Waves and Linear Transport Theory with Applications

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1 Introduction to the Transport Equations

1.1 Movement of one particle

A particle is characterized by its position $X(t)$ and its momentum $P(t)$, which are functions of the time t. In classical mechanics, $P(t) = mV(t)$, where m is the mass of the particle and $V(t)$ its velocity. Moreover, **Newton's laws** apply:

$$\frac{dX(t)}{dt} = \frac{P(t)}{m} = V(t) \quad \text{and} \quad \frac{dP(t)}{dt} = F(X(t)) = -\nabla U(X(t)), \tag{1}$$

where

$$\nabla U = \Big(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \frac{\partial U}{\partial x_3}\Big)^t, \qquad x = (x_1, x_2, x_3).$$

Here F is the force imposed on the particle at position X. We assume that the force derives from a potential U. We specify the initial conditions $X(0) = X_0 \in \mathbb{R}^3$ and $P(0) = P_0 \in \mathbb{R}^3$. Introducing the Hamiltonian of the particle in the force field F

$$H(x,p) = \frac{|p|^2}{2m} + U(x), \tag{2}$$

we obtain **Hamilton's equations**:

$$\left\{\begin{aligned} \frac{dX(t)}{dt} &= \frac{\partial H}{\partial p}(X(t),P(t)) \\ \frac{dP(t)}{dt} &= -\frac{\partial H}{\partial x}(X(t),P(t)) \end{aligned}\right. \quad \text{with} \quad \left\{\begin{aligned} X(0) &= X_0 \\ P(0) &= P_0. \end{aligned}\right. \tag{3}$$

For regular potentials U, the Cauchy problem for this system of two ordinary differential equations is well posed. Its solution gives the trajectory of the particle $t \mapsto X(t)$.

1.2 Equation for the density of particles

When the number of particles becomes large, we replace the particles by a probability of presence at a given point (x, v) in phase space. Let us denote by

$$a(t,x,v) = \textit{ phase space density} \text{ function}, \tag{4}$$

which characterizes the expected particle density at point $(x, v) \in \Omega \times K$ in phase space and time t. Here Ω is the physical domain and K the set of velocities. They are subsets of $\mathbb{R}^3$. Let $d\mathcal{V} = dx\, dv$ be a unit volume in phase space. We have

$$a(t,x,v)d\mathcal{V} = \text{expected number of particles in } d\mathcal{V} \text{ about } (x,v) \text{ at time } t. \tag{5}$$

In the absence of sources and absorption, the number of particles, and the volume they occupy, remain constant in time. Therefore, we obtain

$$\frac{d}{dt}\Big[a(t,X(t),V(t))d\mathcal{V}\Big] = \frac{d}{dt}\Big[a(t,X(t),V(t))\Big]\, d\mathcal{V} = 0. \tag{6}$$

Assuming that the force field $F = -\nabla U$ is known, we deduce from (2) and (3) that (6) can be rewritten as the **Liouville Equation**:

$$\partial_t a(t,x,v) + v\cdot\nabla_x a(t,x,v) - \frac{1}{m}\nabla_x U\cdot\nabla_v a(t,x,v) = 0. \tag{7}$$

When absorption and sources are present, the balance equation becomes

$$\frac{d}{dt}[a(t,X(t),V(t))] = [\text{gains}(t,X(t),V(t))] - [\text{losses}(t,X(t),V(t))]$$

where *gains*(t, x, v) and *losses*(t, x, v) are the density of created and absorbed particles at point (x, v) and time t respectively. These terms characterize the interaction between the particles and the underlying medium. In the linear theory, it is assumed that particles do not interact with one another.

The interactions with the medium are characterized by macroscopic cross sections. The *mean free path* (mfp) describes the mean distance traveled by a particle before interacting with the medium, i.e. before changing position or direction, or being destroyed. It is related to the *total scattering cross section* $\Sigma_t(x, v)$ by

$$(mfp)^{-1} \equiv \Sigma_t(x,v) = \begin{array}{l}\text{probability of interaction per unit distance}\\ \text{traveled by particles at point } (x,v).\end{array} \tag{8}$$

The distance traveled by a particle per unit time is $dx = |v|\,dt$; therefore the frequency of collision is given by $|v|\Sigma_t(x, v)$. The *reaction rate density* describing the number of particles interacting with the medium per unit volume $\mathcal{V}$ and unit time dt is then

$$\text{losses}(t,x,v) \;=\; |v|\Sigma_t(x,v)a(t,x,v). \tag{9}$$

When particles interact with the medium, secondary particles may be emitted in a different direction. Their scattering probability function is defined by

$$f(x,v' \to v)dv = \begin{array}{l}\text{probability that a secondary particle induced by an}\\ \text{incident particle at } (x,v') \text{ be emitted with velocity}\\ \text{in } dv \text{ around } v.\end{array} \tag{10}$$

The number of secondary particles emitted by a collision at (x, v) is given by $c(x, v)$. The *scattering cross section* Σ_s is defined by

$$\Sigma_s(x,v' \to v) = \Sigma_t(x,v')c(x,v')f(x,v' \to v). \tag{11}$$

It describes the number of particles emitted in direction v after an interaction at point (x, v').

A scattering event is characterized by a loss of particles at point (x, v') and a gain at point (x, v). A particle interacting with the medium is either emitted or absorbed. Let us denote by $\Sigma_a(x, v)$, which is also called the real *absorption cross section*, the probability that a particle interact in a unit distance dx at point (x, v) and be not reemitted. Then we have

$$\Sigma_t(x,v) = \Sigma_a(x,v) + \int \Sigma_s(x,v \to v')dv'.$$

More precisely, we obtain using (11) that

$$\Sigma_a(x,v) = (1 - c(x,v))\Sigma_t(x,v). \tag{12}$$

We denote by $s(t, x, v)$ the density of particles created at point (x, v) and time t by a different mechanism than reemission of scattered particles. The gains are given by

$$\text{gains}(t,x,v) \;=\; \int |v'|\Sigma_s(x,v' \to v)a(t,x,v')dv' \;+\; s(t,x,v). \tag{13}$$

We deduce that the general form of the transport equation is

$$\partial_t a + v \cdot \nabla_x a - \frac{1}{m}\nabla_x U \cdot \nabla_v a + |v|\Sigma_t a = \int |v'|\Sigma_s(x, v' \to v) a(t, x, v') dv' + s. \qquad (14)$$

For transport problems posed in the full space $I\!R^3$, we only need to prescribe the initial value

$$a(t = 0, x, v) = a_0(x, v). \qquad (15)$$

For transport problems posed in domains with boundaries, boundary conditions must also be specified. Let be Ω a domain in $I\!R^3$ with a regular boundary $\partial\Omega$. We need to prescribe the amount of particles entering the physical domain Ω. Let be $x_s \in \partial\Omega$. The incoming boundary conditions are given for each time t by

$$a(t, x_s, v) = g(t, x_s, v) \quad \text{on } \Gamma_- = \{(x_s, v) \in \partial\Omega \times K \text{ such that } v \cdot n(x_s) < 0\}. \qquad (16)$$

Here, $n(x_s)$ is the outward normal at point x_s, and g is a given function. We have absorbing boundary conditions when $g = 0$. In general, b may depend on the outgoing flux $a(t, x_s, v)$ for $v \cdot n(x_s) > 0$. We shall come back later to these more complicated boundary conditions.

We consider in this course neutral particles, and assume that the force field $F = -\nabla U$ has a negligible influence. Hence we set $U = 0$. It is convenient for the mathematical analysis to perform the change of variables:

$$\begin{aligned} u(t, x, v) &= a(t, x, v) & q(t, x, v) &= s(t, x, v) \\ \Sigma(x, v) &= |v|\Sigma_t(x, v) & \sigma(x, v', v) &= |v'|\Sigma_s(x, v' \to v). \end{aligned} \qquad (17)$$

We also denote by $\sigma_a = |v|\Sigma_a$ the absorption cross section. With these new variables, the transport equation is given, for all $\tau > 0$, by

$$\begin{aligned} &\partial_t u + v \cdot \nabla_x u + \Sigma u = \int \sigma(x, v', v) u(t, x, v') dv' + q \quad \text{in } (0, \tau) \times \Omega \times K \\ &u(0, x, v) = u_0(x, v) \\ &u(t, x, v) = g(t, x, v) \quad \text{on } (0, \tau) \times \Gamma_-. \end{aligned} \qquad (18)$$

1.3 Integral Formulation

We consider here only vanishing potentials. Hence the trajectories of the particles, which are the characteristics of the transport equation, are now straight lines. We recall that the characteristics $(X(t), V(t))$ are solutions of

$$\left\{ \begin{array}{l} \dfrac{dX(t)}{dt} = V(t) \\ X(t_0) = x \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \dfrac{dV(t)}{dt} = 0 \\ V(t_0) = v. \end{array} \right. \qquad (19)$$

Therefore $V(t) = v$ and $X(t) = x + (t - t_0)v$. Let us denote by

$$S(t,x,v) = \int \sigma(x,v',v)u(t,x,v')dv' + q(t,x,v). \tag{20}$$

The form of the characteristics implies that for every $t_0 \in I\!R$:

$$\frac{d}{dt}u(t,x+(t-t_0)v,v) + (\Sigma u)(x+(t-t_0)v,v) = S(t,x+(t-t_0)v,v). \tag{21}$$

Let us define, for $(x,v) \in \Omega \times K$,

$U(t) = u(t,x+(t-t_0)v,v)$, $\tilde{\Sigma}(t) = \Sigma(x+(t-t_0)v,v)$ and $\tilde{S}(t) = S(t,x+(t-t_0)v,v)$.

