mathbf{W}_t$ therefore requires to be interpreted more carefully. A logical interpretation of the stochastic equation (A.41) is that its solution satisfies the integral equation:

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t b(s, \mathbf{X}_s) ds + \int_0^t \sigma(s, \mathbf{X}_s) d\mathbf{W}_s. \tag{A.47}$$

The first integral $\int_0^t b(s, \mathbf{X}_s) ds$ can be given the usual sense for sufficiently smooth functions $b(s, \mathbf{X}_s)$. The last integral $\int_0^t \sigma(s, \mathbf{X}_s) d\mathbf{W}_s$ however requires interpretation. It can actually be given several non equivalent meanings. Let us generalize the integral to

$$\int_0^t f(s, \omega) d\mathbf{W}_s, \tag{A.48}$$

for $f(s)$ a (matrix-valued) stochastic process. Let (t_k, t_{k+1}) be an interval. Then we have the reasonable **definition**

$$\int_{t_k}^{t_{k+1}} d\mathbf{W}_s = \mathbf{W}_{t_{k+1}} - \mathbf{W}_{t_k}. \tag{A.49}$$

Let $t_1 = 0 < t_1 < \cdots < t_{k+1} = t$ be a partition of $(0,t)$. Then as Riemann sums are used to define Riemann integrals, we can approximate (A.48) as

$$\sum_{m=1}^{k} f(t_m^*)\big(\mathbf{W}_{t_{m+1}} - \mathbf{W}_{t_m}\big), \tag{A.50}$$

where $t_m \leq t_m^* \leq t_{m+1}$. Here however there is a big surprise. The choice of t_m^* matters. The main reason is that $\mathbf{W}_{t_{m+1}} - \mathbf{W}_{t_m}$ has oscillations that on average are of order $\sqrt{t_{m+1} - t_m}$, which is much larger than $t_{m+1} - t_m$, since the latter is equal to the expectation of the *square* of $\mathbf{W}_{t_{m+1}} - \mathbf{W}_{t_m}$. Two choices are famous (with names):

$$\begin{aligned} t_m^* &= t_m && \text{Itô sense,} \\ t_m^* &= \frac{t_m + t_{m+1}}{2} && \text{Stratonowich sense.} \end{aligned} \tag{A.51}$$

As the partition gets more refined and all $t_{m+1} - t_m$ tend to 0 uniformly, the above approximations (A.50) admit limits, which may be *different* depending on the choice of t_m^*. The accepted notation for the limiting integral is

$$\begin{array}{ll} \int_0^t f(s,\omega) d\mathbf{W}_s & \text{Itô integral,} \\ \int_0^t f(s,\omega) \circ d\mathbf{W}_s & \text{Stratonowich integral.} \end{array} \tag{A.52}$$

See [25] for examples of processes $f(t,\omega)$ for which the two definitions of the integrals provide very different results. In the mathematical literature, the Itô integral appears much more frequently. The reason is that the resulting integral is a *martingale*, for which a great deal of estimates are available. The Stratonowich integral may make more sense physically in cases where there is no reason to privilege one direction of time versus the other (forward or backwards). The Itô choice sums terms of the form $f(t_m)(\mathbf{W}_{t_{m+1}} - \mathbf{W}_{t_m})$, where both terms in the product are independent (because Brownian motion has independent increments). If such an independence is deemed correct (such as may be in the stock market), then the Itô choice makes sense. In more symmetric processes, the Stratonowich choice makes sense. Note that although these integrals provide different answers to the meaning of (A.48), we can go from one to the other by relatively simple calculus [25].

A.4 Diffusion Markov Process Limit

The limit theorem obtained in section A.2 may be generalized as follows. Let $q(t)$ be a homogeneous ergodic Markov process in a state space S with infinitesimal generator Q and invariant measure $\bar{p}(q)$. Let $0 < \varepsilon \ll 1$ be a small parameter. Let $F(t,\xi,q,x)$ and $G(t,\xi,q,x)$, from $\mathbb{R} \times \mathbb{R} \times S \times \mathbb{R}^d$ to $\mathbb{R}^d$ be smooth functions in x such that

$$\mathbb{E}_\infty[F(t,\xi,q,x)] = \int_S F(t,\xi,q,x) d\bar{p}(x) = 0.$$

Let $\tau(t)$ be a smooth function from $[0,\infty)$ to $[0,\infty)$ such that $\tau'(t) > 0$.

Let $X^\varepsilon(t)$ satisfy the ODE

$$\begin{aligned} &\frac{dX^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} F(t, \frac{\tau(t)}{\varepsilon}, q(\frac{t}{\varepsilon^2}), X^\varepsilon(t)) + G(t, \frac{\tau(t)}{\varepsilon}, q(\frac{t}{\varepsilon^2}), X^\varepsilon(t)) \\ &X^\varepsilon(0) = X_0. \end{aligned}$$

Then $X^\varepsilon(\cdot)$ converges weakly to a *Diffusion Markov Process* with infinitesimal generator $\bar{Q}$ given by

$$\bar{Q} = \langle \mathbb{E}_\infty[F. \cdot \nabla_x \int_0^\infty ds e^{sQ} F \cdot \nabla_x] \rangle_\xi + \langle \mathbb{E}_\infty[G] \rangle_\xi \cdot \nabla_x.$$

This expression can be simplified by remarking that

$$e^{sQ} F = T^s F = \mathbb{E}[F(t+s) | F(t) = F]$$

independently of t, so that

$$\mathbb{E}_\infty[F \cdot \nabla_x \int_0^\infty ds\, e^{sQ} F \cdot \nabla_x] = \int_0^\infty ds\, \mathbb{E}_\infty[F_j(t,\xi,q(t_0),X) \frac{\partial}{\partial x_j} F_k(t,\xi,q(t_0+s),x) \frac{\partial}{\partial x_k}].$$

This implies that

$$\bar{Q} = \frac{1}{2} a_{jk}(t,x) \frac{\partial^2}{\partial x_j \partial x_k} + b_k(t,x) \frac{\partial}{\partial x_k},$$

where we use summation over repeated indices and

$$\frac{1}{2} a_{jk}(t,x) = \Big\langle \int_0^\infty \mathbb{E}_\infty [F_j(t,\xi,q(t_0),x) F_k(t,\xi,q(t_0+s),x)] ds \Big\rangle_\xi$$
$$b_k(t,x) = \big\langle \mathbb{E}_\infty [G_j(t,\xi,q(t_0),x)] \big\rangle_\xi + \Big\langle \int_0^\infty \mathbb{E}_\infty [F_j(t,\xi,q(t_0),x) \frac{\partial F_k}{\partial x_j}(t,\xi,q(t_0+s),x)] ds \Big\rangle_\xi .$$

It is therefore the generator of a diffusion Markov process X satisfying the Itô equation

$$dX = b(t,x)dt + \sigma(t,x) d\beta_t,$$

where σ is such that $a_{jk} = \sigma_{jl}\sigma_{kl}$.

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