Then (21) can be recast as

$$\frac{dU}{dt}(t) + \tilde{\Sigma}(t)U(t) = \tilde{S}(t). \tag{22}$$

One verifies that the solution of this equation is given for every $t_1 \le t$ by

$$\begin{aligned} U(t) &= U(t_1)\exp[-\int_{t_1}^t \tilde{\Sigma}(s)ds] + \int_{t_1}^t \tilde{S}(s)\exp[-\int_s^t \tilde{\Sigma}(\tau)d\tau]ds \\ &= U(t_1)\exp[-\int_0^{t-t_1} \tilde{\Sigma}(s-t_1)ds] + \int_0^{t-t_1} \tilde{S}(t-s)\exp[-\int_0^s \tilde{\Sigma}(s-\tau)d\tau]ds. \end{aligned} \tag{23}$$

Let us assume that Ω is a convex bounded domain of $I\!R^3$. We define the *travel time* for all $(x,v) \in \Omega \times K$ by

$$t(x,v) = \sup\{t, x - sv \in \Omega \text{ for all } 0 \le s < t\}, \tag{24}$$

i.e. the time needed by a particle to travel from position x at time $t = 0$ to the boundary $\partial\Omega$. The density u is not defined outside Ω. Therefore we can write $u(t,x,v) = u(t,x,v)Y(t(x,v) - t)$ where Y is the **Heaviside** function, i.e.

$$Y(s) = 0 \text{ if } s < 0 \qquad \text{and} \qquad Y(s) = 1 \text{ if } s > 0.$$

By linearity of the transport equation, we can treat separately the initial source term u_0 and the boundary source term g. Let us first assume that $g = 0$. We have

$$U(0) = u_0(x - vt_0, v)Y(t(x,v) - t_0).$$

We deduce from (23) with $t = t_0$ that

$$\begin{aligned} u(t,x,v) = \;& u_0(x-vt,v)Y(t(x,v)-t)\exp[-\int_0^t \Sigma(x-vs,v)ds] \\ &+ \int_0^t \exp[-\int_0^s \Sigma(x-v\tau,v)d\tau]S(t-s,x-vs,v)Y(t(x,v)-s)ds. \end{aligned}$$

Assume now that $u_0 = 0$ and $S = 0$. The influence of g on $u(t, x, v)$ is obtained by choosing $t_0 = t - t(x, v)$ in (23), which corresponds to the time needed by the particle to go from $x - t(x, v)v \in \partial\Omega$ to the point x. We obtain that

$$u(t, x, v) = Y(t - t(x, v)) \exp[\int_0^{t(x,v)} -\Sigma(x - sv, v)ds] g(t - t(x, v), x - t(x, v)v, v).$$

Summarizing the two precedent results, we obtain that the solution of (18) is also a solution of the **Integral Equation**

$$\begin{aligned} u(t, x, v) = \;& u_0(x - vt, v)Y(t(x, v) - t) \exp[-\int_0^t \Sigma(x - vs, v)ds] \\ &+ \int_0^t \exp[-\int_0^s \Sigma(x - v\tau, v)d\tau]S(t - s, x - vs, v)Y(t(x, v) - s)ds \\ &+ Y(t - t(x, v)) \exp[\int_0^{t(x,v)} -\Sigma(x - sv, v)ds]\, g(t - t(x, v), x - t(x, v)v, v), \end{aligned} \tag{25}$$

where S is given by (20).

1.4 Adjoint Equation

It is sometimes useful to consider an adjoint formulation of transport. Let us denote by

$$A(u) \;=\; -v \cdot \nabla_x u - \Sigma u + \int \sigma(x, v', v)u(t, x, v')dv'. \tag{26}$$

We assume to simplify that $\Omega = \mathbb{R}^3$ and that we have no other sources than those given by the initial condition, i.e. $q = 0$ and $g = 0$ with the notations of section 1.3. Then problem (18) reads

$$\begin{aligned} \frac{du}{dt} &= Au \\ u(0) &= u_0. \end{aligned} \tag{27}$$

We define the **adjoint equation** by

$$\begin{aligned} \frac{dw}{dt} &= A^* w \\ w(0) &= w_0. \end{aligned} \tag{28}$$

where A^* is the adjoint operator to A. Let us denote by $(\cdot, \cdot)$ the L^2 scalar product in $\Omega \times K$, defined for every functions $f, g \in L^2(\Omega \times K)$ by

$$(f, g) = \int_{\Omega \times K} f(x, v)g(x, v)dx\, dv. \tag{29}$$

By definition of the adjoint operator, we have $(Au, w) = (u, A^* w)$. Integrations by parts show that

$$A^*(w) \;=\; v \cdot \nabla_x w - \Sigma w + \int \sigma(x, v, v')w(t, x, v')dv' \tag{30}$$

and the adjoint transport equation is given by

$$\begin{aligned}
&\partial_t w - v\cdot\nabla_x w + \sigma_a w = \int \sigma(x,v,v')(w(t,x,v') - w(t,x,v))dv' \quad \text{in } (0,\tau)\times\Omega\times K\\
&w(0,x,v) = w_0(x,v).
\end{aligned} \tag{31}$$

The adjoint transport equation also admits an integral formulation. Replacing v by $-v$ in the definition of the characteristics (19), and (v,v') by (v',v) in the definition of the scattering cross section (17), we obtain that w is a solution of

$$\begin{aligned}
w(t,x,v) = {} & w_0(x+vt,v)\exp[-\int_0^t \Sigma(x+sv,v)]\\
& + \int_0^t ds \exp[-\int_0^s \Sigma(x+\tau v,v)d\tau]\int_K \sigma(x+sv,v,v')w(t-s,x+sv,v')dv'.
\end{aligned} \tag{32}$$

The solution of the adjoint equation is often referred to as the **importance function**. The reason comes from the following property

Proposition 1.1 *Let be u and w smooth solutions of (27) and (28) respectively for $t\in[0,T]$. Then we have*

$$(u_0, w(T)) \;=\; (u(T), w_0). \tag{33}$$

Assume now that w_0 describes a detector of particles. For instance a perfect detector located at position x_0 and detecting only particles with velocity v_0 would be modeled by $w_0(x,v) = \delta_{x_0}\times\delta_{v_0}$. The number of particles detected at time T by the detector w_0 is

$$(u(T), w_0),$$

which is precisely $(u_0, w(T))$ according to the above proposition. We see $w(T)$ can be interpreted as an importance function. It tells us how many particles will be detected at time T by the detector knowing the initial distribution u_0. Notice that the detector is characterized by the solution of the adjoint problem w. The detection corresponding to several initial source distributions is then obtained by solving only one adjoint equation, instead of solving as many forward problems as there are initial conditions. The proof of the proposition is an easy consequence of the properties of the adjoint operator:

Proof Since $(Au,w) = (u,A^*w)$, we have

$$\begin{aligned}
\Big(\frac{du}{dt}(t), w(T-t)\Big) &= (Au(t), w(T-t)) = (u(t), A^*w(T-t))\\
&= \Big(u(t), \frac{dw}{dt}(T-t)\Big) = -\Big(u(t), \frac{d[w(T-t)]}{dt}\Big).
\end{aligned}$$

This implies that $\frac{d}{dt}(u(t), w(T-t)) = 0$. Integrating this equality between 0 and T yields the result. □

1.5 Probabilistic Interpretation and Monte Carlo Methods

We show in this section that the solution of the transport equation can be seen as the expectation of a random process. The random process simulates the behavior of one particle in the medium.

As in the previous section, we assume to simplify that $\Omega = \mathbb{R}^3$ and that we only have initial conditions, i.e. $q = 0$ and $g = 0$ with the notations of section 1.3. We also assume that there is no absorption, $\Sigma_a(x, v) = 0$ or $c(x, v) = 1$.

The particle is characterized by its position $X(t)$ and its velocity $V(t)$. It travels along straight lines until it interacts with the medium. Since there is no absorption, the particle is emitted into a different direction and with a different velocity. This process goes on until final time T.

We have already described the law of reemission into a different velocity after an interaction. For every particle interacting with the medium at (x, v), the probability per unit distance traveled of being emitted into dv about direction v is $f(x, v' \to v)dv$. Since $c(x, v) = 1$, the probability per unit time of being emitted into dv about direction v is found to be $\frac{\sigma(x,v',v)dv}{\Sigma(x,v')}$.

It remains to define the law of free travel through the medium. By definition, a particle travels freely as long as it does not interact with the medium. Let us denote by

$$P_{x_0,v_0}(\tau_1 > t) = \begin{array}{l}\text{probability that a particle starting from } (x_0, v_0)\\ \text{at time } t = 0 \text{ had no collision at time } t.\end{array} \tag{34}$$

Here τ_1 denotes the time of first collision. The frequency of collision is described by $\Sigma = |v|\Sigma_t$. Therefore we have that

$$\begin{aligned} P_{x_0,v_0}(\tau_1 > t + dt) &= P_{x_0,v_0}(\tau_1 > t) - P_{x_0,v_0}(\tau_1 \in dt|\tau_1 > t) \\ &= P_{x_0,v_0}(\tau_1 > t)(1 - \Sigma(x_0 + tv_0, v_0)dt). \end{aligned}$$

This implies that

$$\frac{d}{dt} \ln P_{x_0,v_0}(\tau_1 > t) = -\Sigma(x_0 + tv_0, v_0)$$

or equivalently, since $P_{x_0,v_0}(\tau_1 > 0) = 1$,

$$P_{x_0,v_0}(\tau_1 > t) = \exp[-\int_0^t \Sigma(x_0 + sv_0, v_0)ds]. \tag{35}$$

The law of interaction with the medium is then described by the probability density

$$P_{x_0,v_0}(\tau_1 = t) = \Sigma(x_0 + tv_0, v_0) \exp[-\int_0^t \Sigma(x_0 + sv_0, v_0)ds]. \tag{36}$$

We shall now use the integral formulation of transport to obtain a probabilistic interpretation of the adjoint transport equation. Let be $(X(t), V(t))$ a random process defined as follows. Between two successive shocks, the particles travel along straight lines:

$$\frac{dX(t)}{dt} = V(t) \quad \text{and} \quad \frac{dV(t)}{dt} = 0.$$

The probability density of interaction is given by (36) and the law of change of direction is governed by $\frac{\sigma(x,v,v')dv'}{\Sigma(x,v)}$. One can prove in a very general setting that the random process $Z(t) = (X(t), V(t))$ is Markovian. Roughly speaking, the Markovian property means that the future of the process depends on the past only through the present.

Let us define $z = (x, v)$ and

$$u(t,x,v) = E[u_0(Z(t))/Z(0) = z], \tag{37}$$

which is the conditional expectation of $u_0(X(t), V(t))$ knowing that the particle started at $z = (x, v)$ at time $t = 0$. We can split this expectation in two components, corresponding to whether an interaction occurred or not:

$$\begin{aligned} u(t,x,v) = \quad & E[u_0(Z(t))/Z(0) = z, \tau_1 > t]\, P(\tau_1 > t) \\ & + E[u_0(Z(t))/Z(0) = z, \tau_1 < t]\, P(\tau_1 < t). \end{aligned}$$

Here P stands for $P_{x,v}$ and τ_1 stands for the time of first interaction. The first term is obtained by remarking that the particle traveled freely. With probability $P(\tau_1 > t)$, the position of the particle at time t is $(x + vt, v)$. Therefore, we have

$$E[u_0(Z(t))/Z(0) = z, \tau_1 > t]P(\tau_1 > t) = u_0(x + vt, v)\exp[-\int_0^t \Sigma(x + sv, v_0)].$$

The second term is computed as follows:

$$\begin{aligned} & E[u_0(Z(t))/Z(0) = z, \tau_1 < t]P(\tau_1 < t) \\ = & \int_0^t dsP(\tau_1 \in ds)E[u_0(Z(t))/Z(s) = (x + sv, v)\, \tau_1 = s] \\ = & \int_0^t dsP(\tau_1 \in ds)\int_K P(dv')E[u_0(Z(t))/Z(s) = (x + sv, v),\ \tau_1 = s,\ v(s^+) = v'] \\ = & \int_0^t dse^{-\int_0^s \Sigma(x+\tau v, v)d\tau}\int_K \sigma(x + sv, v, v')E[u_0(Z(t - s))/Z(0) = (x + sv, v')]dv' \\ = & \int_0^t dse^{-\int_0^s \Sigma(x+\tau v, v)d\tau}\int_K \sigma(x + sv, v', v)u(t - s, x + sv, v')dv'. \end{aligned}$$

Here $v(s^+)$ denotes the velocity right after a jump that occurs at time s. Hence u is a solution of the adjoint transport equation (32). The same techniques can be extended to account for volume source terms, boundary conditions and absorption. It is also possible to derive a general probabilistic interpretation of the direct transport equation (18).

Actually, the direct and adjoint equations have analog probabilistic interpretations in many practical cases. Indeed, let us consider that $\sigma(x, v', v) = \sigma(x, -v, -v')$, which is true for isotropic scattering for instance, i.e. $\sigma(x, v', v) = \sigma(x, v' \cdot v)$. Defining the modified process $\tilde{Z}(t) = (\tilde{X}(t), \tilde{V}(t))$ by

$$\frac{d\tilde{X}(t)}{dt} = -\tilde{V}(t) \quad \text{and} \quad \frac{d\tilde{V}(t)}{dt} = 0$$

and denoting by $\tilde{u}(t,x,v) = E[u_0(\tilde{Z}(t))/\tilde{Z}(0) = z]$, we readily check that $\tilde{u}$ is a solution of the direct transport equation (18).

The probabilistic interpretation is suitable for numerical calculations. Let be N realizations $Z_n(t)$, $1 \leq n \leq N$ of the random process Z defined above with initial condition

$$Z_n(0) = (x,v).$$

Then according to the law of large number, we have

$$u(t,x,v) = E[u_0(Z(t))] = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} u_0(Z_i(t)).$$

2 From Schrödinger to Transport equations

2.1 Schrödinger Equation and Wigner Transform

The Schrödinger equation is given by

$$\begin{aligned} & i\partial_t\psi(t,x) + \tfrac{1}{2}\Delta\psi(t,x) - U(x)\psi(t,x) = 0 \\ & \psi(0,x) = \psi_0(x). \end{aligned} \tag{38}$$

Here $U(x)$ is a given real potential. The scaling corresponding to the high frequency regime is

$$t \mapsto \frac{t}{\varepsilon} \quad \text{and} \quad x \mapsto \frac{x}{\varepsilon}.$$

The potential U is split in two parts

$$U(x) = U_0(x) + \varepsilon^{1/2} U_1(\frac{x}{\varepsilon}),$$

where U_0 corresponds to the slowly varying component of the potential and U_1 the rapidly varying one. The constant $\varepsilon^{1/2}$ describes the strength of the fast scale fluctuations.

In the high frequency regime, the solutions of the Schrödinger equation are localized in space and we can obtain an approximate equation for the probability density $|\psi|^2$. This is done mathematically by analyzing the behavior as $\varepsilon \to 0$ of the rescaled Schrödinger equation

$$\begin{aligned} & i\varepsilon\partial_t\psi_\varepsilon(t,x) + \tfrac{\varepsilon^2}{2}\Delta\psi_\varepsilon(t,x) - (U_0(x) + \varepsilon^{1-\alpha}U_1(\tfrac{x}{\varepsilon}))\psi_\varepsilon(t,x) = 0 \\ & \psi_\varepsilon(0,x) = \psi_0(\tfrac{x}{\varepsilon}). \end{aligned} \tag{39}$$

The **Wigner Transform** is adapted to the analysis of the probability density when ε goes to 0. It is defined by

$$W[u](t,x,k) = \frac{1}{(2\pi)^d} \int e^{ik\cdot y} u(t, x - \frac{y}{2}) \overline{u}(t, x + \frac{y}{2}) dy \tag{40}$$

where $\overline{u}$ denotes complex conjugate of u and d is the spatial dimension. In other words, the Wigner transform is the Fourier transform of a two-point correlation function of u. Taking inverse Fourier transform of this equality (for $y = 0$), we obtain the important property

$$\int_{\mathbb{R}^d} W[u](t,x,k)dk = |u|^2. \tag{41}$$

Therefore $W[u]$ can be seen as an angularly resolved probability density. It is not quite correct though because W is not necessarily positive for every k. In the high frequency limit, the solution of the Schrödinger equation oscillates at the fast scale and the correlation function as defined in (40) cannot capture those oscillations. We have to rescale our Wigner transform accordingly. This is done by introducing

$$W_\varepsilon[u](t,x,k) = \frac{1}{(2\pi)^d}\int e^{ik\cdot y} u(t,x-\frac{\varepsilon y}{2})\overline{u}(t,x+\frac{\varepsilon y}{2})dy. \tag{42}$$

For simplicity we denote by $W_\varepsilon = W_\varepsilon[\psi_\varepsilon]$ where ψ_ε is the solution of (39). For general conditions on a sequence u_ε, basically that u_ε oscillates at the scale ε, we can show [5] that $W_\varepsilon[u_\varepsilon]$ converges in the sense of bounded measures to a positive distribution $W(t,x,k)$. Therefore, even though W_ε may not be positive for ε given, it becomes positive in the limit $\varepsilon \to 0$. The aim of this section is to show that W is actually a solution of a transport equation.

2.2 The Liouville Equation

We assume here that $U_1 = 0$. We rewrite (39) as follows

$$\begin{aligned} &\partial_t\psi_\varepsilon(t,x) - \tfrac{i\varepsilon}{2}\Delta\psi_\varepsilon(t,x) + \tfrac{i}{\varepsilon}U(x)\psi_\varepsilon(t,x) = 0 \\ &\psi_\varepsilon(0,x) = \psi_0(\tfrac{x}{\varepsilon}). \end{aligned} \tag{43}$$

¿From (42) we have

$$\partial_t W_\varepsilon = \frac{1}{(2\pi)^d}\int e^{ik\cdot y}[\partial_t\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2}) + \partial_t\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2})\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})]dy.$$

We obtain an equation for W_ε using (43) at points $x-\frac{\varepsilon y}{2}$ and $x+\frac{\varepsilon y}{2}$. We have

$$\partial_t W_\varepsilon + I_1 + I_2 = 0$$

where

$$\begin{aligned} I_1 &= -\frac{i\varepsilon}{2}\frac{1}{(2\pi)^d}\int e^{ik\cdot y}(\Delta\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2}) + \psi_\varepsilon(t,x-\frac{\varepsilon y}{2})\Delta\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2})dy) \\ I_2 &= \frac{i}{\varepsilon}\int e^{ik\cdot y}(U(x-\frac{\varepsilon y}{2})\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2}) + \psi_\varepsilon(t,x-\frac{\varepsilon y}{2})U(x+\frac{\varepsilon y}{2})\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2})dy). \end{aligned}$$

¿From the change of variables $y \mapsto -y$ we see that

$$I_1 = 2Re(-\frac{i\varepsilon}{2}\frac{1}{(2\pi)^d}\int e^{ik\cdot y}\Delta\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2})dy)$$

where Re stands for real part. By integrations by parts, we obtain

$$\begin{aligned} I_1 &= 2Re(-\frac{i\varepsilon}{2}\frac{1}{(2\pi)^d}\int \frac{4}{\varepsilon^2}\Delta[\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})]\, e^{ik\cdot y}\,\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2})dy) \\ &= 2Re(\frac{i\varepsilon}{2}\frac{1}{(2\pi)^d}\int \frac{4}{\varepsilon^2}\nabla[\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})]\cdot\nabla[e^{ik\cdot y}\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2})]dy) \\ &= 2Re(\frac{i\varepsilon}{2}\frac{1}{(2\pi)^d}\int \frac{4}{\varepsilon^2}\nabla[\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})]\cdot(ik)\,e^{ik\cdot y}\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2})dy) \\ &= 2Re(\frac{1}{(2\pi)^d}\int e^{ik\cdot y}k\cdot\nabla\psi_\varepsilon(t,x-\frac{\varepsilon y}{2})\overline{\psi}_\varepsilon(t,x+\frac{\varepsilon y}{2})dy) \\ &= k\cdot\nabla W_\varepsilon. \end{aligned}$$

Here, for instance, $\nabla[\psi_\varepsilon(t, x - \frac{\varepsilon y}{2})]$ means the derivative with respect to the variable y of the composed function $\psi_\varepsilon(t, x - \frac{\varepsilon y}{2})$. This is not to be confused with $\nabla\psi_\varepsilon(t, x - \frac{\varepsilon y}{2})$ which is the value of the function $\nabla\psi_\varepsilon$ at $(t, x - \frac{\varepsilon y}{2})$.

The term I_2 is computed by introducing the Fourier transform of the potential U:

$$\hat{U}(p) = \frac{1}{(2\pi)^d} \int e^{ip\cdot x} U(x) dx. \tag{44}$$

We have $I_2 = I_{21} + I_{22}$, where

$$\begin{aligned} I_{21} &= \frac{i}{\varepsilon} \int e^{ik\cdot y} U(x - \frac{\varepsilon y}{2}) \psi_\varepsilon(t, x - \frac{\varepsilon y}{2}) \overline{\psi}_\varepsilon(t, x + \frac{\varepsilon y}{2}) dy \\ &= \frac{i}{\varepsilon} \int e^{ik\cdot y} \int e^{-ip\cdot(x - \frac{\varepsilon y}{2})} \hat{U}(p) \psi_\varepsilon(t, x - \frac{\varepsilon y}{2}) \overline{\psi}_\varepsilon(t, x + \frac{\varepsilon y}{2}) dp\, dy \\ &= \frac{i}{\varepsilon} \int e^{-ip\cdot x} \hat{U}(p) W_\varepsilon(t, x, k + \frac{\varepsilon p}{2}) dp, \end{aligned}$$

and similarly $I_{22} = -\frac{i}{\varepsilon} \int e^{-ip\cdot x} \hat{U}(p) W_\varepsilon(t, x, k - \frac{\varepsilon p}{2}) dp$. Therefore W_ε is a solution of the following equation

$$\partial_t W_\varepsilon + k \cdot \nabla W_\varepsilon + \frac{i}{\varepsilon} \int e^{-ip\cdot x} \hat{U}(p) [W_\varepsilon(t, x, k + \frac{\varepsilon p}{2}) - W_\varepsilon(t, x, k - \frac{\varepsilon p}{2})] dp = 0. \tag{45}$$

When $\varepsilon \to 0$, we have

$$\frac{1}{\varepsilon} [W_\varepsilon(t, x, k + \frac{\varepsilon p}{2}) - W_\varepsilon(t, x, k - \frac{\varepsilon p}{2})] \to p \cdot \nabla_k W(t, x, k)$$

assuming that W_ε converges in some suitable sense to W. On the other hand, taking inverse Fourier transform yields

$$\int e^{-ip\cdot x} \hat{U}(p) ip\, dp = -\nabla_x U.$$

Therefore we obtain that W_ε converges to a quantity W solution of the **Liouville Equation**:

$$\partial_t W + k \cdot \nabla W - \nabla_x U \cdot \nabla_k W = 0. \tag{46}$$

2.3 Radiative transfer equation

We consider now the case of a fast varying potential of small amplitude ($\alpha = 1/2$). For simplicity, we assume that $U_0 = 0$. Also, U_1 is a mean zero, stationary random function. It models random fluctuations of the underlying potential U_0. The correlation length of this potential is of order one, so as to let the random potential interact fully with the wave function. More precisely we assume that the fluctuations are space homogeneous and isotropic so that there exists a positive real function R such that

$$< U_1(x) U_1(y) >= R(x - y) = R(|x - y|), \tag{47}$$

where $< \cdot, \cdot >$ denotes the statistical averaging over all realizations of the random function. Denoting by $\hat{R}$ the Fourier transform of R, we also check that

$$< \hat{U}_1(p)\hat{U}_1(q) >= \hat{R}(p)\delta(p+q). \tag{48}$$

where δ is the Delta function defined by $< \delta(x), \phi(x) >_{\mathcal{D}'(I\!R^3),\mathcal{D}(I\!R^3)} = \phi(0)$.

As for the computation of I_2 in the previous section, we find that W_ε is a solution of

$$\partial_t W_\varepsilon + k \cdot \nabla W_\varepsilon + \frac{i}{\sqrt{\varepsilon}} \int e^{-ip\cdot\frac{x}{\varepsilon}} \hat{U}(p)[W_\varepsilon(t,x,k+\frac{p}{2}) - W_\varepsilon(t,x,k-\frac{p}{2})]dp = 0. \tag{49}$$

The limit (if any) of the third term in the left-hand side of (49) is not obvious. We want to analyze the behavior of (49) using a perturbation method, which consists in looking at asymptotic expansion in powers of ε. Notice that two different scales play a role here: the slow scale x and the fast scale $y = \frac{x}{\varepsilon}$. Therefore both of them must be present in the expansion. We make the change of variables:

$$W_\varepsilon = W_\varepsilon(t,x,\frac{x}{\varepsilon},k) = W_\varepsilon(t,x,y,k).$$

Assuming that $f(x) = g(x, \frac{x}{\varepsilon})$, we obtain that

$$\nabla f(x) = \nabla_x g(x,\frac{x}{\varepsilon}) + \frac{1}{\varepsilon}\nabla_y g(x,\frac{x}{\varepsilon}).$$

Therefore, using this rule, we find the following equation for $W_\varepsilon(t,x,y,k)$:

$$\partial_t W_\varepsilon + \frac{1}{\varepsilon} k \cdot \nabla_y W_\varepsilon + k \cdot \nabla_x W_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}(W_\varepsilon) = 0, \tag{50}$$

where

$$\mathcal{L}(W) = i \int e^{-ip\cdot y} \hat{U}(p)[W(t,x,y,k+\frac{p}{2}) - W(t,x,y,k-\frac{p}{2})]dp \tag{51}$$

Since there are powers of ε of order 0, $-1/2$ and -1 in (50), we look for an expansion of the form

$$W_\varepsilon(t,x,y,k) = W_0(t,x,y,k) + \sqrt{\varepsilon} W_1(t,x,y,k) + \varepsilon W_2(t,x,y,k) + o(\varepsilon). \tag{52}$$

Let us insert expansion (52) into (50). At the order ε^{-1} we obtain that

$$k \cdot \nabla_y W_0 = 0.$$

This means that W_0 is independent of the fast scale variable y. Since the solution of the Schrödinger equation is deterministic when $U_1 = 0$, we also assume in the limit of small ε that W_0 is deterministic.

The equation at order $\varepsilon^{-1/2}$ yields

$$k \cdot \nabla_y W_1 + \mathcal{L}(W_0) = 0.$$

As we shall see, this equation is easily invertible in Fourier domain. Since the Fourier transform can blow up for certain directions, we regularize this equation by adding a small absorption $\theta > 0$ that we shall send to 0 eventually. After regularization, the equation for W_1 reads

$$k \cdot \nabla_y W_1 + \theta W_1 + \mathcal{L}(W_0) = 0. \tag{53}$$

Taking Fourier transform of this equation with respect to the variable y and with dual variable q yields

$$\tilde{W}_1(t,x,q,k) = \frac{\hat{U}(q)[W_0(t,x,k-\frac{q}{2}) - W_0(t,x,k+\frac{q}{2})]}{k \cdot q + i\theta},$$

where the $\tilde{W}_1$ is the Fourier transform of W_1 with respect to the fast variable:

$$\tilde{W}_1(t,x,q,k) = \frac{1}{(2\pi)^d} \int e^{iq\cdot y} W_1(t,x,y,k)dx.$$

The third and last equation is the term of order 0 in the expansion:

$$\partial_t W_0 + k \cdot \nabla_x W_0 + k \cdot \nabla_y W_2 + \mathcal{L}(W_1) = 0. \tag{54}$$

The fast scale variable y indicates variations caused by fluctuations. By ergodicity, the averaging over realizations, or the integration over a large domain of the gradient with respect to y of any quantity should vanish. Therefore we assume that

$$< k \cdot \nabla_y W_2 >= 0.$$

Averaging then (54) over realizations gives the equation

$$\partial_t W_0 + k \cdot \nabla_x W_0 + < \mathcal{L}(W_1) >= 0. \tag{55}$$

We replace W_1 here by its expression in terms of W_0 to obtain an equation for W_0. Let us consider the first contribution in $\mathcal{L}(W_1)$. We have

$$\begin{aligned}
& < i \int e^{-ip\cdot y} \hat{U}(p) W_1(t,x,y,k+\frac{p}{2})dp > \\
= & < i \int e^{-ip\cdot y} \hat{U}(p) \int e^{-iq\cdot y} \tilde{W}_1(t,x,q,k+\frac{p}{2})dp\, dq > \\
= & < i \int\int e^{-i(p+q)\cdot y} \hat{U}(p)\hat{U}(q) \frac{W_0(t,x,k+\frac{p-q}{2}) - W_0(t,x,k+\frac{p+q}{2})}{(k+\frac{p}{2})\cdot q + i\theta} > \\
= & \; i \int \hat{R}(q) \frac{W_0(k-q) - W_0(k)}{(k-\frac{q}{2})\cdot q + i\theta} dq \; = i \int \hat{R}(k-p) \frac{W_0(p) - W_0(k)}{\frac{1}{2}(|k|^2 - |p|^2) + i\theta} dp
\end{aligned}$$

according to (48). Calculating the second contribution yields

$$< \mathcal{L}(W_1) >= i \int \hat{R}(k-p)[W_0(p) - W_0(k)]\left(\frac{1}{\frac{1}{2}(|k|^2-|p|^2) + i\theta} - \frac{1}{\frac{1}{2}(|k|^2-|p|^2) - i\theta}\right)dp$$

We have that

$$\lim_{\theta\to 0^+} \left(\frac{1}{x+i\theta} - \frac{1}{x-i\theta}\right) = -2i\pi\delta(x).$$

Therefore, we deduce from

$$\int \phi(r)\delta(\frac{r^2-\alpha^2}{2})dr = 2\int\int \phi(r)\delta(r^2-\alpha^2)dr$$

which is true for any test function $\phi \in \mathcal{D}(I\!R)$, that

$$< \mathcal{L}(W_1) >= 4\pi \int \hat{R}(k-p)[W_0(t,x,p) - W_0(t,x,k)]\delta(|k|^2 - |p|^2)dp.$$

We have formally obtained that W_ε converges as $\varepsilon \to 0$ to a function W which is a solution of the **Radiative Transfer Equation**

$$\partial_t W(t,x,k) + k\cdot\nabla_x W(t,x,k) + \Sigma W(t,x,k) = \int \sigma(k,p)W(t,x,p)dp, \qquad (56)$$

where

$$\begin{aligned} \sigma(k,p) &= 4\pi\hat{R}(k-p)\delta(|k|^2-|p|^2) \\ \Sigma(k) &= \int \sigma(k,p)dp. \end{aligned} \qquad (57)$$

When the statistical properties of the fluctuations vary at the large scale x, i.e. $U_1 = U_1(x, \frac{x}{\varepsilon})$, the power spectrum $\hat{R}$ depends on the position x: $\hat{R} = \hat{R}(x, k-p)$ as well as the cross sections σ and Σ.

3 Existence Theory in Hilbert space

We concentrate now on the mathematical analysis of the steady-state transport equation in bounded physical domain. More precisely, the phase space and the coefficients of the transport equations satisfy the following assumptions.

(H1) The domain Ω is a convex bounded open set.

(H2) The velocity space K is a compact subset of $\mathbb{R}^d$ which does not contain 0. Furthermore K is assumed to be the closure of an open set (respectively the union of a finite number of spheres). Its d-dimensional (respectively $d-1$-dimensional) measure is normalized to have $|K| = 1$.

(H3) The cross-sections $\Sigma(x, v)$ and $\sigma(x, v', v)$ are measurable positive bounded functions of their arguments. The scattering kernel is symmetric, i.e.

$$\sigma(x, v', v) = \sigma(x, v, v') \quad \text{a.e. in } \Omega \times K \times K.$$

The absorption cross section Σ_a is non negative, i.e

$$\Sigma_a(x, v) = \Sigma(x, v) - \int_K \sigma(x, v, v')dv' \geq 0 \quad \text{a.e. in } \Omega \times K.$$

Condition (H2) for K is convenient to avoid the difficulties coming from high velocity particles. It also ensures that each particle traveling along straight lines exits the physical domain in a finite time.

We consider in this section solutions of the transport equation in the Hilbert space $L^2(\Omega \times K)$. The source term is $q \in L^2(\Omega \times K)$. Here we assume to simplify that no particles enter the domain. The set of boundary conditions are defined by

$$\Gamma_- = \{(x, v) \in \partial\Omega \times K \,|\, v \cdot n(x) < 0\}, \quad \Gamma_+ = \{(x, v) \in \partial\Omega \times K \,|\, v \cdot n(x) > 0\}. \tag{58}$$

We equip them with two different norms

$$\begin{aligned} \|u\|_{L^2(\Gamma_\pm, |v \cdot n|)} &= \Big(\int_{\Gamma_\pm} |v \cdot n| u^2(x, v) d\sigma dv \Big)^{1/2} \\ \|u\|_{L^2(\Gamma_\pm, d\xi)} &= \Big(\int_{\Gamma_\pm} |v \cdot n| t(x, v) u^2(x, v) d\sigma dv \Big)^{1/2}, \end{aligned} \tag{59}$$

where $d\sigma$ is the surface measure on $\partial\Omega$ and $t(x, v)$ is the travel time defined in (24). Vanishing boundary conditions for the transport equation are given by $u = 0$ on Γ_-. The steady-state transport equation reads then

$$\begin{aligned} & v \cdot \nabla u + \Sigma u = Fu + q \qquad \text{in } \Omega \times K \\ & u = 0 \qquad \text{on } \Gamma_-. \end{aligned} \tag{60}$$

Here we have defined the operator F by

$$Fu(x, v) = \int_K \sigma(x, v', v) u(x, v') dv'. \tag{61}$$

We also rewrite the transport equation as

$$\begin{aligned} & v \cdot \nabla u + \Sigma_a u = Qu + q \qquad \text{in } \Omega \times K \\ & u = 0 \qquad \text{on } \Gamma_-, \end{aligned} \tag{62}$$

where the scattering operator Q is defined by

$$Qu(x,v) = \int_K \sigma(x,v',v)(u(x,v) - u(x,v'))dv' \tag{63}$$

because of Hypothesis (H3). We prove that the transport equation (60) admits a unique solution in the Hilbert space $W^2(\Omega \times K)$, where for $1 \le p \le \infty$, the space $W^p(\Omega \times K)$ is defined by

$$W^p(\Omega \times K) = \{u(x,v) \in L^p(\Omega \times K) \text{ such that } v \cdot \nabla u(x,v) \in L^p(\Omega \times K)\}. \tag{64}$$

One checks that for $1 \le p < \infty$, W^p is a Banach space.

3.1 An a priori estimate

Our result of existence relies on the following a priori estimate.

Lemma 3.1 *Let be $g \in L^2(\Gamma_-, |v \cdot n|)$ and $q \in L^2(\Omega \times K)$. We assume that $u \in W^2(\Omega \times K)$ is a solution of the transport equation*

$$\begin{aligned} & v \cdot \nabla u + \Sigma u = \int_K \sigma(x,v',v)u(x,v')dv' + q \qquad \textit{in } \Omega \times K \\ & u = g \qquad \textit{on } \Gamma_-. \end{aligned} \tag{65}$$

Then we have the following estimate. There exists a constant C independent of Σ_a, g, q and u such that

$$\begin{aligned} & \|u\|_{L^2(\Omega\times K)} + \|v \cdot \nabla u\|_{L^2(\Omega\times K)} + \|u\|_{L^2(\Gamma_+,|v\cdot n|)} \\ \le \quad & C(\|q\|_{L^2(\Omega\times K)} + \|g\|_{L^2(\Gamma_-,|v\cdot n|)}). \end{aligned} \tag{66}$$

Prior to proving this lemma, we need an analog result to the Poincaré inequality for elliptic problems.

Lemma 3.2 *Let $u(x,v) \in W^2(\Omega \times K)$ be such that the restrictions of u on Γ_+ and Γ_- exist and be elements of $L^2(\Gamma_+, d\xi)$ and $L^2(\Gamma_-, d\xi)$ respectively. Furthermore, there exists a constant C, independent of u, such that*

$$\|u\|_{L^2(\Omega\times K)} + \|u\|_{L^2(\Gamma_+,d\xi)} \le C\left(\|v \cdot \nabla u\|_{L^2(\Omega\times K)} + \|u\|_{L^2(\Gamma_-,d\xi)}\right). \tag{67}$$

Remark 3.3 For each function $u(x,v) \in W^2(\Omega \times K)$, the restrictions of u on Γ_+ and Γ_- exist and are elements of $L^2(\Gamma_+, d\xi)$ and $L^2(\Gamma_-, d\xi)$ respectively. Therefore the assumptions given in the lemma are redundant. This result is a trace theorem that can be found in [2] for instance.

Proof of Lemma 3.2. Let us define the following projection on the boundary:

$$\begin{array}{rcl} \Omega \times K & \to & \partial\Omega \\ (x,v) & \mapsto & \overline{x_v} = x - t(x,v)v. \end{array} \tag{68}$$

For any smooth function $u(x,v) \in \Omega \times K$, we have

$$u(x,v) = \int_0^{t(x,v)} v \cdot \nabla u(x - sv, v) ds + u(\overline{x}, v).$$

Since Ω is bounded, we deduce from the Cauchy-Schwartz inequality that

$$|u(x,v)|^2 \leq C \left(\int_0^{t(x,v)} (v \cdot \nabla u)^2 (x - sv, v) ds + |u(\overline{x}, v)|^2 \right). \tag{69}$$

Let us denote by $\chi(x)$ the characteristic function of Ω, i.e. $\chi(x) = 1$ if $x \in \Omega$ and $\chi(x) = 0$ otherwise. Since Ω is bounded by (H1) and $0 \notin K$ by (H2), there exists a constant $t_{max} < \infty$ such that $t(x,v) \leq t_{max}$ for all $(x,v) \in \Omega \times K$. We integrate the first term in the right-hand side of (69) with respect to x and obtain

$$\begin{aligned} & \int_\Omega \int_0^{t(x,v)} |v \cdot \nabla u|^2 (x - sv, v) ds dx \\ = & \int_\Omega \int_0^{t_{max}} \chi(x - sv) |v \cdot \nabla u|^2 (x - sv, v) ds dx \\ = & \int_0^{t_{max}} \int_{\Omega + sv} \chi(y) |v \cdot \nabla u|^2 (y, v) dy ds \\ = & \int_0^{t_{max}} \int_{(\Omega + sv) \cap \Omega} |v \cdot \nabla u|^2 (y, v) dy ds \leq t_{max} \int_\Omega |v \cdot \nabla u|^2 (y, v) dy. \end{aligned} \tag{70}$$

We now integrate with respect to x the second term in the right-hand side of (69). Let $v \in K$ be given and introduce an orthonormal basis $(x_1, \ldots, x_d) = (x', x_d)$ such that x_d be parallel to v. The point on the boundary $\overline{x_v}$ is then only a function of x'. Then we have

$$\int_\Omega |u(\overline{x_v}, v)|^2 dx = \int_{\mathbb{R}^{d-1}} |u(\overline{x}, v)|^2 \left(\int_{\mathbb{R}} \chi(x) dx_n \right) dx' = \int_{\Omega'} |u(\overline{x_v}, v)|^2 t(\overline{x_v}, v) dx',$$

where Ω' is the projection of Ω on $\mathbb{R}^{d-1}$. Because Ω is convex, there exists a diffeomorphism from Ω' to the part of the boundary $\partial\Omega$ defined by $\Gamma_v^- = \{x \in \partial\Omega,\, n(x) \cdot v < 0\}$. Changing the variable x' in $\overline{x_v} \in \Gamma_v^-$ yields $dx' = |v \cdot n| d\sigma$ and

$$\int_\Omega |u(\overline{x_v}, v)|^2 dx = \int_{\Gamma_v^-} |v \cdot n| t(x,v) |u(x,v)|^2 d\sigma. \tag{71}$$

The upper bound for the $L^2(\Omega \times K)$ norm of u is obtained by integrating (70) and (71) with respect to v. We apply the same method to have the upper bound of the $L^2(\Gamma_+, d\xi)$ norm of u. □

Proof of Lemma 3.1. Let us multiply both sides of (65) by u and integrate over $\Omega \times K$. The integrals are well-defined by hypothesis on the regularity of u. We first observe that

$$\int_\Omega \int_K v \cdot \nabla u \, u \, dxdv = -\int_\Omega \int_K v \cdot \nabla u \, u \, dxdv + \int_{\partial\Omega} \int_K (v \cdot n) u^2 \, d\sigma dv$$

and therefore

$$\int_\Omega \int_K v \cdot \nabla u \, u \, dxdv = \frac{1}{2} \int_{\Gamma_+} |v \cdot n| u^2 d\sigma dv - \frac{1}{2} \int_{\Gamma_-} |v \cdot n| g^2 d\sigma dv.$$

We also have that

$$\begin{aligned} & \int_\Omega \int_K [(\Sigma(x,v) - \Sigma_a(x,v)) u^2(x,v) - \int_K \sigma(x,v',v) u(x,v') u(x,v) dv'] dxdv \\ = & \int_\Omega \int_K \int_K [\sigma(x,v,v') u(x,v) - \sigma(x,v',v) u(x,v')] u(x,v) dxdvdv' \\ = & \frac{1}{2} \int_\Omega \int_K \int_K \sigma(x,v',v) [u(x,v) - u(x,v')]^2 dxdvdv' = (Qu,u) \geq 0. \end{aligned} \tag{72}$$

because of (H3). Therefore multiplying (65) by u and integrating yields

$$\begin{aligned} & \int_\Omega \int_K \Sigma_a u^2 dxdv + \frac{1}{2} \int_{\Gamma_+} |v \cdot n| u^2 d\sigma dv + (Qu,u) \\ = & \int_\Omega \int_K qu \, dxdv + \frac{1}{2} \int_{\Gamma_-} |v \cdot n| g^2 d\sigma dv. \end{aligned}$$

We deduce the existence of a constant C such that

$$\begin{aligned} & \|\sqrt{\Sigma_a} u\|^2_{L^2(\Omega\times K)} + \|u\|^2_{L^2(\Gamma_+, |v\cdot n|)} + (Qu,u) \\ \leq \; & C(\|u\|_{L^2(\Omega\times K)} \|q\|_{L^2(\Omega\times K)} + \|g\|^2_{L^2(\Gamma_-, |v\cdot n|)}). \end{aligned} \tag{73}$$

We obtain from equation (65) that

$$\|v \cdot \nabla u\|_{L^2(\Omega\times K)} \leq C(\|\Sigma_a u\|_{L^2(\Omega\times K)} + \|Qu\|_{L^2(\Omega\times K)} + \|q\|_{L^2(\Omega\times K)}). \tag{74}$$

Let us show that $\|Qu\|^2_{L^2(\Omega\times K)} \leq C(Qu,u)$. Indeed

$$(Qu,Qu) = \int_\Omega \int_K \Big(\int_K \sigma(x,v',v) [u(x,v) - u(x,v')] dv' \Big)^2 dxdv,$$

hence the result from (72) and the Cauchy-Schwartz inequality. We deduce from (74) that

$$\|v \cdot \nabla u\|^2_{L^2(\Omega\times K)} \leq C(\|u\|_{L^2(\Omega\times K)} \|q\|_{L^2(\Omega\times K)} + \|q\|^2_{L^2(\Omega\times K)} + \|g\|^2_{L^2(\Gamma_-, |v\cdot n|)}). \tag{75}$$

The Poincaré inequality (67) yields, since $t(x,v)$ is bounded,

$$\|u\|_{L^2(\Omega\times K)} \leq C(\|v\cdot\nabla u\|_{L^2(\Omega\times K)} + \|g\|_{L^2(\Gamma_-,|v\cdot n|)}). \tag{76}$$

Using the estimate (75) for $\|v\cdot\nabla u\|_{L^2(\Omega\times K)}$ in (76), we have

$$\|u\|_{L^2(\Omega\times K)} \leq C(\|q\|_{L^2(\Omega\times K)} + \|g\|_{L^2(\Gamma_-,|v\cdot n|)}).$$

Recalling (73) and (75) completes the proof of the result. □

Remark 3.4 Notice that the Poincaré inequality (67) is not necessary when the absorption cross section Σ_a is uniformly bounded from below by a positive constant. Indeed, we directly obtain an a priori bound for $\|u\|_{L^2(\Omega\times K)}$ from (73). As for elliptic problems however, absorption is not necessary in bounded domains, where the creation of particles, i.e. the source terms q and g, is compensated by enough leakage at the boundary of the domain.

There exists an analogous a priori estimate for the adjoint equation.

Lemma 3.5 *Let be $g \in L^2(\Gamma_+,|v\cdot n|)$ and $q \in L^2(\Omega\times K)$. We assume that $u \in W^2(\Omega\times K)$ is a solution of the transport equation*

$$\begin{aligned} &-v\cdot\nabla u + \Sigma u = \int_K \sigma(x,v',v)u(x,v')dv' + q \qquad \text{in } \Omega\times K \\ &u = g \qquad \text{on } \Gamma_+. \end{aligned} \tag{77}$$

Then we have the following estimate. There exists a constant C independent of Σ_a, g, q and u such that

$$\begin{aligned} &\|u\|_{L^2(\Omega\times K)} + \|v\cdot\nabla u\|_{L^2(\Omega\times K)} + \|u\|_{L^2(\Gamma_-,|v\cdot n|)} \\ \leq\ & C(\|q\|_{L^2(\Omega\times K)} + \|g\|_{L^2(\Gamma_+,|v\cdot n|)}). \end{aligned} \tag{78}$$

Notice that (77) is an adjoint problem to (65) because $\sigma(x,v,v') = \sigma(x,v',v)$ by hypothesis.

3.2 A Theorem of Existence and Uniqueness

The results of existence rely on the theory of closed linear unbounded operators. Let T be a non-bounded linear operator in $L^2(\Omega\times K)$ with domain of definition $D(T) \subset L^2(\Omega\times K)$. To simplify we denote by $X = L^2(\Omega\times K)$. The *Graph* of T is defined by

$$G(T) = \bigcup_{u\in X} (u, Tu) \subset X\times X. \tag{79}$$

We say that the operators T is *closed* if $G(T)$ is closed in $X\times X$. For every set $Y \subset X$, we denote by $\overline{Y}$ its closure in X. Let us recall the following result of functional analysis (see for instance §VII.5 of [9])

Proposition 3.6 *Let $T: D(T) \in X \to X$ a linear non bounded, closed operator with dense domain of definition in X, i.e. $\overline{D(T)} = X$. Then the following properties are equivalent*

(i) T is onto, i.e. $R(T) = \bigcup_{u \in X} Tu = X$.

(ii) There exists a positive constant C such that

$$\|v\|_X \le C\|T^* v\|_X \qquad \text{for all } v \in D(T^*).$$

We are then ready to prove the following result:

Theorem 3.7 *Assume that Hypotheses (H1)-(H3) are satisfied. Let be $q \in L^2(\Omega \times K)$. Then there exists a unique solution to (60) in $W^2(\Omega \times K)$.*

Proof We want to apply Proposition 3.6 for the operator $-A$ defined in (26). The domain of definition of A is defined by

$$D(A) = \{u \in L^2(\Omega \times K) \text{ such that } v \cdot \nabla u \in L^2(\Omega \times K) \text{ and } u = 0 \text{ on } \Gamma_-\}. \tag{80}$$

Here we say that $u = 0$ on Γ_- if $u = 0$ on every compact subset Γ of Γ_-. It is proven in Chapter 21, §2 of [3], that the application of restriction $u \mapsto \gamma(u)$ on every compact subset Γ is well defined for every function $u \in W^2(\Omega \times K)$. Moreover the application is continuous. Therefore $-A$ is a linear operator from $D(A)$ to $L^2(\Omega \times K)$. Moreover $W^2(\Omega \times K) \subset D(A) \subset L^2(\Omega \times K)$. Since $W^2(\Omega \times K)$ is dense in $L^2(\Omega \times K)$, we deduce that $\overline{D(A)} = L^2(\Omega \times K)$.

We now prove that A is closed. Let u_n be a sequence of functions of $D(A)$ converging to u in $L^2(\Omega \times K)$, and such that Au_n converges to f in $L^2(\Omega \times K)$. We deduce from the convergence of Au_n and u_n that $v \cdot \nabla u_n$ converges in $L^2(\Omega \times K)$. By uniqueness of the convergence in the sense of distributions, $v \cdot \nabla u_n$ converges to $v \cdot \nabla u$, and $u \in W^2(\Omega \times K)$. By the continuity of the application γ, $u = 0$ on Γ_-. Therefore $u \in D(A)$. Moreover we clearly have that Au_n converges to Au hence $f = Au$. Therefore $G(A)$ is closed, and so is A.

It remains to check (ii) in Proposition 3.6. We deduce from Lemma 3.5 and (78) that

$$\|u\|_{L^2(\Omega \times K)} \le C\|q\|_{L^2(\Omega \times K)} = C\| - A^* u\|_{L^2(\Omega \times K)}.$$

Therefore (i) is also valid and problem (60) admits a solution in $D(A)$ and therefore in $W^2(\Omega \times K)$. The uniqueness of the solution is a straightforward consequence of the a priori estimate (66). □

3.3 The Evolution Problem

The properties obtained for the operator A also allow to treat the evolution problem. This is the Hille-Yosida theorem [4, 9]. Let us first introduce the definition

Definition 3.8 *A closed linear operator* $A : D(A) \in X \rightarrow X$ *is said dissipative if*

$$(-Au, u) \geq 0 \qquad \text{for all } u \in D(A).$$

A is maximally dissipative if in addition $R(I - A) = X$, *i.e. the operator* $I - A$ *is onto.*

Theorem 3.9 *Let* A *be a maximally dissipative operator. Let be* $u_0 \in D(A)$. *Then the problem*

$$\begin{aligned} \frac{du}{dt} &= Au \quad \text{in } [0, \infty) \\ u(0) &= u_0 \quad \text{(initial condition)} \end{aligned} \tag{81}$$

admits a unique solution

$$u \in C^1([0, \infty); X) \cap C([0, \infty); D(A)).$$

Here $D(A)$ *is equipped with the norm of the graph* $\|u\|_X + \|Au\|_X$. *The solution* u *satisfies*

$$\|u(t)\|_X \leq \|u_0\|_X \quad \text{and} \quad \|\frac{du}{dt}(t)\|_X = \|Au(t)\|_X \leq \|Au_0\|_X \quad \forall t \geq 0.$$

Moreover, if $u_0 \in D(A^k)$ *for* $k \geq 2$, *then*

$$u \in C^{k-j}([0, \infty); D(A^j)) \quad \text{for } 0 \leq j \leq k.$$

It is clear from Lemma 3.1 and Theorem 3.7 that A defined in (26) is maximally dissipative. Therefore the theory applies to the transport equation. An analogous result exists for volume source terms. Let us assume that $q(t, x, v) \in C^1([0, T]; L^2(\Omega \times K)$ for $T > 0$ and $g = 0$. Then problem (18) for $\tau = T$ admits a unique solution

$$u \in C^1([0, T]; L^2(\Omega \times K)) \cap C([0, T]; D(A)).$$

These results can be obtained by directly analyzing the semigroup $T_t = e^{tA}$ instead of its infinitesimal generator A (see [3]).

4 Maximum Principle and L^∞ Theory

In this section, we still assume the Hypotheses (H1) and (H2) given at the beginning of section (3). We replace (H3) by $(\mathrm{H3})_\infty$:

$(\mathrm{H3})_\infty$ The cross-sections $\Sigma(x,v)$ and $\sigma(x,v',v)$ are measurable positive bounded functions of their arguments. There exists a positive constant Σ_0 such that

$$\Sigma(x,v) \geq \Sigma_0 > 0 \quad \text{ a.e. in } \Omega \times K.$$

There exists a constant β such

$$\int_K \sigma(x,v',v)dv' \leq \beta\,\Sigma(x,v), \quad 0 \leq \beta < 1 \quad \text{ a.e. in } \Omega \times K. \tag{82}$$

Notice that the hypothesis (82) is equivalent to assuming that the absorption cross section Σ_a is uniformly bounded from below by a positive constant α:

$$\infty > \Sigma_a(x,v) \geq \alpha > 0 \quad \text{ a.e. in } \Omega \times K. \tag{83}$$

For instance, we can choose $\alpha = (1-\beta)\Sigma_0$. Let us recall the transport problem with non homogeneous boundary condition:

$$\begin{aligned} &Tu = Fu + q \qquad \text{in } \Omega \times K \\ &u = g \qquad \text{on } \Gamma_-, \end{aligned} \tag{84}$$

where the free transport operator T is defined by

$$Tu = v \cdot \nabla u + \Sigma u. \tag{85}$$

We have the following result:

Theorem 4.1 *Assume that (H1), (H2) and* $(H3)_\infty$ *are satisfied,* $q \in L^\infty(\Omega \times K)$ *and* $g \in L^\infty(\Gamma_-)$*. Then (84) admits a unique solution in* $L^\infty(\Omega \times K)$*. Moreover,*

$$\|u\|_{L^\infty(\Omega\times K)} \leq \|g\|_{L^\infty(\Gamma_-)} + \frac{1}{\alpha}\|q\|_{L^\infty(\Omega\times K)}. \tag{86}$$

Proof By linearity of the transport equation, we can treat the source terms q and g successively.

Step 1: Let us first assume that $g = 0$. The free transport equation is given by

$$\begin{aligned} &Tu = h \quad \text{ in } \Omega \times K \\ &u = 0 \quad \text{ on } \Gamma_-. \end{aligned} \tag{87}$$

We define its domain of definition $D(T)$ as the set of functions $u \in L^\infty(\Omega \times K)$ such that $v \cdot \nabla u \in L^\infty(\Omega \times K)$ and $u = 0$ on Γ_-. For $h \in L^\infty(\Omega \times K)$, we use the method of characteristics to solve (87) and obtain

$$u(x,v) = T^{-1}h = \int_0^{t(x,v)} \exp\Big(-\int_0^t \Sigma(x-sv,v)ds\Big)h(x-tv,v)dt,$$

where $t(x,v)$ is defined in (24). We easily check that u and $v\cdot\nabla u$ belong to $L^\infty(\Omega\times K)$. Let us prove that $T^{-1}F$ is a contraction. We have

$$\begin{aligned}|(T^{-1}F\,h)|(x,v) &= |\int_0^{t(x,v)} e^{-\int_0^t \Sigma(x-sv,s)ds}\Sigma(x-tv,t)\frac{Fh}{\Sigma(x-tv,t)}dt| \\ &\leq \|\frac{Fh}{\Sigma}\|_{L^\infty}(\Omega\times V)(1-e^{-\int_0^{t(x,v)}\Sigma(x-sv,s)ds} \leq \|\frac{Fh}{\Sigma}\|_{L^\infty(\Omega\times V)}.\end{aligned}$$

We deduce that $\|T^{-1}\|_{\mathcal{L}(L^\infty(\Omega\times K))} \leq (\Sigma_0)^{-1}$. Moreover

$$|\frac{Fh}{\Sigma}|(x,v) = |\int_K \frac{\sigma(x,v',v)}{\Sigma(x,v)}h(x,v')dv'| \leq \beta\|h\|_{L^\infty(\Omega\times V)}.$$

This proves that

$$\|T^{-1}F\|_{\mathcal{L}(L^\infty(\Omega\times K))} \leq \beta < 1.$$

Therefore, the transport equation (84) with $g=0$ is equivalent to looking for $u\in D(T)$ such that

$$u = T^{-1}Fu + T^{-1}q. \tag{88}$$

Since $T^{-1}F$ is a strict contraction, we obtain that (84) admits a unique solution. Moreover we obtain that

$$\|u\|_{L^\infty(\Omega\times K)} \leq \beta\|u\|_{L^\infty(\Omega\times K)} + \|T^{-1}q\| \leq (1-\beta)^{-1}\Sigma_0^{-1}\|q\|_{L^\infty(\Omega\times K)}.$$

Since $\alpha = (1-\beta)\Sigma_0$, we obtain that

$$\|u\|_{L^\infty(\Omega\times K)} \leq \frac{1}{\alpha}\|q\|_{L^\infty(\Omega\times K)}.$$

Step 2: Let us now assume that $q=0$ and $g\neq 0$. Notice that the uniqueness is easily deduced from Step 1 by linearity of the transport equation. We will construct a solution by the following iterative procedure: $u_0=0$ and for all $n\geq 0$,

$$\begin{aligned}&v\cdot\nabla u_{n+1} + \Sigma u_{n+1} = Fu_n \qquad \text{in } \Omega\times K \\ &u_{n+1} = g \qquad \text{on } \Gamma_-.\end{aligned} \tag{89}$$

We prove by induction that $\|u_n\|_{L^\infty(\Omega\times K)} \leq \|g\|_{L^\infty(\Gamma_-)}$ for all $n\geq 0$. This is obviously true for $n=0$. Let us suppose it for $n-1$. It remains to prove it for n. By the method of characteristics, and mimicking the computations of step 1, we have

$$\begin{aligned}|u^n(x,v)| &= ||\exp(-\int_0^{t(x,v)}\Sigma(x-sv,s)ds)\; g(x-t(x,v)v,v) \\ &+ \int_0^{t(x,v)}\exp(-\int_0^t\Sigma(x-sv,s)ds)\Sigma(x-tv,t)\frac{Fu^{n-1}}{\Sigma(x-tv,t)}dt| \\ &\leq \exp(-\int_0^{t(x,v)}\Sigma(x-sv,s)ds)\;\|g\|_{L^\infty(\Gamma_-)} \\ &\quad +\Big(1-\exp(-\int_0^{t(x,v)}\Sigma(x-sv,s)ds)\Big)\beta\|u^{n-1}\|_{L^\infty(\Omega\times K)}.\end{aligned}$$

We easily deduce the result. □

Here is one way to compute the solution of the transport equation. Let us define the sequence $u_0 = 0$ and for all $n \geq 0$,

$$\begin{aligned} Tu_{n+1} &= Fu_n + q \quad \text{in } \Omega \times K \\ u_{n+1} &= g \quad \text{on } \Gamma_-. \end{aligned} \tag{90}$$

We easily see that u_n converges to the unique solution u of (84). The rate of convergence of the method $\frac{\|u_{n+1}-u_n\|_{L^\infty(\Omega\times K)}}{\|u_n-u_{n-1}\|_{L^\infty(\Omega\times K)}}$ is governed by the constant $(1-\beta)^{-1}$. We easily deduce that the iterative method converges fast if the absorption is important, but may be extremely slow for media with strong scattering.

The scattering operator F and the free transport operator are positive, in the sense that if u_n, q and g are non negative functions, then u_{n_1} defined as the solution to (90) is also non negative. By induction, we easily deduce the following result:

Corollary 4.2 *Assume that q and q are non negative functions. Then the solution u of the transport equation (84) is also non negative.*

5 Averaging Lemma and Applications

5.1 Averaging Lemma

We give now an averaging lemma, which is used to obtain regularity and compactness properties for the solution of transport equations [6, 7].

Lemma 5.1 *Let $f \in \mathcal{S}'(I\!R^d \times I\!R^d)$ such that*

$$f + v \cdot \nabla_x f \in L^2(I\!R^d \times I\!R^d). \tag{91}$$

Then we have that $f \in L^2(I\!R^d \times I\!R^d)$. Moreover

$$\overline{f}(x) = \int_{I\!R^d} f(x,v)\psi(v)dv \in H^{1/2}(I\!R^d) \quad \forall\, \psi \in L^\infty(I\!R^d) \text{ with compact support.} \tag{92}$$

Here $\mathcal{S}'$ stands for the space of distributions which is the dual to the Schwartz space $\mathcal{S}$, given by the smooth functions having all their derivatives decaying at infinity faster than any polynomial. We recall that the Fourier transform is defined for all functions in $\mathcal{S}'$. A definition of the Sobolev spaces H^s is given in (95) below.

Proof Since $f \in \mathcal{S}'$, we can define its Fourier transform $\hat{f}$. Let be $g \in L^2(I\!R^d \times I\!R^d)$ such that $f + v \cdot \nabla_x f = g$. Taking Fourier transform of this equality yields

$$(1 + iv \cdot \xi)\hat{f}(\xi, v) = \hat{g}(\xi, v), \tag{93}$$

both sides of the equation being in $L^2(I\!R^d \times I\!R^d)$. We deduce from (93) that

$$|\hat{f}| = \frac{|\hat{g}|}{\sqrt{1 + (v \cdot \xi)^2}}. \tag{94}$$

This clearly implies that $f \in L^2(I\!R^d \times I\!R^d)$. We denote by $H^s(I\!R^d)$, $s > 0$, the subset of functions $h \in L^2(I\!R^d)$ satisfying

$$\int_{I\!R^d} (1 + |\xi|^2)^s |\hat{h}|^2 d\xi < \infty. \tag{95}$$

So we want to prove that

$$\int_{I\!R^d} (1 + |\xi|^2)^{1/2} |\hat{\overline{f}}|^2 d\xi = \int_{I\!R^d} (1 + |\xi|^2)^{1/2} \Big| \int_{I\!R^d} \hat{f}(\xi, v)\psi(v)dv \Big|^2 d\xi < \infty \tag{96}$$

¿From (94) and using the Cauchy-Schwartz inequality, we ¿have

$$\Big| \int_{I\!R^d} \hat{f}(\xi, v)\psi(v)dv \Big|^2 \le G^2(\xi) \Big(\int_{I\!R^d} \frac{|\psi(v)|^2}{1 + (v \cdot \xi)^2} dv \Big),$$

where

$$G(\xi) = \Big(\int_{I\!R^d} |\hat{g}|^2(\xi, v)dv \Big)^{1/2}$$

By assumption, there exist two constant Φ and R such that ψ is bounded by Φ and its support is included in the ball $B(0,R)$ of radius R centered at 0. Therefore

$$(1+|\xi|^2)^{1/2}\int_{I\!R^d}\frac{|\psi(v)|^2}{1+(v\cdot\xi)^2}dv \le \Phi^2\int_{B(0,R)}\frac{(1+|\xi|^2)^{1/2}}{1+(v\cdot\xi)^2}dv$$

The latter integral is clearly bounded if $|\xi|<1$. It is also bounded if $|\xi|\ge 1$ and $d=1$ since $(1+x^2)^{-1}$ is integrable in $I\!R$. Let us assume that $d\ge 2$ and $\xi\ge 1$. We write $B(0,R)$ in spherical coordinates $(r,\theta_1,\ldots,\theta_{d-1})$. Up to some rotation, that leaves the integral invariant, we assume that $v\cdot\xi=|v||\xi|\cos\theta_1$. Then, there exists a universal constant C_d such that

$$I_\xi=\int_{B(0,R)}\frac{(1+|\xi|^2)^{1/2}}{1+(v\cdot\xi)^2}dv = C_d\int_0^R\int_0^\pi r^{d-1}\frac{(1+|\xi|^2)^{1/2}}{1+r^2|\xi|^2(\cos\theta_1)^2}\sin(\theta_1)d\theta_1dr.$$

By the change of variables $t=\cos(\theta_1)$, we have

$$\int_0^\pi\frac{\sin(\theta_1)d\theta_1}{1+r^2|\xi|^2(\cos\theta_1)^2}=\int_0^1\frac{dt}{1+r^2|\xi|^2t^2}=\frac{1}{r|\xi|}\arctan r|\xi|.$$

This yields, since $|\xi|\ge 1$,

$$I_\xi\le C_d\int_0^R r^{d-2}\sqrt{2}\times 2\pi\le CR^{d-1}$$

¿From this we deduce that I_ξ is bounded uniformly in ξ, hence

$$\int_{I\!R^d}(1+|\xi|^2)^{1/2}|\hat{\overline{f}}|^2d\xi\le C\int_{I\!R^d}G^2(\xi)d\xi=C\|g\|^2_{L^2(I\!R^d\times I\!R^d)}<\infty,$$

where C is a constant which depends only on the function ψ. This completes the proof of the lemma. □

There exists also a version for the evolution problem.

Lemma 5.2 *Let $f\in L^2(I\!R\times I\!R^d\times I\!R^d)$ be a weak solution of*

$$\partial_t f+v\cdot\nabla_x f=g \qquad (x,v)\in I\!R^d\times I\!R^d,\ t\in I\!R \tag{97}$$

where $g\in L^2(I\!R^d\times I\!R^d\times I\!R)$. Moreover let $\psi\in L^\infty(I\!R^d)$ with compact support. Then

$$\overline{f}(t,x)=\int_{I\!R^d}f(t,x,v)\psi(v)dv\in H^{1/2}(I\!R\times I\!R^d). \tag{98}$$

The proof of this lemma is very similar to that of lemma 5.1. We use Fourier transforms both in space and time. We also have an averaging lemma in bounded domains:

Lemma 5.3 *Let Ω be a convex open subdomain in $I\!R^d$. Let f be such that*

$$\|f\|_{L^2(\Omega\times I\!R^d)}+\|v\cdot\nabla_x f\|_{L^2(\Omega\times I\!R^d)}+\|f\|_{L^2(\Gamma_-,|v\cdot n|)}<\infty.$$

Then we have

$$\overline{f}(x)=\int_{I\!R^d}f(x,v)\psi(v)dv\in H^{1/2}(\Omega)\quad\forall\,\psi\in L^\infty(I\!R^d)\ \textit{with compact support.} \tag{99}$$

The result follows from the application of lemma 5.1 to a function whose restriction to Ω with $\overline{f}$. This is the aim of the following extension lemma.

Lemma 5.4 *Let us define first*

$$W^2_-(\Omega \times I\!R^d) = \{u(x,v) \in W^2(\Omega \times I\!R^d) \ s.t. \ \|u\|_{L^2(\Gamma_-,|v\cdot n|)} < \infty\}. \tag{100}$$

There exists a continuous extension operator

$$\mathcal{E} : W^2_-(\Omega \times I\!R^d) \to W^2(I\!R^d \times I\!R^d), \qquad i.e. \ (\mathcal{E}u)_{|\Omega\times I\!R^d} = u. \tag{101}$$

The proof of these results can be found in [2, 6] and is not given here.

5.2 Transport Equation and Compactness Property

We shall now apply the averaging lemma to derive an important compactness property. Let us first start with a corollary:

Corollary 5.5 *Let be Ω a convex bounded domain in $I\!R^d$. Let be $\sigma(v,v')$ in $L^\infty(I\!R^d \times I\!R^d)$ with compact support. Let be $\mathcal{U}$ the family of functions u such that*

$$\|u\|_{W^2(\Omega\times I\!R^d)} + \|u\|_{L^2(\Gamma_-,|v\cdot n|)} \le 1$$

and $\mathcal{F}$ the family of functions $f \in L^2(\Omega \times I\!R^d)$ defined by

$$f(x,v) = \int_{I\!R^d} \sigma(v,v')u(x,v')dv'. \tag{102}$$

Then $\mathcal{F}$ is relatively compact in $L^2(\Omega \times I\!R^d)$.

Proof We want to use the Riesz-Fréchet-Kolmogorov Theorem [1]. Let us denote by τ_h the translation in phase space,

$$\tau_h f = f(x+h_x, v+h_v), \qquad \text{where} \quad h=(h_x,h_v) \in I\!R^d \times I\!R^d. \tag{103}$$

u is prolongated outside Ω using the extension lemma 5.4, so that τ_h is defined for every $h \in I\!R^d \times I\!R^d$. We want to prove that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all h of norm $|h| < \delta$, and for all $f \in \mathcal{F}$, we have

$$\|\tau_h f - f\|_{L^2(\Omega\times I\!R^d)} \le \varepsilon. \tag{104}$$

Then the Riesz-Fréchet-Kolmogorov Theorem implies the relative compactness of the family $\mathcal{F}$ in $L^2(\Omega \times I\!R^d)$.

To simplify, we denote by $\|\cdot\| = \|\cdot\|_{L^2(\Omega\times I\!R^d)}$ and compute

$$\begin{aligned}\|\tau_h f - f\| &= \| \textstyle\int_{I\!R^d} \sigma(v+h_v,v')u(x+h_x,v')dv' - \int_{I\!R^d} \sigma(v,v')u(x,v')dv'\| \\ &\le \| \textstyle\int_{I\!R^d} \Big(\sigma(v+h_v,v') - \sigma(v,v')\Big)u(x+h_x,v')dv'\| \\ &\quad + \| \textstyle\int_{I\!R^d} \sigma(v,v')\Big(u(x+h_x,v') - u(x,v')\Big)dv'\|.\end{aligned}$$

The second term is $\|f(x+h_x,v)-f(x,v)\|$. Let be a fixed v. According to the averaging lemma (5.3), we have

$$\|f(x,v)\|_{H^{1/2}(\Omega)} \leq C$$

where C is a constant independent of $u \in \mathcal{U}$. Therefore we deduce from the compactness embedding from $H^{1/2}(\Omega)$ into $L^2(\Omega)$ that for every $\eta > 0$ there exists a δ so that for all $|h_x| < \delta$ and $f \in \mathcal{F}$,

$$\int_\Omega \Big(f(x+h_x,v) - f(x,v)\Big)^2 \leq \eta.$$

Since σ has a compact support, we easily deduce that

$$\|f(x+h_x,v) - f(x,v)\| \leq C\eta.$$

This proves the convergence of the second term. It remains to consider

$$\begin{aligned} & \| \textstyle\int_{I\!R^d} \Big(\sigma(v+h_v,v') - \sigma(v,v')\Big) u(x+h_x,v')dv' \| \\ \leq \quad & C\|u\| \, \|\sigma(v+h_v,v') - \sigma(v,v')\|_{L^2(I\!R^d \times I\!R^d)} \end{aligned}$$

Now since $\sigma \in L^2(I\!R^d \times I\!R^d)$ because it is of compact support, we know that [1]

$$\|\sigma(v+h_v,v') - \sigma(v,v')\|_{L^2(I\!R^d \times I\!R^d)} \to 0$$

when $|h_v|$ tends to 0. Therefore the condition (104) is satisfied and the family $\mathcal{F}$ relatively compact in $L^2(\Omega \times I\!R^d)$. $\Box$

Let us come back to the transport equation

$$\begin{aligned} & Tu = Fu + q \qquad \text{in } \Omega \times K \\ & u = 0 \qquad \text{on } \Gamma_-. \end{aligned} \tag{105}$$

We assume here that Ω is a bounded open convex subset of $I\!R^d$ and that K is a closed subset of $I\!R^d$. The domain of definition of T is $D(T) = D(A)$, defined by (80). Then we have the following result

Theorem 5.6 *Let T and F be given by (85) with domain $D(T)$ and (61). We assume that*

$$\sigma(x,v',v) = \sum_{i=1}^{I} c_i(x)\sigma_i(v',v), \tag{106}$$

where $I \in I\!N$, $c_i \in L^\infty(\Omega)$ and $\sigma_i \in L^\infty(K \times K)$ with compact support, for $i = 1, \dots, I$. We also assume that the cross section Σ is uniformly bounded from below by $\Sigma_0 > 0$.

Then the operators $T^{-1}F$ and FT^{-1} are compact in $L^2(\Omega \times K)$.

Proof The operators $T^{-1}F$ and FT^{-1} are clearly bounded in $L^2(\Omega \times K)$. The adjoint of $T^{-1}F$ in $L^2(\Omega \times K)$ is given by $F^*(T^*)^{-1}$, where T^* and F^* are defined by

$$\begin{cases} T^*u = -v \cdot \nabla u + \Sigma u \\ u = 0 \quad \text{on} \quad \Gamma_+ \end{cases}, \qquad F^*u = \sum_{i=1}^{I} c_i(x) \int_K \sigma_i(v, v')u(x, v')dv'.$$

We will prove that FT^{-1} and $F^*(T^*)^{-1}$ are compact. The result follows from the fact that if an operator K is compact, its adjoint operator K^* is also compact.

Assume that $\sigma(x, v', v) = \sigma_i(v', v)$ for some $1 \leq i \leq I$. Let us consider FT^{-1}. Let be g in the unit ball of $L^2(\Omega \times K)$. We denote by $u = T^{-1}g$. Therefore,

$$v \cdot \nabla_x u + \Sigma u = g.$$

This implies that u and $v \cdot \nabla u$ are in $L^2(\Omega \times K)$ since Σ is uniformly bounded from below by $\Sigma_0 > 0$. We use Corollary 5.5 to obtain that the family $Fu = FT^{-1}g$ for u in the unit ball of $L^2(\Omega \times K)$ is relatively compact in $L^2(\Omega \times K)$. This is the definition of the compactness of FT^{-1}. Since the multiplication operator $c_i(x)(\cdot)$ is continuous, we obtain that FT^{-1} is compact as the finite sum of compact operators.

We notice that both operators F and F^* satisfy (106). Moreover, we also deduce from the equation $u = (T^*)^{-1}$ that u and $v \cdot \nabla u$ are in $L^2(\Omega \times K)$. The proof is then completed as above. □

An important property of positive compact operators is the following [3, 8]

Theorem 5.7 (Krein Rutman.) *Let be T a compact operator in $L^2(\Omega \times K)$ such that for all non negative functions $u \geq 0$, we have $Tu \geq 0$ is also non negative. Then there exists $\psi \geq 0$ and $\lambda > 0$ such that $T\psi = \lambda\psi$. Moreover λ is the unique eigenvalue associated with a positive eigenvector. At last, λ is simple and every other eigenvalue μ of T satisfies $|\mu| \leq \lambda$.*

This theorem is a powerful result that allows us to obtain an optimal result of existence for the transport equations:

Theorem 5.8 *Let T and F satisfy the hypotheses of Theorem 5.6. Then the problem*

$$\begin{aligned} Tu &= \mu Fu + q \qquad \text{in } \Omega \times K \\ u &= 0 \qquad \text{on } \Gamma_- \end{aligned} \tag{107}$$

admits a unique solution for all q if and only if $\mu \notin \sigma(T^{-1}F)$, the (point-)spectrum of $T^{-1}F$. Denote by λ the largest positive eigenvalue of $T^{-1}F$. Then for all $\mu < \lambda^{-1}$, (107) admits a unique solution. Moreover the solution u is non negative if q is non negative. It is given by the Neumann series

$$u = \sum_{k=0}^{\infty} (\mu T^{-1}F)^k T^{-1}q. \tag{108}$$

Proof This theorem is simply an application of the Fredholm alternative for positive compact operators, satisfying the hypothesis of Krein Rutman's theorem. □

Remark 5.9 Assume that the medium is conservative or dissipative, i.e. that Hypothesis (H3) is satisfied. We also assume (H1) and (H2) to simplify. Then the a priori estimate (65) holds, and for every $\mu \leq 1$, solutions of

$$u = \mu T^{-1} F u$$

satisfy $u = 0$. This implies that the largest eigenvalue of $T^{-1}F$ satisfies $\lambda < 1$. Even for a bounded conservative medium, i.e. without pure absorption $\Sigma_a = 0$, there is no possible equilibrium due to some leakage at the boundary of the domain. In order to obtain an equilibrium, particles must be created to compensate for the leakage. The eigenvalue λ indicates how much creation is necessary to obtain equilibrium. In this sense, this is a criticality constant for the domain Ω.

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